

BIOMATHEMATICS TRAINING PROGRAM

The Locally Optimal Combination of Certain
Multivariate Test Statistics

by

Katherine Lane Monti

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Most overall tests of hypotheses combining $s \geq 2$ independent statistics are based on the p-values of the individual tests. Van Zwet and Oosterhoff considered procedures for combining tests of univariate hypotheses which are based on the values of the (asymptotically exponentially distributed) statistics themselves. This dissertation deals with combinations of certain multivariate test statistics with distributions which are not in the exponential family.

Many of the classical GLMM analysis of variance statistics have F distributions and can be transformed to beta-distributed random variables. It is shown that if the statistics to be combined have beta distributions which are noncentral under the alternative hypothesis, then a type D test, which provides a locally most powerful combination procedure for testing the null hypothesis does exist. The form of the test is found for the balanced case where the design matrices are proportional. For the unbalanced case, a test which has the best average local power is found; this test is termed a type F test.

It is shown that, for local alternatives, if there are a large number of small-sample statistics to be combined, the type D combination procedure is more efficient than Fisher's method according to Bahadur asymptotic relative efficiency. It is also shown that under the same assumptions, the type F combination procedure is more efficient than Fisher's according to an average measure of Bahadur

asymptotic relative efficiency.

The exact distribution of the overall type D and type F statistics is given if $s = 2$. As s increases, the exact distribution of the statistics becomes more difficult to calculate. The Central Limit Theorem does apply, however, and the accuracy of applying the Central Limit Theorem to find approximate critical values for the overall test statistic is discussed.

The method of applying the type D and type F tests is illustrated in a numerical example based on some pharmaceutical data.

THE LOCALLY OPTIMAL COMBINATION OF CERTAIN
MULTIVARIATE TEST STATISTICS

by

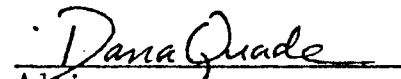
Katherine Lane Monti

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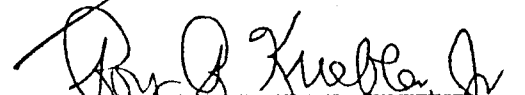
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CHAPTER I

INTRODUCTION

1.1. Statement of the Problem and Outline of the Research

Let Q_1, \dots, Q_s be s independent statistics for testing the same null hypothesis against the same alternative. The problem is to combine the results of these s tests into a single overall test for the common testing problem. Most combination procedures available in the literature are based not on the statistics Q_i but on their probability integral transforms P_i , where $P_i = \Pr(Q_i \geq q_i | H_0)$ where q_i is the observed value of Q_i for $i = 1, \dots, s$. Those combination procedures which have been based on the Q_i 's are mainly limited to the case of univariate hypotheses with test statistics distributed in the one-parameter exponential family. The aim of this dissertation is to extend these latter results to certain multivariate and multiparametric testing problems. It is assumed throughout that the distributions of the statistics are continuous.

Let the hypothesis be

$$H_0: \lambda \sim \lambda_0 \quad (1.1.1)$$

with alternative

$$H_1: \lambda \neq \lambda_0 .$$

Statistics used in the testing of this hypothesis include Hotelling's T^2 statistics and statistics used in conjunction with the general linear

model. If $\tilde{X} = \tilde{A}\tilde{\xi} + \tilde{\epsilon}$, where \tilde{X} is an $N \times m$ matrix of observations, \tilde{A} is an $N \times q$ design matrix, $\tilde{\xi}$ is a $q \times m$ matrix of unknown parameters, and $\tilde{\epsilon}$ is an $N \times m$ error matrix, then the general linear hypothesis is

$$H_0: \tilde{C}\tilde{\xi}\tilde{M} = \tilde{0}$$

with alternative

$$H_1: \tilde{C}\tilde{\xi}\tilde{M} \neq \tilde{0},$$

where \tilde{C} and \tilde{M} are appropriate $g \times q$ and $m \times u$ matrices and $\tilde{0}$ is a $g \times u$ matrix of zeros. If $g=p$ and $u=1$ or $g=1$ and $u=p$, then $\tilde{C}\tilde{\xi}\tilde{M}$ is a $p \times 1$ or $1 \times p$ vector, respectively. In either case, the null hypothesis is of the form specified above. Now assume that \tilde{C} and \tilde{M} are of full rank and let

$$\tilde{H} = \tilde{M}'\tilde{X}'\tilde{A}(\tilde{A}'\tilde{A})^{-1}\tilde{C}'[\tilde{C}(\tilde{A}'\tilde{A})^{-1}\tilde{C}']^{-1}\tilde{C}(\tilde{A}'\tilde{A})^{-1}\tilde{A}'\tilde{X}\tilde{M}$$

and

$$\tilde{E} = \tilde{M}'\tilde{X}'[\tilde{I}-\tilde{A}(\tilde{A}'\tilde{A})^{-1}\tilde{A}']\tilde{X}\tilde{M}.$$

Statistics commonly used in testing (1.1.1) are based on the characteristic roots λ_i , $i = 1, \dots, [\min(g, u)]$ of $|\tilde{H} - \lambda\tilde{E}|$: Roy's largest root ($\max_i \lambda_i$); the Hotelling-Lawley T_0^2 ($\sum_i \lambda_i$); and Wilks' Λ ($\prod_i (1 + \lambda_i)^{-1}$). In the current problem there is only one root, λ , so that Roy's statistic and T_0^2 are identical. Under the null hypothesis and assumptions of normality, $(N-q)\lambda/g = F$ is an F-distributed variate with g and $N-q$ degrees of freedom. The above model can, for the type of hypothesis being considered, be restated as $H_0: \tilde{C}\tilde{\xi}\tilde{M}\tilde{a} = \tilde{0}$ versus $H_1: \tilde{C}\tilde{\xi}\tilde{M}\tilde{a} \neq \tilde{0}$, where \tilde{a} is a non-null vector. Then the common test statistic is $F(\tilde{a}) = (N-q)/g[\tilde{a}'\tilde{H}\tilde{a}/\tilde{a}'\tilde{E}\tilde{a}]$, which has an F distribution with g and $N-q$ degrees of freedom. (Anderson [1958], Morrison [1967]).

Since the F distribution is difficult to work with, it is assumed that the statistics have been transformed to beta-distributed random variables. If F has an F distribution with p and q degrees of

freedom, then $U = (pF/q)/(1+pF/q)$ has a beta distribution with parameters $p/2$ and $q/2$. If F is noncentrally distributed with parameter θ , U has a noncentral beta distribution with the same noncentrality parameter. If F has a noncentral F distribution with noncentrality parameter θ and degrees of freedom p and q , then the density of F is given by

$$\begin{cases} \frac{pe^{-\theta/2}}{q} \sum_{r=0}^{\infty} \frac{\theta^r (pf/q)^{p/2+r-1}}{2^r r! \beta(p/2+r, q/2) (1+pf/q)^{(p+q)/2+r}}, & f > 0; \\ 0, & f \leq 0. \end{cases}$$

When $\theta=0$, the density reduces to that of a central F distribution, given by

$$(p/q)^{p/2} f^{p/2-1} / [\beta(p/2, q/2) (1+pf/q)^{(p+q)/2}]$$

for positive values of f . If F has a noncentral distribution and U is as defined above, then U has a noncentral beta distribution with noncentrality parameter θ and parameters $p/2$ and $q/2$. The density of U is given by

$$\beta_u^\theta = \begin{cases} e^{-\theta/2} \sum_{r=0}^{\infty} \theta^r u^{p/2+r-1} (1-u)^{q/2-1} / [2^r r! \beta(p/2+r, q/2)], & 0 \leq u \leq 1; \\ 0, & \text{otherwise.} \end{cases} \quad (1.1.2)$$

When $\theta=0$, U has a central beta distribution and density

$$\beta_u^0 = u^{p/2-1} (1-u)^{q/2-1} / \beta(p/2, q/2), \quad 0 \leq u \leq 1. \quad (1.1.3)$$

In this dissertation some locally most powerful combinations of beta-distributed statistics used in tests of (1.1.1) are developed. The concepts and notions of type D and type F tests, which maximize different measures of average curvature of the power surface at λ_0 , are

presented in Section 1.3.

In Chapter II the type D test is determined for designs which are balanced as defined in Section 2.3, with details for the two-sample case.

In Chapter III the type F test, which can be applied in the case of an unbalanced design, is determined. It is also shown that if there are a large number of finite-sample statistics to be combined, the test based on the type D region is more efficient than Fisher's test when the comparison is made on the basis of Bahadur asymptotic relative efficiency. Furthermore, it is shown that the type F test is more efficient than Fisher's test when the comparison is made on the basis of a measure of average Bahadur asymptotic relative efficiency. A section mentioning some extensions concludes this chapter.

The two-sample results discussed in Chapter II are difficult to extend to cases involving even a moderately large number of statistics to be combined. However, the overall statistic is a linear combination of the independent statistics to be combined, and the Central Limit Theorem does apply. Finite-sample approximations to this asymptotic result are studied in Chapter IV.

Chapter V includes an example of applications of the theory to some clinical data. Some tables of critical values for the two-sample case are also presented. A final summary chapter concludes the dissertation.

1.2. Review of the Literature

Recall that the problem is to combine the results of s independent tests into a single overall test for the common testing problem.

Let Q_i , $i = 1, \dots, s$, be the test statistics and let P_i , $i = 1, \dots, s$, be their probability integral transforms, as defined in the preceding section. Many of the combination procedures in the literature are based only on the P_i 's. Fisher [1932, pp. 99-101] proposed

$$\chi = -2 \log P_1 \dots P_s = -2 \sum_{i=1}^s \log P_i$$

as an overall test statistic. Under the null hypothesis, χ is distributed as a chi-squared random variable with $2s$ degrees of freedom. Independently of Fisher, K. Pearson [1933] proposed

$$\chi' = -2 \sum_{i=1}^s \log (1-P_i)$$

as an appropriate statistic. Further discussion of these methods is given by E. S. Pearson [1938], Wallis [1942] and Oosterhoff [1969].

Tippett [1931] proposed that the null hypothesis be rejected if the smallest of the P_i 's is sufficiently small. Wilkinson [1951] later extended this test, suggesting that the k -th smallest of the P_i 's be used as the statistic for the overall test, where k is a pre-determined value.

Lipták [1958] proposed combinations of the form

$$\sum_{i=1}^s w_i \phi^{-1}(P_i)$$

where ϕ^{-1} is the inverse of an arbitrary function ϕ and the w_i are suitable weights. Lipták suggested that one natural choice of ϕ is the normal distribution function. If ϕ is taken to be the exponential function, so that ϕ^{-1} is the logarithmic function, this test reduces to a weighted version of Fisher's test, proposed earlier by Good [1955] and applied by Zelen and Joel [1959] in the case where the Q_i are distributed according to an F distribution and $s = 2$. Lancaster [1961] also

discussed adding weights to Fisher's method.

Birnbaum [1954] compared the methods of Fisher, Pearson, and Wilkinson. He found that no method is optimal for all combination problems. However, he showed that if the test statistics have distributions belonging to the exponential class, then Pearson's method and Wilkinson's method with $k > 2$ are inadmissible. (Admissibility is defined and discussed in Section 2.2.) He also noted that while Tippett's method is admissible, Fisher's method seems to be somewhat more uniformly sensitive to commonly chosen alternatives.

Naik [1969] compared Fisher's method to Tippett's method, and Wilk and Shapiro [1968] compared Fisher's method to the mean of the normal transform for empirical studies. In both papers it was noted that Fisher's method is not necessarily optimal for all testing problems.

Littell and Folks [1971] compared Fisher's method, Tippett's method, Wilkinson's method with $k=s$, and Lipták's method, with Φ defined as the normal distribution function, using Bahadur relative efficiency. Assuming that there were a fixed number of large-sample statistics to be combined, the authors showed that Fisher's method is generally at least as efficient as the other methods. However, for some testing problems, not all methods can be distinguished on the basis of Bahadur efficiency and in these cases other methods may be superior. In a later paper, Littell and Folks [1973] showed that, under the same assumption, Fisher's method is at least as good as any other method on the basis of Bahadur relative efficiency.

The problem of finding optimal combination procedures is still open, however, since it is often the case that researchers have sample

sizes insufficient to satisfy the asymptotic arguments of Bahadur efficiency. Furthermore, as noted by Littell and Folks, Bahadur efficiency cannot distinguish Fisher's method from some other methods for certain testing problems.

All of the tests discussed thus far are based on the probability integral transforms of the s test statistics which are to be combined. Various justifications for using the transforms rather than the test statistics themselves have been proposed. Birnbaum suggested that (i) the values of Q_1, \dots, Q_s may not be known to the statistician, (ii) the distributions of the Q_i , which are not necessarily the same for all i , may not be known to the statistician, (iii) although the values and distributions of the statistics may be known, there may be no available overall test based on the values of the statistics themselves, or (iv) available methods may be difficult to apply. Usually, however, the statistician does have the relevant information concerning the statistics to be combined. If the available methods which are based on the values of the statistics themselves are difficult to apply, the statistician may prefer a simpler method based on the probability integral transforms if the loss in power is small. The arguments that combination procedures which do utilize the values of the statistics do not exist may be losing its validity as research in this area proceeds.

Van Zwet and Oosterhoff [1967] and Oosterhoff [1969] proposed procedures for combining Q_1 and Q_2 which are based on the statistics themselves. These authors considered the cases when the distributions are either normal or Student's- t distributions, but noted that their techniques could be applied to derive results if the statistics have distributions belonging to the one-parameter exponential family of

distributions. They proposed three overall tests and compared them with Fisher's test, finding none of the four procedures to be uniformly superior for the problems considered.

