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REFORMULATION OF REISSNER - MINDLIN PLATE EQUATIONS FOR COMPUTATIONAL EFFICIENCY

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1. ABSTRACT

Finite Elements based on Reissner-Mindlin Plate Theory are becoming more popular than the ones based on Kirchoff-Poisson Plate Theory because of the obvious advantages. These advantages include applicability for a larger range of thickness and inclusion of rotary inertia in dynamics. However such formulations have been plagued by a number of computational problems like slow convergence, shear locking, rank deficiency etc. This article first points out the absence of transpose relation between the $[\psi]$ and $[\partial]$ operator matrices in the governing differential equation $[\psi][D][\partial]\{u\} + \{f\} = 0$ for Timoshenko beam and Reissner-Mindlin plates. Secondly, it contributes a superposition concept by which the state of stress and deformation in a general Reissner-Mindlin plate can be obtained by superposing two primitive states: *Pure Shear state* and *Pure Moment (Bending and Twisting) state.* This concept restores the above mentioned transpose relation and also eliminates the 'shear-locking' problem.

2. INTRODUCTION

The classical theories for beams and plates associated with Euler, Bernoulli, Lagrange, Kirchoff and Poisson (1953) have long history. The important assumptions in these theories are that normal stress in the thickness direction is negligible and a normal to the middle plane remains straight and normal to the deflected middle surface after loading. The assumption regarding the normal restricts the use of these theories to the case of thin beams and plates where the transverse shear deformation can be assumed to be negligible. Refinements to these theories including the effects of shear deformation were provided by Timoshenko (1921) in the case of beams and Reissner (1945) and Mindlin (1951) in the case of plates.

The early finite elements for beams and plates were based on the classical theories. The motivation behind the considerable research efforts in the recent years for using the refined theories are to make the elements applicable for a larger range of thickness/span ratios and also to obviate the analytical difficulties connected with the C^1 continuity of shape functions required in the classical theories by using independent displacement functions for w and θ.

The work of Hughes et. al. (1977) marks one of the earliest attempts at deriving shear-deformable finite elements for beams and plates. The interest was centered on developing elements with minimum number of nodes and degrees of freedom (dofs). Retaining the engineering degrees of freedom w and θ_γ ($\gamma = 1$ for beams and 1,2 for plates) at the nodal points, the authors developed finite elements based on linear shape functions for beams and bilinear shape functions for quadrilateral plate elements. However this attempt was not an unmitigated success. They encountered with the computational malfunction called shear-locking. They suggested heuristically the technique of reduced integration of shear stiffness coefficients to overcome the problem. The technique worked alright for beams. In the case of plates it was found that situations may arise wherein the element with reduced integration may lead to singular structure matrices. A spectral analysis of the quadrilateral plate stiffness matrix showed five zero eigen-values-two more than the usual three rigid body modes. The element is thus shown to be rank-deficient and not reliable in an automated analysis situation.

An astounding amount of research effort has been expended since then on the various aspects of the shear-locking phenomenon including means to overcome it. A stream of researchers put an efforts aimed at developing a shear-deformable element, which in the thin limit will reflect the behaviour as predicted by the classical theories.

The present article describes a technique by which the shear-locking phenomenon is eliminated by attacking the malady at the very source of its origin, namely, by avoiding the direct interaction of shear and flexural stiffnesses in the same stiffness matrix. The technique christened here as Strain-Mode Partitioning Technique (SMPT), in a mathematical sense, can be considered as reformulation of fourth-order differential equations of Timoshenko beam theory or Reissner-Mindlin plate theory to a set of lower order equations by decomposition technique.