Van Elteren [1958] considered linear combinations of two-sample Wilcoxon tests, giving attention to the properties of tests based on two different sets of weights.

Bhattacharya [1961] compared three procedures for combining independent chi-squared statistics for sampling experiments. He found Tippett's method to be, in general, inferior to Fisher's method and to the overall test based on the addition of the s independent chi-squared statistics, but no uniform preference between the latter two methods was observed. The addition of independent chi-squared statistics for contingency table tests to obtain a test over several control groups is a well known procedure (see, for example, Hill [1971]). Mantel [1963] has proposed extensions of the Mantel-Haenszel procedure to various related contingency table problems.

Schaafsma [1968] studied certain aspects of linear combinations of normally distributed statistics. Oosterhoff [1969] reported that Davies also considered procedures to combine normally distributed statistics, investigating β -optimal size- α tests to test the variance of the normal distribution.

In the papers mentioned above, it has generally been assumed that the statistics to be combined have continuous distributions. In this dissertation consideration will be limited to the continuous case. Aspects of the discontinuous case have been discussed by Wallis [1942], Lancaster [1949], E. S. Pearson [1950], and Kincaid [1962].

1.3. Introduction to Type D and Type F Critical Regions

As mentioned earlier, for certain multivariate testing problems type D and type F critical regions provide two locally most powerful tests found by maximizing different measures of the curvature of the power surface at λ_0 . In this section the preliminary notions concerning the type D and type F critical regions are presented.

1.3.1. Type D Critical Regions

Uniformly most powerful tests do not exist for many testing problems, but often locally most powerful tests can be found. For testing $H_0: \lambda = \lambda_0$ against $H_1: \lambda \neq \lambda_0$, where λ is a one-dimensional parameter, the locally best unbiased critical region has been described by Neyman and Pearson [1936]. This so-called type A region of an unbiased test is found by maximizing the curvature of the power curve at the point $\lambda = \lambda_0$, for a fixed size of the test.

Generalizing to a multiparametric testing problem, $H_0: \lambda = \lambda_0$ versus $H_1: \lambda \neq \lambda_0$ where λ is a p -dimensional vector, it is reasonable to attempt to find the locally best unbiased region by maximizing the curvature of the surface of the power function at the point $\lambda = \lambda_0$. Among the class of all unbiased tests of a given size α , the test which has the critical region found by simultaneously maximizing the curvature at every cross-section at the point $\lambda = \lambda_0$ does not exist. Neyman and Pearson [1938] proposed a type C region, for the case $p = 2$. A type C region for a testing problem is defined to be the region with the best local power along a given family of concentric ellipses with the same shape and orientation. If the concentric ellipses are circles, then the type C region is said to be regular; otherwise it is nonregular. However, to construct a type C region it is necessary to know the relative

importance locally of type II errors, and this is not usually known. Furthermore, type C regions are not invariant under transformations which are one-to-one and twice differentiable. Therefore, transformations of regular regions may be nonregular.

To overcome these objections, Isaacson [1951] proposed a type D region, which is defined as the unbiased critical region of size α which maximizes the Gaussian curvature of the power surface at the point $\lambda = \lambda_0$. The type D region can be constructed without knowledge of the type II errors and is invariant under one-to-one, twice differentiable transformations. Geometrically, a type D critical region provides intuitively appealing justifications for its adoption as an extension of a type A critical region. First, the type A critical region maximizes the curvature of the power curve at the point specified in the null hypothesis and the type D region maximizes an average measure of the curvature of the power surface at the point specified in the null hypothesis. Also, since a type A region maximizes the curvature of the power function in an infinitesimal neighborhood of $\lambda = \lambda_0$, the horizontal chord intersecting the power curve at a height $\alpha + \epsilon$ is minimized among all chords so constructed for unbiased tests of size α . Similarly, among all unbiased critical regions of size α , the type D critical region minimizes the area of the region on a horizontal plane at a height of $\alpha + \epsilon$ which is defined by the intersection of the plane and the power surface.

For simplicity, the definition of a type D critical region will be presented in the two-parameter case. The extension to any finite number of parameters is immediate. Adapting the notation of Isaacson, for any region w in the set of all possible critical regions for a test,

the power function is given by

$$B(\underline{\lambda}|\underline{w}) = \Pr(U \in \underline{w}|\underline{\lambda})$$

where $\underline{U} = (U_1, U_2)$ is a sample point in a 2-dimensional sample space.

Here, each U_i is distributed as a beta with parameters b and a and non-centrality $\theta = f(\underline{\lambda} - \underline{\lambda}_0)$ for some function f . Notice that testing

$H_0: \underline{\lambda} = \underline{\lambda}_0$ is equivalent to testing $H_0: \underline{\lambda} - \underline{\lambda}_0 = \underline{0}$, if the alternative

in each instance is that equality does not hold. If the criteria used

in determining a type D region are invariant under translation, there

is no loss in generality in assuming that $\underline{\lambda}_0 = \underline{0}$. Thus the size of the

test associated with the critical region \underline{w} is given by

$$B(\underline{\lambda}_0|\underline{w}) = B((0,0)'|\underline{w}) = \Pr(\underline{U} \in \underline{w} | (0,0)').$$

Now define

$$B_i = B_i(\underline{w}) = \left. \frac{\partial [B((\lambda_1, \lambda_2)'|\underline{w})]}{\partial \lambda_i} \right|_{(\lambda_1, \lambda_2) = (0,0)}, \quad i = 1, 2,$$

$$B_{ij} = B_{ij}(\underline{w}) = \left. \frac{\partial^2 [B((\lambda_1, \lambda_2)'|\underline{w})]}{\partial \lambda_i \partial \lambda_j} \right|_{(\lambda_1, \lambda_2) = (0,0)}, \quad i, j = 1, 2,$$

assuming that the derivatives exist. A necessary condition for unbiasedness is that

$$B_1(\underline{w}) = B_2(\underline{w}) = 0. \quad (1.3.1.1)$$

The Gaussian curvature of the power surface at $\underline{\lambda} = \underline{0}$ is given by

$$\frac{B_{11}(\underline{w}) B_{22}(\underline{w}) - B_{12}^2(\underline{w})}{1 + B_1^2(\underline{w}) + B_2^2(\underline{w})} = \frac{B_{11}(\underline{w}) B_{22}(\underline{w}) - B_{12}^2(\underline{w})}{1 + 0 + 0} = \det(\underline{B}_{\underline{w}}),$$

the determinant of $\underline{B}_{\underline{w}}$, where

$$\underset{\sim}{B}_w = \begin{pmatrix} B_{11}(w) & B_{12}(w) \\ B_{12}(w) & B_{22}(w) \end{pmatrix}.$$

The restriction (1.3.1.1) guarantees that a relative extremum exists at $\lambda = \underline{0}$. In the two-dimensional case, the extremum is a minimum if the Gaussian curvature is positive and if the curvature in all directions is positive, i.e., if $\det(\underset{\sim}{B}_w) > 0$ and $B_{ii} > 0$, $i = 1, 2$ (Taylor and Mann [1972, p. 227]). Notice that these two restrictions are necessary and sufficient conditions that $\underset{\sim}{B}_w$ be positive definite. Together, the conditions that $B_1 = B_2 = 0$ and $\underset{\sim}{B}_w$ is positive definite imply that the test is unbiased. The type D critical region maximizes the Gaussian curvature, $\det(\underset{\sim}{B}_w)$, among all regions w which provide unbiased tests of a specified size. Since the Gaussian curvature is invariant under translation, the above translation of λ_0 to $\underline{0}$ is not restrictive. To summarize:

Definition 1.3.1.1. (Isaacson [1951]). A region w_0 is said to be an unbiased critical region of type D for testing the null hypothesis that $\lambda = \underline{0}$ if:

$$(i) B(0|w_0) = \alpha; \quad (1.3.1.2)$$

$$(ii) B_i(w_0) = 0, \quad i = 1, 2; \quad (1.3.1.3)$$

$$(iii) \underset{\sim}{B}_{w_0} \text{ is positive definite;} \quad (1.3.1.4)$$

$$(iv) \det(\underset{\sim}{B}_{w_0}) \geq \det(\underset{\sim}{B}_w) \text{ for any other region } w \text{ satisfying (i) - (iii).} \quad (1.3.1.5)$$

Returning to the discussion of the geometrical interpretation, consider the power surfaces of size- α unbiased tests of a hypothesis H_0

and the set of regions on a horizontal plane at a height of $\alpha + \epsilon$ which are defined by the intersection of the horizontal plane and the power surface. It will be shown that the region which is formed by the intersection of the plane and the power surface of the test based on the type D critical region has minimal area among all regions in the set. Consider the Taylor series expansion of the power function around $\underline{0}$. Ignore terms of third and higher orders and assume that the derivatives of third order exist. Then

$$\begin{aligned} B((\lambda_1, \lambda_2)' | w) &= B((0, 0)' | w) + \lambda_1 B_{11} + \lambda_2 B_{22} + \frac{1}{2}(\lambda_1^2 B_{11} + 2\lambda_1 \lambda_2 B_{12} + \lambda_2^2 B_{22}) \\ &= \alpha + \frac{1}{2}(\lambda_1^2 B_{11} + 2\lambda_1 \lambda_2 B_{12} + \lambda_2^2 B_{22}) \\ &= \alpha + \frac{1}{2} \lambda' \underset{\sim w}{B} \lambda. \end{aligned}$$

If λ_1 and λ_2 are in an infinitesimal neighborhood of $\underline{0}$, the terms excluded from the expansion are negligible. Since $\underset{\sim w}{B}$ is positive definite, $\epsilon = \frac{1}{2}\delta = \frac{1}{2}\lambda' \underset{\sim w}{B} \lambda > 0$. For a fixed value of ϵ , the area of the ellipse defined by $\delta = \lambda' \underset{\sim w}{B} \lambda$ is given by $\pi\delta / \sqrt{B_{11}B_{22} - B_{12}^2} = \pi\delta / \sqrt{\det(\underset{\sim w}{B})}$. For fixed ϵ , this area is minimized when $\det(\underset{\sim w}{B})$ is maximized. By (1.3.1.5), w_0 is the critical region which maximizes $\det(\underset{\sim w}{B})$ and thus minimizes the area of the ellipse.

To extend the definition to any arbitrary number, p , of parameters, define λ and $\underline{0}$ as $p \times 1$ vectors, increase the range of the index i in (1.3.1.3) to p , and recognize $\underset{\sim w}{B}$ and $\underset{\sim w_0}{B}$ as $p \times p$ matrices. The only point which requires further explanation is the verification that $\underset{\sim w_0}{B}$ being positive definite assures that a minimum exists at the extremum point $\lambda_{\underline{0}} = \underline{0}$. A $p \times p$ matrix $\underset{\sim w_0}{B}$ is positive definite if and only if, for every nonzero p -vector λ , $\lambda' \underset{\sim w_0}{B} \lambda > 0$. Assume that $\lambda = \underline{0}$

is not a point minimum of the power surface. Then there exists a point $\lambda^* = (\lambda_1^*, \dots, \lambda_p^*)$ in an infinitesimal neighborhood of 0 such that

$$B(\lambda^* | w_0) < B(0 | w_0). \quad (1.3.1.6)$$

By a Taylor series expansion, ignoring terms of third and higher orders and assuming that the third order derivatives exist,

$$\begin{aligned} B(\lambda^* | w_0) &= B(0 | w_0) + 0 + \frac{1}{2} (\sum_{i=1}^p \lambda_i^* B_{ii}(w_0) \lambda_i^* + 2 \sum_{\substack{i,j=1 \\ i < j}}^p \lambda_i^* B_{ij}(w_0) \lambda_j^*) \\ &= B(0 | w_0) + \frac{1}{2} (\lambda^*{}' B_{w_0} \lambda^*) \\ &< B(0 | w_0) \end{aligned}$$

by (1.3.1.3) and by assumption (1.3.1.6). The inequality in the last two lines implies that $\lambda^*{}' B_{w_0} \lambda^* < 0$, which is not the case since B_{w_0} is by assumption positive definite. Thus the assumption that $B(\lambda^* | w_0)$ does not attain a minimum at the point $\lambda = 0$ is false, which is the desired conclusion. Therefore, B_{w_0} being positive definite is sufficient to show that the unbiased test achieves a minimum at $\lambda = 0$.

1.3.2. Type F Critical Regions

The type F test is based on the concept of best average power, defined by Wald [1943] as follows:

Definition 1.3.2.1. A critical region w_0 for a testing problem

$H_0: \lambda = 0$ versus $H_1: \lambda \neq 0$ is said to have the uniformly best average power with respect to a family of surfaces $R_c(\lambda)$ and a weight function $\Delta(\lambda)$, if, for any other critical region w of the same size,

$$\int_{R_c(\lambda)} \Pr(w_0 | \lambda) \Delta(\lambda) dA \geq \int_{R_c(\lambda)} \Pr(w | \lambda) \Delta(\lambda) dA$$

where $\int_{R_c(\lambda)} \psi(\lambda) dA$ is the surface integral of $\psi(\lambda)$ over $R_c(\lambda)$.

If the surface is a sphere of radius r , so that $\lambda' \lambda = r^2$, and the function $\Delta(\lambda)$ is a constant for all λ , then it can be shown (in Weyl [1939] and more simply in Roy et al. [1971, p. 25]) that

$$\int_{\lambda' \lambda = r^2} \lambda' B_w \lambda dA \propto \text{tr}(B_w). \quad (1.3.2.7)$$

This implies that if a deviation from the null hypothesis is equally likely in any direction, given $\lambda' \lambda = r^2$, so that there is a uniform distribution over the points on the p -dimensional sphere of radius r with center at the origin, then the expected value of $\lambda' B_w \lambda$, the power of the test at λ , is proportional to the trace of B_w . Since the trace of a positive definite matrix is positive, to maximize the expected or average power of the test subject to the restriction $\lambda' \lambda = r^2$, one needs only to maximize $\text{tr}(B_w)$ with respect to w . Following the notation of Neyman and Pearson and Isaacson, a critical region which provides the test with the best average local power will be termed a type F critical region. (Isaacson [1951] extended the notion of a type D critical region to include nuisance parameters and called such a region a type E region.) To summarize, the following definition is offered:

Definition 1.3.2.2. A region w_0 is said to be an unbiased critical region of type F for testing $H_0: \lambda = 0$ against $H_1: \lambda \neq 0$ if w_0 satisfies

(i) - (iii) of Definition 1.3.1.1 and if

(iv) $\text{tr}(B_{w_0}) \geq \text{tr}(B_w)$ for every other critical region w satisfying (i) - (iii).

Although both the type D and the type F tests provide tests which are locally most powerful on the average, by some means of averaging, to be consistent with Wald's terminology, a type D test will be referred to as a locally most powerful test while a type F test will be termed a test which is locally most powerful on the average.

CHAPTER II

TYPE D CRITICAL REGIONS FOR COMBINING BETA-DISTRIBUTED STATISTICS

2.1. Introduction

In Chapter I a critical region w_0 was defined as an unbiased critical region of type D for testing the null hypothesis that $\lambda = 0$ if w_0 satisfied four conditions. In this chapter the concept of permissibility of a test is introduced and the form of the critical region of permissible tests is considered. By use of the concept of permissibility, it is next shown that a type D critical region does exist for the test procedure combining s independent statistics, U_1, \dots, U_s , where U_1 has a beta distribution. The exact form of the type D critical region is found under the assumption that the design is balanced in the sense that the noncentrality parameters are proportional with known proportionality constants. Under this assumption, expressions for finding critical values of size- α type D tests for combining two statistics are given. Finally, applications of these results to some F-distributed statistics which arise in analysis are mentioned.

2.2. Permissible Tests and Critical Regions

There are two conditions which any reasonable test which combines statistics must satisfy. The first is that the combining test should depend on all of the statistics to be combined. The second condition, given by Birnbaum [1954, p. 562], requires that for every

point in the critical region of the test, every other point which represents a more extreme deviation from the null hypothesis must also be in the critical region. The term "permissible test" is introduced to mean any test which satisfies these two conditions and is formally defined as follows:

Definition 2.2.1. Let U_i , $1 \leq i \leq s$, be s independent statistics for testing a null hypothesis H_0 against an alternative H_1 . Let W be the sample space, a subset of s -dimensional Euclidean space. If H_0 is rejected for large values of U_i , then a test of size α , $0 < \alpha < 1$, for combining U_1, \dots, U_s , with its corresponding critical region w , is permissible if:

(i) w depends on each U_i , $1 \leq i \leq s$, in the sense that

$\Pr(\underline{U} = (U_1, \dots, U_i, \dots, U_s) \in w | U_i; \underline{U} \in W)$ is not constant for all values of U_i ; and

(ii) for any arbitrary point (u_1, \dots, u_s) in w , every point in the set

$$S = \{\underline{u}' = (u'_1, \dots, u'_s) : u'_i \geq u_i, 1 \leq i \leq s; \underline{u}' \in W\} \quad (2.2.1)$$

is also in w .

If H_0 is rejected for small values of U_i , then the definition holds when the inequality in (2.2.1) is reversed.

Since any reasonable combination procedure is permissible, permissibility cannot be used to distinguish effectively between competing procedures. Birnbaum noted, however, that more restrictive conditions which would narrow the number of choices cannot be specified without further assumptions. Having assumed that the densities of the

statistics to be combined all belong to the one-parameter exponential family, he considered properties of the acceptance regions of admissible tests, where admissibility is defined as follows:

Definition 2.2.2. (Birnbaum [1954]). A test of a hypothesis is admissible if there is no test of that hypothesis with the same level of significance which, without being less sensitive to any possible alternative hypothesis, is more sensitive to at least one alternative.

Birnbaum [1955] showed that if the densities of the statistics belong to the exponential family, a necessary condition for the admissibility of the test is that the acceptance region be convex. Since the non-central beta distribution is not a member of the one-parameter exponential family with respect to the noncentrality parameter, Birnbaum's conclusion on the convexity of the acceptance region does not apply. However, the following characterization of permissible critical regions does apply and will be useful in subsequent derivations:

Theorem 2.2.1. Let U_1, \dots, U_s be s independent statistics such that each U_j is distributed continuously over a finite or an infinite interval I_j . Let w be a size α , $0 < \alpha < 1$, critical region for a test $\Psi(U_1, \dots, U_s) = \Psi(\underline{U})$ and assume that this test is permissible. Also assume that the boundary between the rejection and acceptance regions is included in the rejection region. Then for any i , $1 \leq i \leq s$, the boundary between the acceptance and rejection regions can be written

$$u_i = f_i(u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_s) \quad u_j \in I_j, \quad 1 \leq j \leq s,$$

where f_i is a function which, in each of its $(s-1)$ arguments, is non-increasing and is strictly decreasing at at least one point.

Proof. If $\alpha = 0$ or $\alpha = 1$, the f_i are not necessarily as specified; hence the restriction that $0 < \alpha < 1$. Also, since the U_i are continuous, there is no loss in generality in assuming that the boundary is included in the rejection region.

Since the test is permissible, it is clear from Definition 2.2.1 part (ii) that if, for any j , $\underline{u}^{(1)} = (u_1, \dots, u_j, \dots, u_s) \notin w$ but $\underline{u}^{(1)} \in W$ and $\underline{u}^{(2)} = (u_1, \dots, u'_j, \dots, u_s) \in w$, it must be the case that $u'_j > u_j$.

First let $i = 1$. Now assume that the boundary between the acceptance and rejection regions is increasing with respect to some argument, say u_2 . Then it is possible to find $\underline{u}^{(3)} = (u_1, u_2, \dots, u_s) \notin w$, $\underline{u}^{(3)} \in W$, and $\underline{u}^{(4)} = (u_1, u'_2, \dots, u_s) \in w$ where $u'_2 < u_2$. In this instance the test is not permissible; therefore, the boundary cannot increase with u_2 . By the same argument, the boundary does not increase in u_j , $3 \leq j \leq s$. Next assume that the boundary does not depend on u_j for some j , $2 \leq j \leq s$. Then part (i) of Definition 2.2.1 is not satisfied. Therefore, the boundary must depend on each u_j , $2 \leq j \leq s$. Since the boundary depends on each u_j , $2 \leq j \leq s$ and is nonincreasing in each u_j , $2 \leq j \leq s$, it must decrease in each u_j , $2 \leq j \leq s$.

This completes the proof if $i = 1$. Since the above argument can be repeated for $i = 2, \dots, s$, the theorem holds.

2.3. The Existence of a Type D Critical Region for Combining s Independent Statistics with Beta Distributions

In this section the following theorem is proved:

Theorem 2.3.1. Assume that U_i , $1 \leq i \leq s$, are s independent statistics and that each U_i has a beta distribution with parameters b and a_i and

noncentrality parameter θ_i . Also assume that the noncentrality parameters are of the form $\theta_i = \lambda' V_i \lambda$, where each V_i , $1 \leq i \leq s$, is a $p \times p$ positive definite matrix and λ is a $p \times 1$ vector. If each U_i is a statistic for the testing problem defined by (1.1.1), then an unbiased critical region of type D does exist for the overall test of H_0 which combines U_1, \dots, U_s .

Proof. To verify the existence of a type D critical region for the particular problem specified in the theorem, it must be shown that the four conditions (1.3.1.2) - (1.3.1.5) hold for this problem. The first condition specifies the size of the test. Let

$$W = \{(u_1, \dots, u_s) : 0 \leq u_i \leq 1, 1 \leq i \leq s\} \quad (2.3.1)$$

and let w be a critical region, $w \subset W$, such that $B(\underline{0}|w) = \alpha$. Since the densities of the U_i are continuous in W , this condition can be satisfied for any predetermined value of α , $0 < \alpha < 1$. Once the form of the critical region is known, its exact boundary can be found for every such α . The first condition can thus be satisfied.

To verify (1.3.1.3) for the particular problem of this theorem it must be shown that

$$\begin{aligned} B_i(w) &= \left[\partial \int_w \prod_{h=1}^s (\beta_{u_h}^{\theta_h} du_h) / \partial \lambda_i \right] \Big|_{\lambda=0} \\ &= \left\{ \sum_{j=1}^s \left[\partial \int_w \prod_{h=1}^s (\beta_{u_h}^{\theta_h} du_h) / \partial \theta_j \times (\partial \theta_j / \partial \lambda_i) \right] \right\} \Big|_{\lambda=0} = 0, \quad 1 \leq i \leq p, \end{aligned} \quad (2.3.2)$$

where $\beta_{u_h}^{\theta_h}$ is defined in (1.1.2) with θ_h of the form specified in the theorem. For each $i = 1, \dots, p$ and $j = 1, \dots, s$,

$$\begin{aligned} \partial \theta_j / \partial \lambda_i &= \partial (\lambda' v_j \lambda) / \partial \lambda_i = \partial \left(\sum_{m'=1}^p \sum_{m=1}^p \lambda_m \lambda_{m'} v_{mm'}^{(j)} \right) / \partial \lambda_i \\ &= 2 \sum_{m=1}^p \lambda_m v_{mi}^{(j)}, \end{aligned} \quad (2.3.3)$$

where $v_{mm'}^{(j)} = v_{m'm}^{(j)}$ is the mm' -th element of V_j . For any i and j , (2.3.3) is zero if $\lambda = 0$. Therefore, (2.3.2) holds and $B_i(w) = 0$ for all $i = 1, \dots, p$ for all critical regions w .

The third condition to be satisfied requires that the matrix

$$B_{\tilde{w}} = \left(\left[\partial^2 B(\tilde{\lambda} | w) / (\partial \lambda_i \partial \lambda_{i'}) \right] \Big|_{\tilde{\lambda}=0} \right)$$

be positive definite. In the problem being considered here, the ii' -th element of $B_{\tilde{w}}$ is

$$\left[\partial^2 \left\{ \int_w \prod_{h=1}^s (\beta_{u_h}^{\theta_h} du_h) \right\} / (\partial \lambda_i \partial \lambda_{i'}) \right] \Big|_{\tilde{\lambda}=0}. \quad (2.3.4)$$

By expanding the densities $\beta_{u_h}^{\theta_h}$ in their series representations given by (1.1.2), it can be shown (see Appendix A, Lemma A.1) that (2.3.4) equals

$$\begin{aligned} & \left\{ \frac{\partial^2}{\partial \lambda_i \partial \lambda_{i'}} \left[\sum_{r_1=0}^{\infty} \dots \sum_{r_s=0}^{\infty} e^{-\theta_1/2} (\theta_1/2)^{r_1} / (r_1! \beta(b+r_1, a_1)) \right. \right. \\ & \quad \times \dots \times e^{-\theta_s/2} (\theta_s/2)^{r_s} / (r_s! \beta(b+r_s, a_s)) \\ & \quad \left. \left. \times \int_w \prod_{j=1}^s (u_j^{b+r_j-1} (1-u_j)^{a_j-1} du_j) \right] \right\} \Big|_{\tilde{\lambda}=0} \quad (2.3.5) \end{aligned}$$

$$= \left\{ \frac{\partial^2}{\partial \lambda_i \partial \lambda_{i'}} g[w; b; (\theta_i, r_i, a_i), 1 \leq i \leq s] \right\} \Big|_{\tilde{\lambda}=0} \quad (2.3.6)$$

where $g = g[w; b; (\theta_i, r_i, a_i), 1 \leq i \leq s]$ is the term in square brackets in (2.3.5). Upon differentiating, (2.3.6) becomes

$$\left[\frac{\partial}{\partial \lambda_i} \left(\sum_{j=1}^s \frac{\partial g}{\partial \theta_j} \right) \times \frac{\partial \theta_j}{\partial \lambda_i} \right] \Big|_{\lambda=0} + \left[\sum_{j=1}^s \frac{\partial g}{\partial \theta_j} \times \frac{\partial}{\partial \lambda_i} \left(\frac{\partial \theta_j}{\partial \lambda_i} \right) \right] \Big|_{\lambda=0}. \quad (2.3.7)$$

From (2.3.3), the first term of (2.3.7) is zero. Also from (2.3.3),

$$\frac{\partial}{\partial \lambda_i} \left(\frac{\partial \theta_j}{\partial \lambda_i} \right) = \frac{\partial}{\partial \lambda_i} (2 \sum_{m=1}^p \lambda_m v_{mi}^{(j)}) = 2 v_{ii}^{(j)}. \quad (2.3.8)$$