3. TONTI DIAGRAM ANALYSIS OF DIFFERENTIAL OPERATORS IN TIMOSHENKO BEAM AND REISSNER-MINDLIN PLATE THEORIES

It is well-known that problems in Structural Mechanics can be formulated in terms of a governing differential equation of the form

$$[\psi][D][\partial]\{u\} + \{f\} = 0 \tag{1}$$

where $[\psi]$ and $[\partial]$ are matrices of differential operators, $[D]$ is a constitutive matrix, $\{u\}$ is the vector of field displacements and $\{f\}$ is the vector of field forces. $[\partial]$ represents the strain-displacement relation

$$\{\epsilon\} = [\partial]\{u\} \tag{2}$$

$[\psi]$ represents the stress-force relation (called equilibrium equation)

$$[\psi]\{\sigma\} = \{f\} \tag{3}$$

The transformations in equations (1), (2) and (3) can be represented in the form of a diagram, called Tonti Diagram (1974 by Tonti, Oden & Reddy) as shown in Fig.1.

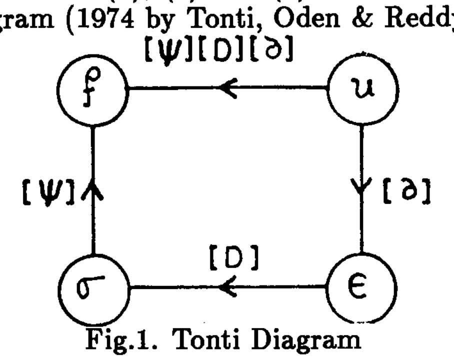


Fig.1. Tonti Diagram

In many problems, a positive or negative transpose relation exists between $[\psi]$ and $[\partial]$. It is well known that in solid elasticity problems

$$[\psi] = -[\partial]^T \tag{4}$$

In case of Kirchoff-Poisson plate theory, this relation can be shown as

$$[\psi] = [\partial]^T \tag{5}$$

That such a relation however does not exist in the case of Timoshenko Beam or Reissner-Mindlin plate theory, will be shown presently.In these theories, lateral deflection w and rotations of the normal θ_γ are the field displacements and the corresponding field forces are distributed load q and body moment m_γ.
where $\gamma = 1$ in the case of Timoshenko beams and
$\gamma = 1, 2$ in the case of Reissner-Mindlin plates

Hence, we can write

$$\Phi = \nabla w - \Theta \tag{6}$$

$$\rho = -L\Theta \tag{7}$$

$$\nabla^T S = -q \tag{8}$$

$$L^T M = S + m_\gamma \tag{9}$$

where S = Shear forces
M = Bending and twisting moments

So

$$\begin{Bmatrix} \Phi \\ \rho \end{Bmatrix} = \begin{bmatrix} \nabla & -I \\ 0 & -L \end{bmatrix} \begin{Bmatrix} w \\ \Theta \end{Bmatrix} \tag{10}$$

and

$$\begin{Bmatrix} q \\ m_\gamma \end{Bmatrix} = \begin{bmatrix} -\nabla^T & 0 \\ -I & L^T \end{bmatrix} \begin{Bmatrix} S \\ M \end{Bmatrix} \tag{11}$$

It can be seen from (10) and (11) there is no transpose retation between the two transformations. Later by changing the field variables it is shown here that the relation can be brought back.

4. MATHEMATICAL BASIS OF REFORMULATION CONCEPT

From (10) and (11) it is possible to write governing differential equations of the Reissner-Mindlin Plate theory as

$$\alpha \nabla^T [\nabla w - \Theta] = -q \tag{12}$$

$$\alpha(\nabla w - \Theta) + (L^T D L)\Theta = -m_\gamma \tag{13}$$

where $\alpha = kGA$ or kGt
k = Shear area factor
G = Shear modulus
A = Area of cross-section of the beam
t = Thickness of the plate

In this reformulation concept, we split the transverse deflection w into a shear component w_s and a bending component w_b as

$$w = w_s + w_b \tag{14}$$

The point of departure in this concept is to make the shear deflection w_s and the normal rotation θ_γ as the primary field variables, as against w and θ_γ usually adopted.

The suggested departure will enable a decoupling of the w and θ terms in the governing differential equations (12) and (13) as will be shown presently.

The bending deflection w_b is related to the normal rotation θ_γ by the Kirchoff compatibility condition.

$$\nabla w_b = \Theta \tag{15}$$

Now equations (12) and (13) reduce respectively to

$$(\nabla^T \nabla) w_s = -\frac{q}{\alpha} \tag{16}$$

$$(L^T DL)\Theta = -\alpha \nabla w_s - m_\gamma \tag{17}$$

These reformulated equations (16) and (17) followed by equation (15) in that sequential order form the basis of this reformulation concept. Equations (16) and (17) are respectively of the Poisson and Navier type.

The terms $\alpha \nabla w_s$ may be recognised as the shear forces S in the plate. Hence by carrying out finite element analysis on the weak forms of these equations, shear forces are obtained in the first stage. These shear forces along with given m_γ form the right-hand sides of finite element equations in the second stage based on equations (16) and (17). The second stage analysis will yield Θ and M.

The calculation of w_b can be done as a part of post-processing from equation (15). It is also seen that the transpose relation not observed in the Reissner-Mindlin plate theory is restored back in this concept.

5. PHYSICAL MEANING OF THE REFORMULATION CONCEPT

This technique can be physically interpreted as an application of the principle of superposition. Considering a beam or plate with its middle plane horizontal and subjected to vertical loads q, the state of deformation and stress in the plate can be obtained by superposing the following two primitive states. The Fig. 2 illustrates the reformulation concept with cantilever beam as an example.

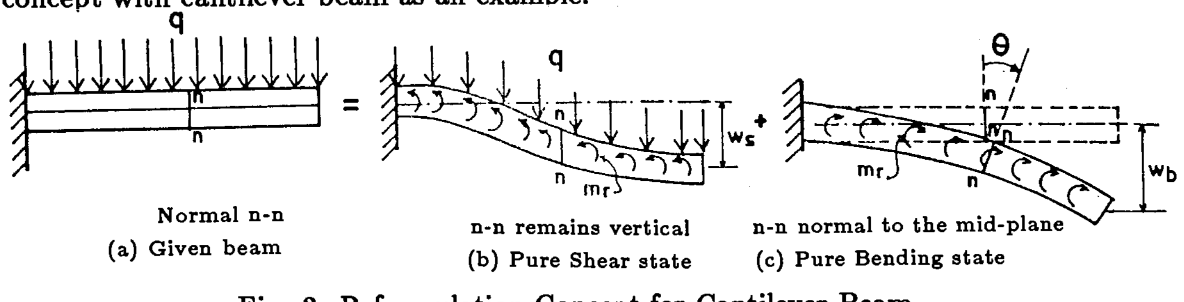


Fig. 2. Reformulation Concept for Cantilever Beam

i) Pure shear state

All normals are restrained to remain vertical by externally applied restraint moments $m_{r\gamma}$ in the domain of the beam or plate. The given transverse loads q are applied. Beams undergoing such a mode of deformation have been known in literature as shear beams and the concept can be easily extended to plates. Whatever deflection is produced now is shear deflection w_s. Internally in the plate only shear strain Φ and associated shear stress resultants S are produced. Θ being zero, the bending strain ρ and the associated stress resultants M are also zero. It is easy to see that

$$\Phi = \nabla w_s; \qquad S = \alpha \nabla w_s; \qquad \nabla S = -q \quad \text{and} \quad \nabla^T \nabla w_s = -\frac{q}{\alpha} \tag{18}$$

From the equations of equilibrium of moments applied to this stage, and assuming the positive direction for restraint moment $m_{r\gamma}$ as opposite to that of m_γ, it can be shown that

$$m_{r\gamma} = S = \alpha \nabla w_s \tag{19}$$

If the prescribed loads include body-moments m_γ, then it gets added to the right-hand side in eqn.(19).