It remains to evaluate $[\partial g / \partial \theta_j] \Big|_{\lambda=0}$, $1 \leq j \leq s$. Ignoring for the

moment the integral, consider

$$\begin{aligned} & \frac{\partial}{\partial \theta_j} \left[\sum_{r_j=0}^{\infty} \frac{e^{-\theta_j/2} (\theta_j/2)^{r_j} u_j^{b+r_j-1} (1-u_j)^{a_j-1}}{r_j! \beta(b+r_j, a_j)} \right] \Big|_{\lambda=0} \\ &= \left[\sum_{r_j=0}^{\infty} (-1/2) \frac{e^{-\theta_j/2} (\theta_j/2)^{r_j} u_j^{b+r_j-1} (1-u_j)^{a_j-1}}{r_j! \beta(b+r_j, a_j)} \right. \\ & \quad \left. + \sum_{r_j=1}^{\infty} e^{-\theta_j/2} (r_j/2) (\theta_j/2)^{r_j-1} u_j^{b+r_j-1} (1-u_j)^{a_j-1} \right] \Big|_{\lambda=0} \\ &= [-1/2 + 1/2 \left(\frac{a_j+b}{b} \right) u_j] \beta_{u_j}^0 \end{aligned} \quad (2.3.9)$$

since

$$[(a_j+b)/b] / \beta(b, a_j) = 1 / \beta(b+1, a_j). \quad (2.3.10)$$

In light of (2.3.5) - (2.3.9), (2.3.4) can be rewritten

$$\sum_{j=1}^s v_{ii'}^{(j)} \int_w \phi_j \prod_{h=1}^s (\beta_{u_h}^0 du_h)$$

where $\phi_j = -1 + (a_j + b)u_j/b$. Therefore

$$\begin{aligned} B_{\sim w} &= \sum_{j=1}^s v_{\sim j} \int_w \phi_j \prod_{h=1}^s (\beta_{u_h}^0 du_h) \\ &= \sum_{j=1}^s v_{\sim j} \times C_j(w) \end{aligned} \quad (2.3.11)$$

$$\text{where } C_j(w) = \int_w \phi_j \prod_{h=1}^s (\beta_{u_h}^0 du_h).$$

In order to show that $B_{\sim w}$ is positive definite, it suffices to show that the coefficients $C_j(w) > 0$, $1 \leq j \leq s$, since the matrices $v_{\sim j}$ are by assumption positive definite. Let $j = 1$ and restrict consideration to the two-sample case. Then it can be shown that

$$C_1(w) = \int_w \beta_{u_1}^0 \beta_{u_2}^0 \phi_1 du_1 du_2 > 0$$

whenever w is a permissible critical region. By rewriting (2.3.11), it can be seen that

$$\begin{aligned} C_1(w) &= \int_w \beta_{u_1}^0 \beta_{u_2}^0 (-1 + (a_1 + b)u_1/b) du_1 du_2 \\ &= \int_{[u_2]} \beta_{u_2}^0 \left[\int_{[u_1|u_2]} (\beta_{u_1}^0 - \beta_{u_1}^{0*}) du_1 \right] du_2 \end{aligned} \quad (2.3.12)$$

where $\beta_{u_1}^{0*}$ denotes the density of a random variable U_1 with a central beta distribution with parameters $(b+1)$ and a_1 , where $[u_2]$ denotes the lower bound of u_2 , and where $[u_1|u_2]$ denotes the lower bound of u_1 given u_2 . If w is permissible, then by Theorem 2.2.1, the region of integration can be expressed in this manner as long as $0 < \alpha < 1$. Assume that $[u_1|u_2] \equiv 0$. Then the inner integral of (2.3.12) becomes the integral over the range of positive density of the difference of two

density functions, which is zero. Since $a_1 + b > b > 0$, $-1 + (a_1+b)u_1/b$ is a strictly increasing function of u_1 which attains its minimum at $u_1 = 0$ if $0 \leq u_1 \leq 1$. It can, therefore, be concluded that

$$I(u_2) = \int_{[u_1|u_2]}^1 \beta_{u_1}^0 \phi_1 du_1 \geq \int_0^1 \beta_{u_1}^0 \phi_1 du_1 = \int_0^1 (\beta_{u_1}^0 - \beta_{u_1}^{0*}) du_1 = 0$$

with strict equality only if $[u_1|u_2] \equiv 0$. However, if $0 < \alpha < 1$ and w is a permissible critical region, $[u_1|u_2]$ must exceed zero for values u_2 in some interval. It thus follows that for any permissible critical region w of size α , $0 < \alpha < 1$, $I(u_2) > 0$. Since $\beta_{u_2}^0$ is positive in $([u_2], 1)$,

$$C_1(w) = \int_{[u_2]}^1 \beta_{u_2}^0 I(u_2) du_2 > 0.$$

Similarly $C_2(w) > 0$. Therefore, $B_{\sim w}$ is positive definite when $s = 2$. The demonstration that these coefficients are positive in the general s -sample case follows in identical fashion.

Finally, it must be shown that there exists a size- α critical region w_0 such that

$$\begin{aligned} \det(B_{\sim w_0}) &= \det[\sum_{i=1}^s V_{\sim i} \times C_i(w_0)] \\ &\geq \det[\sum_{i=1}^s V_{\sim i} \times C_i(w)] = \det(B_{\sim w}) \end{aligned}$$

for all other size- α critical regions w . Since all matrices $B_{\sim w}$ of the form $\sum_{i=1}^s V_{\sim i} \times C_i(w)$ are positive definite, $\det(B_{\sim w})$ is a real positive-values function of $B_{\sim w}$. Also, $B_{\sim w}$ is not a constant for all w since $C_i(w)$ is not constant. It follows that $\det(B_{\sim w})$ can be maximized and, therefore, the final condition can be satisfied. This completes the proof of Theorem 2.3.1.

Although it has been shown that a type D critical region does exist, it is not in general possible to find this region. Since the determinant function is not a linear operator, there is no simple method of finding

$$\max_w [\det(\sum_{i=1}^s \tilde{V}_i \times C_i(w))]. \quad (2.3.13)$$

In Section 2.4 it will be shown that the type D critical region can be found if the design is balanced in the sense that $\tilde{V}_i = \gamma_i V$, $1 \leq i \leq s$, where the γ_i are scalars. The case of an unbalanced design will be considered in the third chapter.

2.4. The Type D Critical Region: Balanced Design

In this section the following theorem, specifying the form of the type D critical region in the case of a balanced design, is proved:

Theorem 2.4.1. If the conditions of Theorem 2.3.1 hold and if $\tilde{V}_i = \gamma_i V$, $1 \leq i \leq s$, where the γ_i are known positive constants, then the type D critical region of size α for the procedure combining U_1, \dots, U_s is given by

$$w_0 = \{(u_1, \dots, u_s) : \sum_{i=1}^s \gamma_i (a_i + b) u_i / b \geq k'; 0 \leq u_i \leq 1, 1 \leq i \leq s\}, \quad (2.4.1)$$

where k' depends on α .

Proof. Let $C_w = \sum_{i=1}^s \gamma_i C_i(w)$. Then by (2.3.11), if $\tilde{V}_i = \gamma_i V$,

$$\begin{aligned} \max_w [\det(B_w)] &= \det(V) \times [\max_w (\sum_{i=1}^s \gamma_i C_i(w))]^p \\ &= \det(V) \times [\max_w (C_w)]^p. \end{aligned}$$

Since γ_i and $C_i(w)$ are all positive, C_w is positive. Thus the type D critical region is found by maximizing C_w . This maximum can be found by applying the Neyman-Pearson Fundamental Lemma, stated here without proof.

Lemma 2.4.1. (The Neyman-Pearson Fundamental Lemma). Suppose $m + 1$ given integrable functions g_0, g_1, \dots, g_m are defined in an s -dimensional space. Consider the set of all regions w for which the following hold:

$$\int_w g_i \prod_{j=1}^s du_j = c_i, \quad i = 1, \dots, m \quad (2.4.2)$$

where the c_i are m given constants. If w_0 is a region which satisfies (2.4.2) and if

$$\begin{cases} g_0 \geq \sum_{i=1}^m k_i g_i & \text{in } w_0, \\ g_0 \leq \sum_{i=1}^m k_i g_i & \text{outside } w_0, \end{cases} \quad (2.4.3)$$

for m chosen constants k_i , then w_0 has the property that

$$\int_{w_0} g_0 \prod_{j=1}^s du_j \geq \int_w g_0 \prod_{j=1}^s du_j$$

for any region w which satisfies (2.4.2). (Isaacson [1951]).

Let $m = 1$, $g_0 = \left(\prod_{j=1}^s \beta_{u_j}^0 \right) \left(\sum_{i=1}^s \gamma_i \phi_i \right)$, and $g_1 = \prod_{j=1}^s \beta_{u_j}^0$, so that $c_1 = \alpha$.

If $k_1 = \left(-\sum_{i=1}^s \gamma_i \right) + k'$, then $g_0 \geq k_1 g_1$ if and only if

$\sum_{i=1}^s \gamma_i (a_i + b) u_i / b \geq k'$. Since the reverse inequalities also hold

simultaneously, (2.4.3) holds for k_1 as defined above. Therefore, the type D critical region w_0 , the critical region which maximizes C_w , is

as specified in the theorem. This completes the proof.

2.5. The Type D Critical Region for a Size- α Combination Procedure in the Two-Sample Case

In the previous section the form of the type D critical region w_0 was found. Given α , the problem is to determine k' in (2.4.1) such that

$$\int_{w_0} \prod_{j=1}^s (\beta_{u_j}^{\circ} du_j) = \alpha.$$

In this section this problem is handled in the two-sample case. For simplicity replace k' by $k = k'b/[\gamma_2(a_2+b)]$ so that

$$w_0 = \{(u_1, u_2): fu_1 + u_2 \geq k; 0 \leq u_i \leq 1, i = 1, 2\}$$

where $f = [\gamma_1(a_1+b)]/[\gamma_2(a_2+b)]$. The purpose is to find k such that for a given α ,

$$\int_{fu_1 + u_2 \geq k} u_1^{b-1}(1-u_1)^{a_1-1} u_2^{b-1}(1-u_2)^{a_2-1} / [\beta(b, a_1)\beta(b, a_2)] du_1 du_2 = \alpha. \quad (2.5.1)$$

By Taylor series expansions of the terms in the integrand, several different expressions for the evaluated integral are found. All of the expressions involve double summations which have finite or infinite numbers of terms depending on which, if any, of the parameters a_1 , a_2 , and b are integers. It is of practical interest to use the expression which has the fewest infinite summations for a given set of parameters, and it can be seen that each of the different expressions found is the most useful in certain instances.

At the onset it is useful to notice that the type D critical region is one of three different types, illustrated in Figure 2.5.1. The boundary between the acceptance and rejection regions is a segment of the line $U_1 = (k-U_2)/f$ or, equivalently, $U_2 = k - fU_1$.

For simplicity, in the following discussion let $\beta_i = \beta(b, a_i)$ and let W be as defined in (2.3.1) with $s = 2$. Also, if x is a real positive number, denote

$$x^* = \begin{cases} x, & \text{if } x \text{ is an integer;} \\ \infty, & \text{otherwise.} \end{cases}$$

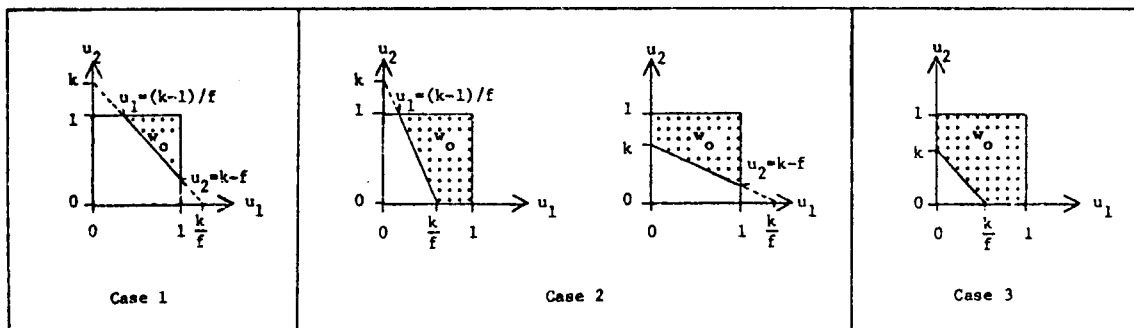


Fig. 2.5.1. Type D Critical Regions for Combining Two Beta-Distributed Statistics

Again for simplicity, some of the lemmas in Appendix A are used repeatedly without specific reference. In particular, Lemma A.2 justifies the interchanging of the order of iterated integration and Lemma A.3 justifies interchanging the order of summation and integration for this problem. The expansions and results of other lemmas are also used without reference.

Expressions relating k and α are now found for each of the three types of area of integration.

2.5.1. Case 1

From Figure 2.5.1 it can be seen that in Case 1, $0 \leq k-f < 1$, $0 \leq (k-1)/f < 1$, $1 \leq k/f$, and $1 \leq k$. Under the assumption that w_0 is such that these conditions hold, an expression for (2.5.1) which is

useful when b is an integer is first found. It is later shown that this expression is also useful if none of the parameters, a_1 , a_2 , and b , is an integer. Expressions which are more practical when a_1 , a_2 , or both are integers are also found.

First write

$$\begin{aligned} \int_{w_0} \beta_{u_1}^{\circ} \beta_{u_2}^{\circ} du_1 du_2 &= 1 - \int_{W-w_0} \beta_{u_1}^{\circ} \beta_{u_2}^{\circ} du_1 du_2 \\ &= 1 - \left[\int_0^{(k-1)/f} \beta_{u_1}^{\circ} \left(\int_0^1 \beta_{u_2}^{\circ} du_2 \right) du_1 \right. \\ &\quad \left. + \int_0^1 \beta_{u_1}^{\circ} \left(\int_0^{k-fu_1} \beta_{u_2}^{\circ} du_2 \right) du_1 \right]. \end{aligned} \quad (2.5.1.1)$$

In the first term in brackets, the inner integral is unity, leaving

$$\int_0^{(k-1)/f} \beta_{u_1}^{\circ} du_1 = I_{(k-1)/f}(b, a_1) \quad (2.5.1.2)$$

where $I_{(k-1)/f}(b, a_1)$ is the incomplete beta function ratio. (Tables available by Pearson [1968]. For other references see Johnson and Kotz, Vol. 2 [1970, pp. 48-51].) The remaining term in (2.5.1.1) can be evaluated by setting $u_2^{b-1} = (1-(1-u_2))^{b-1}$ and proceeding as follows:

$$\begin{aligned} &\frac{1}{\beta_1 \beta_2} \int_0^1 \beta_{u_1}^{b-1} (1-u_1)^{a_1-1} \int_0^{k-fu_1} \beta_{u_2}^{b-1} (1-u_2)^{a_2-1} du_2 du_1 \\ &= \frac{1}{\beta_1 \beta_2} \int_0^1 \beta_{u_1}^{b-1} (1-u_1)^{a_1-1} \\ &\quad \times \sum_{i=0}^{(b-1)^*} \binom{b-1}{i} (-1)^i \int_0^{k-fu_1} (1-u_2)^{a_2+i-1} du_2 du_1 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\beta_1 \beta_2} \sum_{i=0}^{(b-1)^*} \binom{b-1}{i} \frac{(-1)^i}{a_2+i} \\
&\quad \times \int_{(k-1)/f}^1 u_1^{b-1} (1-u_1)^{a_1-1} [1-(1-(k-fu_1))^{a_2+i}] du_1.
\end{aligned} \tag{2.5.1.3}$$

The integral remaining to evaluate in (2.5.1.3) can be considered as the difference of two integrals, the first of which is

$$\int_{(k-1)/f}^1 u_1^{b-1} (1-u_1)^{a_1-1} du_1 = \beta_1 [1 - I_{(k-1)/f}(b, a_1)]. \tag{2.5.1.4}$$

Using Lemma A.6 the second integral can be evaluated as follows:

$$\begin{aligned}
&\int_{(k-1)/f}^1 u_1^{b-1} (1-u_1)^{a_1-1} (1-k+fu_1)^{a_2+i} du_1 \\
&= \sum_{j=0}^{(b-1)^*} \binom{b-1}{j} (-1)^j \int_{(k-1)/f}^1 (1-u_1)^{a_1+j-1} \left(u_1 - \frac{k-1}{f}\right)^{a_2+i} f^{a_2+i} du_1 \\
&= f^{a_2+i} \sum_{j=0}^{(b-1)^*} \binom{b-1}{j} (-1)^j \left(1 - \frac{k-1}{f}\right)^{a_2+i+a_1+j} \beta(a_1+j, a_2+i+1) \\
&= (1-k+f)^{a_2+i} \sum_{j=0}^{(b-1)^*} \binom{b-1}{j} (-1)^j \left(1 - \frac{k-1}{f}\right)^{a_1+j} \beta(a_1+j, a_2+i+1).
\end{aligned} \tag{2.5.1.5}$$

Since both $(k-f)$ and $(k-1)/f$ are between zero and one in Case 1, neither $(1-k+f)$ nor $(1 - \frac{k-1}{f})$ increases as its exponent increases. Combine (2.5.1.4) and (2.5.1.5) and apply Lemma A.5 to show that (2.5.1.3) is equal to

$$\begin{aligned}
1 - I_{(k-1)/f}(b, a_1) &= \frac{1}{\beta_1 \beta_2} \sum_{i=0}^{(b-1)^*} \binom{b-1}{i} \frac{(-1)^i}{a_2+i} (1-k+f)^{a_2+i} \\
&\times \sum_{j=0}^{(b-1)^*} \binom{b-1}{j} (-1)^j \left(1 - \frac{k-1}{f}\right)^{a_1+j} \beta(a_1+j, a_2+i+1). \quad (2.5.1.6)
\end{aligned}$$

These series converge in light of Lemma A.7. Substitute (2.5.1.2) and (2.5.1.6) into (2.5.1.1) and cancel terms where possible. Then it is seen that (2.5.1.1) can be written

$$\begin{aligned}
\frac{1}{\beta_1 \beta_2} \sum_{i=0}^{(b-1)^*} \binom{b-1}{i} \frac{(-1)^i}{a_2+i} (1-k+f)^{a_2+i} \\
\times \sum_{j=0}^{(b-1)^*} \binom{b-1}{j} (-1)^j \left(1 - \frac{k-1}{f}\right)^{a_1+j} \beta(a_1+j, a_2+i+1). \quad (2.5.1.7)
\end{aligned}$$

If b is not an integer but either a_1 or a_2 or both are integers, another expansion of (2.5.1) may have fewer terms and thus be more useful. Consider again the final term in (2.5.1.1). It will now be evaluated by expanding the $(1-u_i)^{a_i}$ terms rather than the u_i^{b-1} terms as before:

$$\begin{aligned}
\frac{1}{\beta_1 \beta_2} \int_{(k-1)/f}^1 u_1^{b-1} (1-u_1)^{a_1-1} \int_0^{k-fu_1} u_2^{b-1} (1-u_2)^{a_2-1} du_2 du_1 \\
= \frac{1}{\beta_1 \beta_2} \int_{(k-1)/f}^1 u_1^{b-1} (1-u_1)^{a_1-1} \\
\times \sum_{i=0}^{(a_2-1)^*} \binom{a_2-1}{i} (-1)^i \int_0^{k-fu_1} u_2^{b+i-1} du_2 du_1
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\beta_1 \beta_2} \sum_{i=0}^{(a_2-1)^*} \binom{a_2-1}{i} \frac{(-1)^i}{b+i} \int_{(k-1)/f}^1 u_1^{b-1} (1-u_1)^{a_1-1} (k-fu_1)^{b+i} du_1 \\
&= \frac{1}{\beta_1 \beta_2} \sum_{i=0}^{(a_2-1)^*} \binom{a_2-1}{i} \frac{(-1)^i}{b+i} \sum_{j=0}^{(a_1-1)^*} \binom{a_1-1}{j} (-1)^j \\
&\quad \times \int_{(k-1)/f}^1 u_1^{b+j-1} (k-fu_1)^{b+i} du_1. \tag{2.5.1.8}
\end{aligned}$$

Let $y = fu_1/k$ then apply Lemma A.5 to find that the remaining integral in (2.5.1.8) equals

$$k^{b+i} \left\{ (k/f)^{b+j} [\beta_{f/k}(b+j, b+i+1) - \beta_{1-1/k}(b+j, b+i+1)] \right\}. \tag{2.5.1.9}$$

where $\beta_x(c, d) = \beta(c, d) \times I_x(c, d)$ is the incomplete beta function. From the proof of Lemma A.7 the term in square brackets is bounded and even if the summation over j is infinite, that series converges. However, since $k \geq 1$ in Case 1, a_2 must be integral to assure the convergence of the series summed over i . By combining (2.5.1.1), (2.5.1.2), (2.5.1.8), and (2.5.1.9) and applying Lemma A.9, a second expansion for the integral over the critical region is found:

$$\begin{aligned}
I_{1-(k-1)/f}(a_1, b) &= \frac{1}{\beta_1 \beta_2} \sum_{i=0}^{a_2-1} \binom{a_2-1}{i} \frac{(-1)^i}{b+i} k^{b+i} \\
&\quad \times \sum_{j=0}^{(a_1-1)^*} \binom{a_1-1}{j} (-1)^j (k/f)^{b+j} \\
&\quad \times [\beta_{f/k}(b+j, b+i+1) - \beta_{1-1/k}(b+j, b+i+1)]. \tag{2.5.1.10}
\end{aligned}$$

If a_2 is not an integer but a_1 is an integer, the order of integration of U_1 and U_2 can be reversed. Essentially the same steps are

performed to find that, for integral a_1 , (2.5.1.1) can be written

$$I_{1-(k-f)}(a_2, b) = \frac{1}{\beta_1 \beta_2} \sum_{i=0}^{a_1-1} \binom{a_1-1}{i} \frac{(-1)^i}{b+i} (k/f)^{b+i} \sum_{j=0}^{(a_2-1)^*} \binom{a_2-1}{j} (-1)^j \times [k^{b+j} \{\beta_{1/k}(b+j, b+i+1) - \beta_{1-f/k}(b+j, b+i+1)\}]. \quad (2.5.1.11)$$

By Lemma A.7, the term in square brackets is bounded and the j -indexed series converges.

If neither a_1 nor a_2 is an integer, other expressions for

$$\int_{w_0} \beta_{u_1}^0 \beta_{u_2}^0 du_1 du_2 = \int_{fu_1+u_2 \geq k} \beta_{u_1}^0 \beta_{u_2}^0 du_1 du_2 \quad (2.5.1.12)$$

can be found. They are, however, no improvement over using (2.5.1.7) when $(b-1)^* = \infty$.

Three expressions, (2.5.1.7), (2.5.1.10), and (2.5.1.11), have been found which, given a_1 , a_2 , b , and f , can be used to find the k such that (2.5.1.12) equals some predetermined value α whenever w_0 is of the form considered in Case 1. It is not, in general, possible to solve for k in terms of α , so iterative solutions must be relied on.

The results of this section are formalized in the following theorem:

Theorem 2.5.1.1. For each $i = 1, 2$, let $\beta_{u_i}^0$ be the density of a random variable U_i which is distributed according to a central beta distribution with parameters b and a_i . Let w_0 be a critical region, given by

$$w_0 = \{(u_1, u_2): fu_1 + u_2 \geq k; \quad 0 \leq u_i \leq 1, \quad i = 1, 2\}$$

where f is a known constant. If, for a given $\alpha \in (0, 1)$, (2.5.1.12) equals α and if $k \geq 1$ and $k/f \geq 1$, then alternative evaluations of (2.5.1.12) are given by (2.5.1.7), (2.5.1.10) and (2.5.1.11). If either

(i) b is an integer and at least one of (a_1, a_2) is not an integer, or
(ii) neither a_1 , a_2 or b is an integer, then (2.5.1.7) is the most useful of the three expressions. If b is not an integer but either a_1 or a_2 is an integer, then (2.5.1.11) and (2.5.1.10) are more useful than (2.5.1.7); expressions (2.5.1.10) and (2.5.1.11) both apply if a_1 and a_2 are both integers. If all three parameters are integers, all of the three expressions are applicable and all have a finite number of terms.

2.5.2. Case 2

In Case 2 there are essentially two subcases, as illustrated in Figure 2.5.1. Only the first alternative will be considered here, since the second subcase can be transformed into the first by resubscripting and proceeding accordingly. "Case 2" will now be used to refer specifically to the first subcase, where $0 \leq (k-1)/f < 1$, $0 \leq k/f \leq 1$, and $1 \leq k$.

In this case

$$\int_{w_0} \beta_{u_1}^0 \beta_{u_2}^0 du_1 du_2 = 1 - \left[\int_0^{(k-1)/f} \beta_{u_1}^0 \int_0^1 \beta_{u_2}^0 du_2 du_1 \right. \\ \left. + \int_{(k-1)/f}^{k/f} \beta_{u_1}^0 \int_0^{k-fu_1} \beta_{u_2}^0 du_2 du_1 \right].$$

This expression is exactly (2.5.1.1) except that k/f replaces 1 as the upper limit of the outer integral in the last term. The first term in brackets was evaluated in the previous section, so it remains only to evaluate the second term in brackets and combine terms. Since the techniques of integration used do not differ from those in the previous section, the details are omitted.

Corresponding to (2.5.1.7) is the expression

$$\begin{aligned}
 I_{1-k/f}(a_1, b) &+ \frac{1}{\beta_1 \beta_2} \sum_{i=0}^{(b-1)^*} \binom{b-1}{i} \frac{(-1)^i}{a_2+i} (1-k+f)^{a_2+i} \\
 &\times \sum_{j=0}^{(b-1)^*} \binom{b-1}{j} (-1)^j \left(1 - \frac{k-1}{f}\right)^{a_1+j} \\
 &\times \beta_{1/(1-k+f)}(a_2+i+1, a_1+j). \tag{2.5.2.1}
 \end{aligned}$$

In Case 2, $0 \leq (k-1)/f < 1$ so $(1 - (k-1)/f)^{a_1+j}$ does not increase with j .

However, $k/f \leq 1$ so $1 \leq (1-k+f)$ and thus $(1-k+f)$ may increase with i .

By Lemma A.8 the i -indexed series converges anyway. Corresponding to

(2.5.1.10) and (2.5.1.11) respectively are

$$\begin{aligned}
 I_{1-(k-1)/f}(a_1, b) &- \frac{1}{\beta_1 \beta_2} \sum_{i=0}^{(a_2-1)^*} \binom{a_2-1}{i} \frac{(-1)^i}{b+i} k^{b+i} \\
 &\times \sum_{j=0}^{(a_1-1)^*} \binom{a_1-1}{j} (-1)^j (k/f)^{b+j} \beta_{1/k}(b+i+1, b+j) \tag{2.5.2.2}
 \end{aligned}$$

and

$$\begin{aligned}
 1 - \frac{1}{\beta_1 \beta_2} \sum_{i=0}^{(a_1-1)^*} \binom{a_1-1}{i} \frac{(-1)^i}{b+i} (k/f)^{b+i} \\
 \times \sum_{j=0}^{(a_2-1)^*} \binom{a_2-1}{j} (-1)^j k^{b+j} \beta_{1/k}(b+j, b+i+1). \tag{2.5.2.3}
 \end{aligned}$$

Lemmas A.8 and A.7 can be used to show the convergence of these series

when $k/f \leq 1$ and $k \geq 1$. To summarize:

Theorem 2.5.2.1. Let U_i , $\beta_{u_i}^0$, and w_0 be defined as in Theorem 2.5.1.1.