(ii) Pure moment (bending and twisting) state

In this state the restraint moments $m_{r\gamma}$ in (19) which were artificially introduced in the first stage are applied in the opposite direction on the plate. The plate normals are assumed to satisfy Kirchoff normality condition. Hence no shear strains or shear stresses are produced in this state. Equations expressing equilibrium of moments can be shown to be as in (17).

6. NUMERICAL TESTS

(i) Beam element

Numerical experiments were made on beams using linear beam element (SMPL2). The tip loaded cantilever which has been studied extensively in the literature has been taken for illustration.

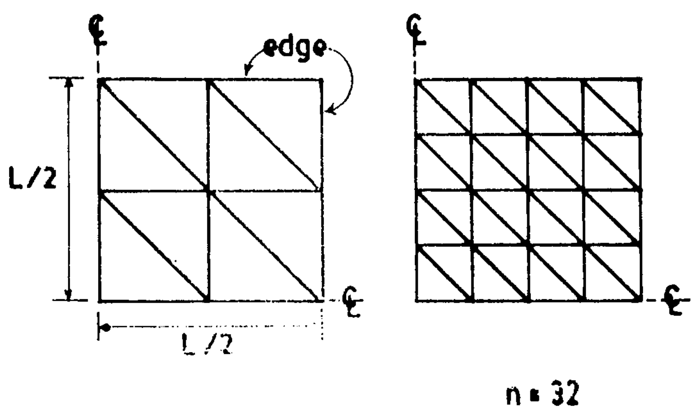
$E = 1000, \quad G = 375, \quad k = \frac{5}{6}, \quad A = 1.0, \quad I = \frac{1}{12},$
$L = 4$ for thick beam and $L = 40$ for thin beam

The study showed the equivalence of SMPL2 element with that of Hughes (1977) in its selectively integrated form and with the Tessler's T1CC4 element (1981). The normalised values of tip displacement and support moment for different discretisations are given in Table 1.

Table 1. Results of Tip-Loaded Cantilever using SMPL2 element

No. of	Thick Beam case		Thin Beam case	
Elems	Tip Disp.	Sup. Moment	Tip Disp.	Sup. Moment
1	0.7619	0.5000	0.7501	0.5000
2	0.9405	0.7500	0.9375	0.7500
4	0.9851	0.8750	0.9844	0.8750
8	0.9963	0.9375	0.9961	0.9375
16	0.9991	0.9688	0.9990	0.9688

(ii) Plate element



$L = 10.0 \quad t = 0.1, 1.0, \quad k = \frac{5}{6}, \quad E = 10.92E05$ and $\nu = 0.3$

Fig. 3. Mesh pattern for square plate under uniformly distributed pressure

Standard patch tests were also carried out on plate problems using linear 3 node triangular plate bending element (SMPP3). This element passes through the patch tests efficiently.

Simply Supported Square Plates under uniformly distributed pressure.

Square plates of size $L \times L$ and thickness t were analysed with values of $\frac{L}{t}$ equal to 10 and 100 under uniformly distributed pressure load. Taking advantage of symmetry a quadrant of the plate of size $\frac{L}{2} \times \frac{L}{2}$ was discretised into triangular elements as shown in Fig. 3. The results of the analysis are shown in Table 2. For the case $\frac{L}{t} = 100$, the results are also compared with the results from the triangular elements reported by Tessler et.al.(1985) and Hughes et.al.(1982).

Table 2. Normalised values of central displacement and bending moment for simply supported square plate under uniformly distributed pressure load.

No.	Thin plate ($L/t = 100$)						Thick($L/t = 10$)	
of	Hughes		Tessler		SMPP3		SMPP3	
Elem.	Disp.	B.M.	Disp.	B.M.	Disp.	B.M.	Disp.	B.M.
8	0.681	0.555	0.806	0.651	0.751	0.762	0.754	0.767
32	0.773	0.539	0.891	0.766	0.933	0.936	0.932	0.943
128	0.827	0.580	0.982	0.953	0.984	0.983	0.982	0.990

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