If, for a given $\alpha \in (0, 1)$, (2.5.1.12) equals α and if $0 \leq k/f \leq 1$ and

$k \geq 1$, then alternative evaluations for (2.5.1.12) are given by (2.5.2.1),

(2.5.2.2) and (2.5.2.3). None of these expressions restricts any of the

parameters to integral values, although (2.5.2.1) has the fewest terms if b is an integer and at least one of a_1 and a_2 is not an integer, (2.5.1.2) has the fewest terms if only a_2 is an integer and (2.5.1.3) has the fewest terms if only a_1 is an integer.

As before, iterative solutions of k must be used since k cannot be found explicitly for a given α .

2.5.3. Case 3

In Case 3, $0 \leq k \leq 1$ and $0 \leq k/f \leq 1$. Also in this case, the integral in (2.5.1) can be written

$$\begin{aligned}
& \frac{1}{\beta_1 \beta_2} \int_0^{k/f} u_1^{b-1} (1-u_1)^{a_1-1} \int_{k-fu_1}^1 u_2^{b-1} (1-u_2)^{a_2-1} du_2 du_1 \\
& + \frac{1}{\beta_1 \beta_2} \int_{k/f}^1 u_1^{b-1} (1-u_1)^{a_1-1} \int_0^1 u_2^{b-1} (1-u_2)^{a_2-1} du_2 du_1 \\
& = \frac{1}{\beta_1 \beta_2} \int_{1-k/f}^1 y_1^{a_1-1} (1-y_1)^{b-1} \int_0^{1-k+f(1-y_1)} y_2^{a_2-1} (1-y_2)^{b-1} dy_2 dy_1 \\
& + \frac{1}{\beta_1} \int_0^{1-k/f} y_1^{a_1-1} (1-y_1)^{b-1} dy_1 \\
& = \frac{1}{\beta_1 \beta_2} \int_{(k'-1)/f}^1 y_1^{a_1-1} (1-y_1)^{b-1} \int_0^{k'-fy_1} y_2^{a_2-1} (1-y_2)^{b-1} dy_2 dy_1 \\
& + \frac{1}{\beta_1} \int_0^{(k'-1)/f} y_1^{a_1-1} (1-y_1)^{b-1} dy_1 \tag{2.5.3.1}
\end{aligned}$$

where $k' = (1-k+f)$. If (2.5.3.1) and the term in square brackets in (2.5.1.1) are compared it is seen that they are identical except that

(i) U_i has a beta distribution with parameters b and a_i while Y_i has a beta distribution with parameters a_i and b and (ii) k has been replaced by k' . Let

$$\int_{w_0^{(i)}} \beta_{u_1}^0 \beta_{u_2}^0 du_1 du_2 = I_i [k; (b, a_1), (b, a_2), f]$$

where $I_i [k; (b, a_1), (b, a_2), f]$ denotes an expansion of the integral if $w_0 = w_0^{(i)}$ is a critical region which is covered in Case i, given the parameters of the beta distributions and f . Then the above comparison of (2.5.3.1) and (2.5.1.1) reveals that

$$I_3 [k'; (a_1, b), (a_2, b), f] = 1 - I_1 [k; (b, a_1), (b, a_2), f].$$

The appropriate substitutions in (2.5.1.7) and (2.5.1.10) give rise to the following expressions, which are valid if the critical region is of the form considered in Case 3:

$$1 - \frac{1}{\beta_1 \beta_2} \sum_{i=0}^{(a_2-1)^*} \binom{a_2-1}{i} \frac{(-1)^i}{b+i} k^{b+i} \times \sum_{j=0}^{(a_1-1)^*} \binom{a_1-1}{j} (-1)^j (k/f)^{b+j} \beta(b+j, b+i+1); \quad (2.5.3.2)$$

$$I_{1-k/f}(a_1, b) + \frac{1}{\beta_1 \beta_2} \sum_{i=0}^{b-1} \binom{b-1}{i} \frac{(-1)^i}{a_2+i} (1-k+f)^{a_2+i} \times \sum_{j=0}^{b-1} \binom{b-1}{j} (-1)^j \left(1 - \frac{k-1}{f}\right)^{a_1+j} \times \left[\beta_{f/(1-k+f)}(a_1+j, a_2+i+1) - \beta_{(f-k)/(1-k+f)}(a_1+j, a_2+i+1) \right]. \quad (2.5.3.3)$$

Since $0 \leq k \leq 1$ and $0 \leq k/f \leq 1$, neither a_2 nor a_1 in (2.5.3.2) is restricted to integer values. However, since $(1-k+f)$ and $(1-(k-1)/f)$ may exceed one, b is restricted to integer values in (2.5.3.3). Although (2.5.1.11) also has a corresponding expression in Case 3, it also requires that b be integral. What is needed is an expression which will improve on (2.5.3.2) when a_1 , but not a_2 , is an integer. It is given by

$$1 - \frac{1}{\beta_1 \beta_2} \sum_{i=0}^{(a_1-1)^*} \binom{a_1-1}{i} \frac{(-1)^i}{b+i} (k/f)^{b+i} \\ \times \sum_{j=0}^{(a_2-1)^*} \binom{a_2-1}{j} (-1)^j k^{b+j} \beta(b+j, b+i+1). \quad (2.5.3.4)$$

The results of this section are now summarized:

Theorem 2.5.3.1. Let U_i , $\beta_{u_i}^0$, and w_0 be as defined in Theorem 2.5.1.1. If, for a given $\alpha \in (0,1)$, (2.5.1.12) equals α and if $0 \leq k \leq 1$ and $0 \leq k/f \leq 1$, then alternative expressions for (2.5.1.12) are given by (2.5.3.2), (2.5.3.3), and (2.5.3.4). Expression (2.5.3.3) is applicable only if b is an integer. Neither (2.5.3.2) nor (2.5.3.4) requires that a_1 and/or a_2 be integral, but, for nonintegral b , if a_2 but not a_1 is an integer, then (2.5.3.2) has the fewest number of terms while if a_1 but not a_2 is an integer, (2.5.3.4) has the fewest terms.

As before, k cannot in general be solved for explicitly in terms of α .

2.6. The s-Sample Case

In Section 2.5, discussion is limited to the two-sample case. As the number of statistics to be combined increases, the number of cases to be considered increases and the mathematics of evaluating

the integrals becomes increasingly cumbersome. For small values of s , say $s = 3$ and $s = 4$, it is probably not unreasonable to seek solutions, although none are sought here. As s increases much further, the labor involved in evaluating the necessary integrals becomes extremely tedious. However, since the overall statistic for combining s independent statistics U_1, \dots, U_s is

$$Z = \sum_{i=1}^s Z_i = \sum_{i=1}^s f_i U_i$$

where $f_i = \gamma_i(a_i+b)/b$, it can be shown that the Central Limit Theorem holds. The approximate normality of Z will be studied in Chapter IV.

2.7. A Note on Applications

As noted in Chapter I, statistics with F distributions can be transformed into beta-distributed random variables. Therefore, the results of this chapter can be applied to the statistics discussed in Section 1.1. If, for example, there are s independent Hotelling's T^2 statistics to be combined, where $(N_i - p) T_i^2 / [(N_i - 1)p], i = 1, \dots, s$, has an F distribution with p and $N_i - p$ degrees of freedom, then it follows from the material in Section 1.1 that $U_i = [T_i^2 / (N_i - 1)] / [1 + T_i^2 / (N_i - 1)]$ has a beta distribution with parameters $p/2$ and $(N_i - p)/2$. The results above can be applied by setting $b = p/2$ and $a_i = (N_i - p)/2$ and calculating f accordingly. In many actual experiments the number of variates may be quite small in comparison to the sample size so that $b = [p/2] < [(N_i - p)/2] = a_i$. If this is the case and $s = 2$, then for integral b the expressions derived in Section 2.5.1 - 2.5.3 which have summations extending to $(b-1)$ have fewer terms than the other expressions. In particular, if $b = 1$, then $b - 1 = 0$ and expressions

(2.5.1.7), (2.5.2.1), and (2.5.3.3) have only one term. In this instance, k can be solved for explicitly in (2.5.1.7). Of course, for other sets of parameters the other expressions presented in Sections 2.5.1 - 2.5.3 may have fewer terms and thus be preferable.

The results of this chapter also apply to the other statistics mentioned in Section 1.1. Again, if $s = 2$, the most useful expression for finding k , given α , can be chosen on the basis of the parameters of those statistics.

CHAPTER III

FURTHER CONSIDERATIONS IN THE LOCALLY OPTIMAL COMBINATION OF BETA-DISTRIBUTED STATISTICS

3.1. Introduction

In this chapter the theoretical results of Chapter II are extended. In Section 3.2 a test which is locally most powerful on the average is found in the case of an unbalanced design; the locally most powerful test based on a type D critical region is in general not available for this case. In Section 3.3 it is shown that when there are a large number of small-sample statistics to be combined, the locally most powerful test found in Chapter II is more efficient than Fisher's test according to Bahadur asymptotic relative efficiency, for local alternatives to the null hypothesis. It is also shown that, for local alternatives, the type F test is more efficient than Fisher's test according to an average measure of Bahadur asymptotic relative efficiency. This result is of particular interest in light of the work of Littell and Folks [1971,1973] which was discussed in Chapter I. In the final section several other extensions of the results of the second chapter are considered.

3.2. The Type F Critical Region: Unbalanced Design

The case of an unbalanced design is considered in this section. It is assumed that, under H_1 , the s independent statistics U_1, \dots, U_s to

be combined have noncentral beta distributions with noncentrality parameters $\theta_i = \lambda' V_i \lambda$, $1 \leq i \leq s$, as before. It is no longer assumed, however, that the design matrices V_i are proportional. It is known from Theorem 2.3.1 that a type D critical region does exist, but it is not in general possible to find the type D critical region since it is not in general possible to find

$$\max_w [\det(B_w)] = \max_w [\det(\sum_{i=1}^s V_i \times C_i(w))]$$

where B_w and $C_i(w)$ are defined in (2.3.11).

Although it is not always possible to find the type D critical region in the case of an unbalanced design, it is possible to find the type F critical region, which provides a test which is locally most powerful on the average, as discussed in Chapter I. From Definition 1.3.2.1 and (2.3.11) it follows that the type F critical region is given by

$$\begin{aligned} \max_w [\text{tr}(B_w)] &= \max_w [\text{tr}(\sum_{i=1}^s V_i \times C_i(w))] \\ &= \max_w [\sum_{i=1}^s \text{tr}(V_i) \times C_i(w)] \\ &= \max_w [\sum_{i=1}^s \gamma_i C_i(w)] \end{aligned}$$

where $\gamma_i = \text{tr}(V_i)$. This maximum has already been found in Section 2.4 and the desired critical region w_0 is defined by (2.4.1) with γ_i appropriately redefined. The results of Sections 2.5 - 2.7 also apply.

If the design is indeed balanced, so that $V_i = \delta_i V$, $1 \leq i \leq s$,

then

$$\begin{aligned} \max_w [\text{tr}(\underline{B}_{\sim w})] &= \max_w [\sum_{i=1}^S \text{tr}(\underline{V}_{\sim i}) \times C_i(w)] \\ &= \text{tr}(\underline{V}) \times \max_w [\sum_{i=1}^S \delta_i C_i(w)]. \end{aligned}$$

Since $\sum_{i=1}^S \delta_i C_i(w)$ is exactly the term maximized in Section 2.4 to find the type D region, the type F region is identical to the type D region if the design is balanced.

It is of interest to attempt to find other combination procedures which are related to the type D and type F tests. In order to investigate this issue and to further compare the type D and type F tests, it is useful to consider the geometrical interpretations of the characteristic roots, $\ell_1(w), \dots, \ell_p(w)$, of $\underline{B}_{\sim w}$. Restrict consideration to the two dimensional case ($p=2$) and let \underline{v}_1 and \underline{v}_2 be the two ($p \times 1$) characteristic vectors associated with roots $\ell_1(w)$ and $\ell_2(w)$. Assume that $\ell_1(w) > \ell_2(w)$ and that \underline{v}_1 and \underline{v}_2 are normalized so that $\underline{v}_i' \underline{v}_{\sim i} = 1$, $i = 1, 2$. The vectors form a basis of a two-dimensional Euclidean space. Since $\underline{B}_{\sim w}$ is positive definite symmetric, that basis is orthogonal (Graybill [1969, p. 47]). Let $\underline{\lambda}$ be a vector with coordinates in terms of the usual axes. Now consider the ellipse, mentioned in Section 1.3.1,

$$\underline{\lambda}' \underline{B}_{\sim w} \underline{\lambda} = \epsilon, \quad (3.2.1)$$

defined by the approximate intersection of the power surface and a cross-sectional horizontal plane at a height of $\alpha + \epsilon$. First let $\underline{\lambda}$ be a vector along the \underline{v}_1 -axis, so that $\underline{\lambda} = q_1 \underline{v}_1$, where q_1 is defined so that (3.2.1) holds. Then

$$\epsilon = \underline{\lambda}' \underline{B}_{\sim w} \underline{\lambda} = (q_1 \underline{v}_1)' \underline{B}_{\sim w} q_1 \underline{v}_1 = q_1^2 \underline{v}_1' \ell_1(w) \underline{v}_1 = \ell_1(w) q_1^2,$$

since $\underline{B}_{\underline{w}} \underline{v}_1 = \ell_1(w) \underline{v}_1$, whence $q_1 = \sqrt{\epsilon/\ell_1(w)}$. Similarly, if $\underline{\lambda} = q_2 \underline{v}_2$, $q_2 = \sqrt{\epsilon/\ell_2(w)}$. By definition, $\ell_1(w) \geq \ell_2(w)$, and therefore $q_1 \leq q_2$, implying that the larger the characteristic root, the shorter the axis of the ellipse in the direction of the associated characteristic vector. This relationship is pictorially displayed by superimposing the ellipse (3.2.1) on the $\underline{v}_1 \underline{v}_2$ -axes, as shown in Figure 3.2.1.

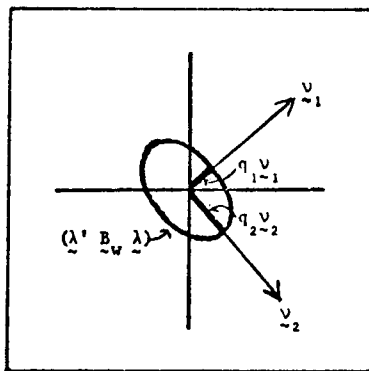


Fig. 3.2.1. The Ellipse Formed by the Approximate Intersection of the Power Surface and a Horizontal Plane at a Height of $\alpha + \epsilon$.

Notice that the type D critical region is given by

$$\max_{\underline{w}} [\det(\underline{B}_{\underline{w}})] = \max_{\underline{w}} \left(\prod_{i=1}^p \ell_i(w) \right)$$

since the determinant of a matrix is equal to the product of its

characteristic roots. By maximizing $\prod_{i=1}^p \ell_i(w)$, one minimizes

$\prod_{i=1}^p \sqrt{\epsilon/\ell_i(w)}$ for fixed ϵ , and, therefore, the type D critical region

provides the test which minimizes the product of the lengths of the

axes of the ellipse $\underline{\lambda}' \underline{B}_{\underline{w}} \underline{\lambda}$. On the other hand, the type F region

maximizes $\text{tr}(\underline{B}_{\underline{w}}) = \sum_{i=1}^p \ell_i(w)$. For fixed p , the type D critical region

is found by maximizing the geometric mean of the characteristic roots,

and the type F critical region is found by maximizing the arithmetic mean of the characteristic roots.

In the literature there is one other combination of the characteristic roots which is in common usage, one which would provide a minimax test by minimizing the longest axis of the ellipse $\tilde{\lambda}' B \tilde{\lambda}$. This test would maximize the curvature of the power surface along the axis where the curvature is the least by maximizing the minimum root. However, for this test it is necessary to find

$$\max_w \left\{ \min_{\tilde{\lambda}} \left[\frac{\tilde{\lambda}' (\sum_{i=1}^s \tilde{V}_i \times C_i(w)) \tilde{\lambda}}{\tilde{\lambda}' \tilde{\lambda}} \right] \right\} \quad (3.2.2)$$

(Roy et al. [1970, p. 26]), which is not in general possible in the case of an unbalanced design. If the design is balanced, so that $\tilde{V}_i = \gamma_i \tilde{V}$, $1 \leq i \leq s$, then this minimax test coincides with the type D test. This can be shown by rewriting (3.2.2) as

$$\max_w (\sum_{i=1}^s C_i(w) \gamma_i) \times \min_{\tilde{\lambda}} \left(\frac{\tilde{\lambda}' \tilde{V} \tilde{\lambda}}{\tilde{\lambda}' \tilde{\lambda}} \right). \quad (3.2.3)$$

The second term in (3.2.3) is independent of w , and the solution to the first term is exactly the type D critical region which was found in Section 2.4 for the balanced case.

3.3. Bahadur Efficiency

In this section it is shown that, for local alternatives to the null hypothesis, if there are a large number of finite-sample beta-distributed statistics to be combined, the type D critical region provides a test which is superior to Fisher's test according to Bahadur relative efficiency. It is also shown that the type D critical region

provides a test which is superior to Fisher's test according to a measure of average Bahadur asymptotic relative efficiency. Littell and Folks [1971,1973] have shown that Fisher's test is in general at least as efficient as every other test when there are a small number of large-sample statistics to be combined. However, other tests have been shown to be superior to Fisher's test for certain instances in which the two tests cannot be distinguished on the basis of Bahadur efficiency.

Define the testing problem as $H_0: \lambda = \lambda_0$ versus $H_1: \lambda \neq \lambda_0$. For $s = 1, 2, \dots$, let T_s be a real valued statistic which depends on the first s observations, $\{q_1, \dots, q_s\}$, of an infinite sequence. Assume that large values of T_s lead to rejection of H_0 . Also assume that the sequence $\{T_s\}$ is a standard sequence, which is defined as follows:

Definition 3.3.1. (Bahadur [1960]). A sequence $\{T_s\}$ is said to be a standard sequence for testing H_0 if and only if $\{T_s\}$ satisfies the following conditions:

- (i) there exists a continuous distribution function F such that

$$\lim_{s \rightarrow \infty} \Pr_{\lambda_0} (T_s < z) = F(z) \quad \text{for every } z;$$

- (ii) there exists a constant a , $0 < a < \infty$, such that

$$-2 \log [1 - F(z)] = az^2 [1 + o(1)] \quad \text{as } z \rightarrow \infty;$$

- (iii) there exists a function $b(\lambda)$, $0 < b(\lambda) < \infty$, such that for each $\lambda \neq \lambda_0$,

$$\lim_{s \rightarrow \infty} \Pr_{\lambda} (|T_s/s^{1/2} - b(\lambda)| > z) = 0 \quad \text{for every } z > 0.$$

If $\{T_s\}$ satisfies these conditions, then

$$c(\lambda) = \begin{cases} 0, & \lambda = \lambda_0, \\ a[b(\lambda)]^2, & \lambda \neq \lambda_0, \end{cases}$$

is said to be the approximate slope of $\{T_s\}$. To compare the asymptotic efficiencies of two standard sequences $\{T_s^{(1)}\}$ and $\{T_s^{(2)}\}$ with slopes $c_1(\lambda)$ and $c_2(\lambda)$, respectively, Bahadur [1960] suggested $\phi_{12} = c_1(\lambda)/c_2(\lambda)$ as a measure of asymptotic relative efficiency. If $\phi_{12}(\lambda) > 1$, the first sequence is more successful than the second; if $\phi_{12}(\lambda) < 1$, the second is more successful than the first; and if $\phi_{12}(\lambda) = 1$, the two sequences are equally successful according to this measure of efficiency.

To compare the asymptotic relative efficiency of the test based on the type D critical region to Fisher's test for the testing problem which has been discussed in this dissertation, the normalized statistics are used. If U_1, \dots, U_s are s beta-distributed statistics to be combined, the locally most powerful test was found in Chapter II to be based on the statistic $\sum_{i=1}^s [\gamma_i (a_i+b) U_i / b]$ where a_i , b , and γ_i are defined in Section 2.3. Let $X_i = (\gamma_i (a_i+b) U_i / b) - E_{\lambda_0}(\gamma_i (a_i+b) U_i / b) = (\gamma_i (a_i+b) U_i / b) - \gamma_i$ so that $E_{\lambda_0}(X_i) = 0$. Since the U_i , and hence the X_i , are independent, $\text{Var}_{\lambda_0}(\sum_{i=1}^s X_i) = \sum_{i=1}^s \text{Var}_{\lambda_0}(X_i)$. Therefore, the normalized overall test statistic for the locally most powerful test, based on the type D critical region, can be written

$$\begin{aligned} T_s^{(1)} &= \left\{ \sum_{i=1}^s [X_i - E_{\lambda_0}(X_i)] \right\} / \left[\sum_{i=1}^s \text{Var}_{\lambda_0}(X_i) \right]^{1/2} \\ &= \left[\sum_{i=1}^s X_i \right] / \left[\sum_{i=1}^s \text{Var}_{\lambda_0}(X_i) \right]^{1/2}. \end{aligned} \quad (3.3.1)$$

Now let $Y_i = -2 \log \int_{U_i}^1 \beta_z^0 dz$, where β_z^0 is defined in (1.1.3). Then

$\sum_{i=1}^S Y_i$ is Fisher's overall test statistic for this problem. Define

$$T_s^{(2)} = \left\{ \sum_{i=1}^S [Y_i - E_o(Y_i)] \right\} / \left[\sum_{i=1}^S \text{Var}_o(Y_i) \right]^{1/2}. \quad (3.3.2)$$

The asymptotic relative efficiency of $\{T_s^{(1)}\}$ to $\{T_s^{(2)}\}$ is the asymptotic relative efficiency of the locally most powerful test to Fisher's test, as long as the conditions of Definition 3.3.1 are satisfied. The first condition can be satisfied since both $T_s^{(1)}$ and $T_s^{(2)}$ are asymptotically normal, so that $F = \Phi$ in (i), where Φ is the standard normal distribution function. The Central Limit Theorem holds with respect to $T_s^{(1)}$ since the X_i are bounded and $\lim_{s \rightarrow \infty} \text{Var}_o(\sum_{i=1}^S X_i) =$

$\lim_{s \rightarrow \infty} \sum_{i=1}^S \gamma_1^2 a_i^2 / [b(a_i+b+1)] \rightarrow \infty$ (Fisz [1963, pp. 206-207]). Under H_o ,

$\sum_{i=1}^S Y_i \sim \chi_{2s}^2$, and the asymptotic normality of $T_s^{(2)}$ is well established (Johnson and Kotz, Vol. 1, [1970, p. 176]). The second condition is satisfied when $F = \Phi$ with $a = 1$ in (ii) (Bahadur [1960]). It remains only to show that (iii) holds for $T_s^{(1)}$ and $T_s^{(2)}$.

Consider first the sequence $\{T_s^{(1)}\}$. Under H_1 , $\sum_{i=1}^S X_i/s \rightarrow \sum_{i=1}^S E_\lambda(X_i)/s$ in probability as $s \rightarrow \infty$. The Markov Law of Large Numbers (Fisz [1963, pp. 216-217]) assures this since $E_\lambda(X_i^2) = O(1)$ so that $\sum_{i=1}^S E_\lambda(X_i^2)/s^2 \rightarrow 0$ as $s \rightarrow \infty$. Furthermore, since $\theta_i = \gamma_i \theta$, it can be shown that for local alternatives

$$\begin{aligned} \sum_{i=1}^S E_\lambda(X_i) &= \sum_{i=1}^S \gamma_i \frac{a_i+b}{b} \left[e^{-\theta_i/2} \left[\frac{b}{a_i+b} + \frac{b+1}{a_i+b+1} \frac{\theta_i}{2} + o(\theta_i^2) \right] - \frac{b}{a_i+b} \right] \\ &= \sum_{i=1}^S \gamma_i \frac{a_i+b}{b} \left[[1-\theta_i/2 + o(\theta_i^2)] \left[\frac{b}{a_i+b} + \frac{b+1}{a_i+b+1} \frac{\theta_i}{2} + o(\theta_i^2) \right] - \frac{b}{a_i+b} \right] \\ &= \sum_{i=1}^S [\gamma_i (\theta_i/2)] \left[\frac{b+1}{a_i+b+1} \frac{a_i+b}{b} - 1 \right] + o(\theta_i^2) \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^s (\theta_i/2) [\gamma_i a_i / [b(a_i+b+1)]] + o(\theta^2) \\
&= \theta/2 \sum_{i=1}^s \gamma_i^2 a_i / [b(a_i+b+1)] + o(\theta^2). \tag{3.3.3}
\end{aligned}$$

Also,

$$\begin{aligned}
\sum_{i=1}^s \text{Var}_{\underset{\sim}{o}}(X_i) &= \sum_{i=1}^s \gamma_i^2 \left(\frac{a_i+b}{b}\right)^2 \text{Var}_{\underset{\sim}{o}}(U_i) \\
&= \sum_{i=1}^s \gamma_i^2 \left(\frac{a_i+b}{b}\right)^2 \frac{a_i b}{(a_i+b)^2 (a_i+b+1)} \\
&= \sum_{i=1}^s \gamma_i^2 a_i / [b(a_i+b+1)]. \tag{3.3.4}
\end{aligned}$$

In light of (3.3.3) and (3.3.4) it can be seen that, for local alternatives to the null hypothesis, so that terms of order θ^2 and higher order are ignored,

$$\begin{aligned}
b_1(\lambda) &= s^{-1/2} E_{\lambda} (T_s^{(1)}) \underset{\sim}{\sim} s^{-1/2} \theta/2 [\sum_{i=1}^s \gamma_i^2 a_i / [b(a_i+b+1)]]^{1/2} \\
&= s^{-1/2} \theta/2 [\sum_{i=1}^s \text{Var}_{\underset{\sim}{o}}(X_i)]^{1/2} \\
&= s^{-1/2} \theta/2 [\text{Var}_{\underset{\sim}{o}}(\sum_{i=1}^s X_i)]^{1/2}. \tag{3.3.5}
\end{aligned}$$

Notice that $0 < b_1(\lambda)$ for X_i as defined in this problem.

Next consider the sequence $\{T_s^{(2)}\}$. Under H_1 , $\sum_{i=1}^s Y_i / s \xrightarrow{P} \sum_{i=1}^s E_{\lambda}(Y_i) / s$, again by the Markov Law of Large Numbers since, for this problem, $E_{\lambda}(Y_i^2) = o(1)$. In light of (1.1.2), the density of a non-central beta-distributed random variable U_i with parameters b and a_i and noncentrality θ_i can be written as

$$\begin{aligned}
&e^{-\theta_i/2} [\beta_{u_i}^o(b, a_i) + \theta_i/2 \beta_{u_i}^o(b+1, a_i) + o(\theta_i^2)] \\
&= \beta_{u_i}^o(b, a_i) + \theta_i/2 [(a_i+b)u_i/b - 1] \beta_{u_i}^o(b, a_i) + o(\theta_i^2),
\end{aligned}$$

using the exponential expansion. Therefore,

$$\begin{aligned}
\sum_{i=1}^S E_{\tilde{\lambda}}(Y_i) &= \sum_{i=1}^S \int_0^1 (-2 \log \int_{u_i}^1 \beta_z^0 dz) \\
&\quad \times [\beta_{u_i}^0(b, a_i) + \theta_i/2((a_i+b)u_i/b-1) \beta_{u_i}^0(b, a_i) + o(\theta_i^2)] du_i \\
&= \sum_{i=1}^S E_{\tilde{0}}(Y_i) + \frac{\theta}{2} \sum_{i=1}^S \int_0^1 [-2 \log \int_{u_i}^1 \beta_z^0 dz] \\
&\quad \times [\gamma_i(a_i+b)u_i/b - \gamma_i] \beta_{u_i}^0(b, a_i) du_i + o(\theta^2) \\
&= \sum_{i=1}^S E_{\tilde{0}}(Y_i) + \theta/2 \sum_{i=1}^S E_{\tilde{0}}(Y_i X_i) + o(\theta^2). \tag{3.3.6}
\end{aligned}$$

Substitute (3.3.6) into (3.3.2) and take the expectation to find, for local alternatives, that

$$b_2(\tilde{\lambda}) = s^{-1/2} [\theta/2 \sum_{i=1}^S E_{\tilde{0}}(Y_i X_i)] \times [\text{Var}_{\tilde{0}}(\sum_{i=1}^S Y_i)]^{-1/2}. \tag{3.3.7}$$

$E_{\tilde{0}}(Y_i X_i) > 0$ for all i since $E_{\tilde{0}}(Y_i X_i) - E_{\tilde{0}}(Y_i) E_{\tilde{0}}(X_i) = E_{\tilde{0}}(Y_i X_i)$ and since as U_i increases both X_i and Y_i increase. Therefore, $b_2(\tilde{\lambda}) > 0$.

By combining (3.3.5) and (3.3.7), it is seen that the asymptotic relative efficiency of the locally most powerful test to Fisher's test, for local alternatives, is

$$\begin{aligned}
\phi_{12}(\tilde{\lambda}) &= b_1^2(\tilde{\lambda})/b_2^2(\tilde{\lambda}) \\
&= \text{Var}_{\tilde{0}}(\sum_{i=1}^S X_i) \times \text{Var}_{\tilde{0}}(\sum_{i=1}^S Y_i) / [\sum_{i=1}^S E_{\tilde{0}}(Y_i X_i)]^2, \tag{3.3.8}
\end{aligned}$$

as long as the limits taken exist. (In the context of experimental designs, there are general models where the limits taken in (3.3.8), in light of (3.3.5) and (3.3.7), exist. See Sen [1969] for details.) Also

$$\begin{aligned}
E_{\tilde{0}}(\sum_{i=1}^S Y_i \sum_{i=1}^S X_i) &= E_{\tilde{0}}(\sum_{i=1}^S Y_i X_i + \sum_{i=1}^S \sum_{\substack{j=1 \\ i \neq j}}^S Y_i X_j) \\
&= \sum_{i=1}^S E_{\tilde{0}}(Y_i X_i) + \sum_{i=1}^S \sum_{\substack{j=1 \\ i \neq j}}^S E_{\tilde{0}}(Y_i) E_{\tilde{0}}(X_j) \\
&= \sum_{i=1}^S E_{\tilde{0}}(Y_i X_i), \tag{3.3.9}
\end{aligned}$$

since Y_i and X_j are independent if $i \neq j$ and $E_{\tilde{0}}(X_j) = 0$ for all j .

Combine (3.3.8) and (3.3.9) to see that

$$\phi_{12}(\lambda) = \text{Var}_{\tilde{0}}(\sum_{i=1}^S X_i) \times \text{Var}_{\tilde{0}}(\sum_{i=1}^S Y_i) / E_{\tilde{0}}^2(\sum_{i=1}^S X_i \sum_{i=1}^S Y_i) \geq 1. \tag{3.3.10}$$

There is equality whenever Y_i and X_i are linearly related. Since X_i is a linear function of U_i and Y_i is a function of U_i which is never linear, the inequality is strict. Therefore,

$$\phi_{12}(\lambda) > 1. \tag{3.3.11}$$

This result implies that if there are a large number of finite-sample statistics to be combined, the test based on the type D critical region is more successful than Fisher's test according to asymptotic relative efficiency, for local alternatives to H_0 .

It should be noted that the asymptotic relative efficiency based on approximate slopes can be misleading. It has been noted (for example, by Bahadur [1967]) that tests which are identical may not have identical approximate slopes so that $\phi_{12}(\lambda)$ is not 1. This dilemma does not occur if $\phi_{12}(\lambda)$ is defined as the ratio of exact rather than approximate slopes. Assume that the exact distribution function of T_S , F_S , is such that (ii) in Definition 3.3.1 can be replaced by (ii)*:

(ii)* (1) $\lim_{s \rightarrow \infty} F_s(z) = F(z)$ and (2) there exists a function f on $(0, \infty)$ into $(0, \infty)$ such that, for any given sequence $\{v_s\}$ of positive constants v_s such that $\lim_{s \rightarrow \infty} \{v_s^2/s\} = z$, where $0 < z < \infty$,

$$2/s \log[1 - F_s(v_s)] = -f(z)[1 + o(1)].$$

Then $c^*(\lambda) = f[b(\lambda)^2]$ is the exact slope of $\{T_s\}$. If $\phi_{12}^*(\lambda) = c_1^*(\lambda)/c_2^*(\lambda)$, then $\phi_{12}^*(\lambda) = 1$ if the tests are indeed identical. However, $c^*(\lambda)$ is often difficult to find, as it is for the sequence $\{T_s^{(1)}\}$, and hence the use of approximate measures here. Bahadur [1967] suggests, though, that for local alternatives to the null hypothesis, $\phi_{12}(\lambda)$ is likely to be a good approximation of $\phi_{12}^*(\lambda)$. Since the results above are based on the assumption that the alternatives are local and considering the inherent properties of the test based on the type D region, it is reasonable to assume that the conclusion of the superior efficiency of this test over Fisher's test, for local alternatives, is not an artifact.

If the design is not balanced, the final steps in (3.3.3) and (3.3.6), and hence the measure of relative efficiency given above, do not apply. However, similar results can be found for comparisons of the type F test and Fisher's test. Let $\gamma_i = \text{tr}(V_i)$. Then the approximate slope of $\{T_s^{(1)}\}$, for local alternatives, is

$$b_1(\lambda) \doteq s^{-1/2} \left\{ \sum_{i=1}^s (\theta_i/2) \gamma_i a_i / [b(a_i + b + 1)] \right\} \\ \times \left\{ \sum_{i=1}^s \gamma_i^2 a_i / [b(a_i + b + 1)] \right\}^{-1/2}. \quad (3.3.12)$$

This result is analogous to (3.3.5) and follows immediately from (3.3.3)

and (3.3.4); the θ_i have not been factored out since the assumption that $\theta_i = \gamma_i \theta$ no longer holds. Similarly, in light of (3.3.7), the approximate slope for $\{T_s^{(2)}\}$ can be written as

$$b_2(\lambda) = s^{-1/2} \sum_{i=1}^s (\theta_i / 2\gamma_i) E_{\tilde{0}}(Y_i | X_i) \times [\text{Var}_{\tilde{0}}(\sum_{i=1}^s Y_i)]^{-1/2}. \quad (3.3.13)$$

For the unbalanced case, the Bahadur asymptotic relative efficiency, for local alternatives, is $b_1^2(\lambda)/b_2^2(\lambda)$ where $b_1(\lambda)$ and $b_2(\lambda)$ are given in (3.3.12) and (3.3.13) respectively. Unfortunately, this ratio is difficult to evaluate for the unbalanced case. Consider instead the ratio of the squares of the average approximate slopes, where the average is found by integrating over the sphere $\lambda' \lambda = r^2$. The definition

$$\bar{\phi}_{12}(\lambda) = \bar{b}_1^2(\lambda) / \bar{b}_2^2(\lambda) \quad (3.3.14)$$

is introduced as a measure of average Bahadur asymptotic relative efficiency, where

$$\bar{b}_i(\lambda) = \int_{\lambda' \lambda = r^2} b_i(\lambda) dA.$$

Notice that this averaging is the same as that given in Definition 1.3.2.1. To evaluate $\bar{b}_1(\lambda)$ and $\bar{b}_2(\lambda)$, first apply (1.3.2.7) to find

$$\int_{\lambda' \lambda = r^2} \theta_i dA = \int_{\lambda' \lambda = r^2} \lambda' V_{\tilde{i}} \lambda dA = \delta(r) \times \text{tr}(V_{\tilde{i}}) = \delta(r) \times \gamma_i \quad (3.3.15)$$

where $\delta(r)$ is a function of r . Apply this integration to the right-hand side of (3.3.12) and recall (3.3.4) to find that, for local alternatives,

$$\begin{aligned} \bar{b}_1(\lambda) &= s^{-1/2} (\delta(r)/2) \{ \sum_{i=1}^s \gamma_i^2 a_i / [b(a_i + b + 1)] \} \\ &\quad \times \{ \sum_{i=1}^s \gamma_i^2 a_i / [b(a_i + b + 1)] \}^{-1/2} \end{aligned}$$

$$= s^{-1/2}(\delta(r)/2)[\text{Var}_{\underset{\sim}{0}}(\sum_{i=1}^s X_i)]^{1/2}. \quad (3.3.16)$$

Similarly, from (3.3.13),

$$\bar{b}_2(\lambda) = s^{-1/2}(\delta(r)/2)\sum_{i=1}^s E_{\underset{\sim}{0}}(Y_i | X_i) \times [\text{Var}_{\underset{\sim}{0}}(\sum_{i=1}^s Y_i)]^{-1/2} \quad (3.3.17)$$

for local alternatives. As before, in the present context

$0 < b_i(\lambda) < \infty$, $i = 1, 2$. It follows from (3.3.14), (3.3.16)-(3.3.17)

and (3.3.8)-(3.3.11) that locally,

$$\bar{\phi}_{12}(\lambda) = \bar{b}_1^2(\lambda)/\bar{b}_2^2(\lambda) = b_1^2(\lambda)/b_2^2(\lambda) > 1.$$

Therefore, according to the above measure of average Bahadur asymptotic relative efficiency, the type F test is superior to Fisher's test for local alternatives.

As before, Bahadur asymptotic relative efficiency is not necessarily an accurate comparison of two tests. Furthermore, the processes of averaging used above may introduce other inaccuracies or difficulties in interpretation. Nevertheless, in light of the method of derivation of the type F critical region, the above result is of interest.

3.4. Extensions to Other Problems

Locally optimal combination procedures for tests of hypotheses specifying the value of a vector of parameters have been found in this and the previous chapter. It is possible to find analogous procedures for combining certain tests of a null hypothesis of the form

$$H_0: \mu_{q \times r} = \underset{\sim}{0}, \quad (3.3.4.1)$$

even though the optimality of such procedures is not necessarily extended from previous results.

The likelihood ratio statistic, Λ , for tests of (3.3.4.1) is often written $U_{q,r,t}$, where q, r , and t are appropriate parameters. Let $U_{q,r,t}^{(i)}$, $1 \leq i \leq s$, be s independent statistics for testing a common hypothesis; q and r need not be indexed. Gupta [1971] showed that in the noncentral linear case (that is, when the alternative has unit rank), $U_{q,r,t}^{(i)}$ is distributed as $\prod_{j=1}^q X_{ij}$ where X_{ij} has a central beta distribution $\beta_{x_{ij}}^0((r+1-j)/2, t_i/2)$, $2 \leq j \leq q$, and $(1 - X_{i1})$ has a noncentral beta density, so that X_{i1} has density

$$\begin{aligned} & e^{-\theta_i/2} \sum_{h=0}^{\infty} \theta_i^h x_{i1}^{t_i/2-1} (1-x_{i1})^{r/2+h-1} / [2^h h! \beta(t_i/2, r/2+h)] \\ & = e^{-\theta_i/2} \sum_{h=0}^{\infty} (\theta_i/2)^h \beta_{x_{i1}}^0(t_i/2, r/2+h)/h!, \quad 0 \leq x_{i1} \leq 1, \end{aligned}$$

where θ_i is the noncentrality parameter. (See Gupta for details of calculating θ_i . For the present discussion it is sufficient to note that θ_i is dependent on $\mu' V_{i1}^* \mu$ for some appropriate design matrix V_{i1}^* .) Under H_0 , X_{i1} has density $\beta_{x_{i1}}^0(r/2, t_i/2)$, $1 \leq i \leq s$. Therefore, if $g_{u_i}^\theta(r; t_i/2)$ denotes the noncentral density of $U_i = U_{q,r,t}^{(i)} = \prod_{j=1}^q X_{ij}$, where $r' = [r/2, (r+1)/2, \dots, (r+1-q)/2]$, then

$$g_{u_i}^\theta(r; t_i/2) = e^{-\theta_i/2} \sum_{h=0}^{\infty} \left(\frac{\theta_i}{2}\right)^h \frac{1}{h!} g_{u_i}^0(r/2+h, (r+1)/2, \dots, (r+1-q)/2; t_i/2),$$

$$0 \leq u_i \leq 1.$$

Recall that the type D and type F test statistics are of the form

$$\sum_{i=1}^s \gamma_i (a_i + b) u_i / b = \sum_{i=1}^s \gamma_i \beta_{u_i}^0(b+1, a_i) / \beta_{u_i}^0(b, a_i), \quad (3.3.4.2)$$

where γ_i is a weight dependent on the design matrix V_i and $\beta_{u_i}^o(b, a_i)$ is the density of a beta-variate U_i with parameters b and a_i .

Analogously define the following as a test statistic for combining independent statistics U_i for testing (3.3.4.1):

$$\sum_{i=1}^s \gamma_i \frac{g_{u_i}^o(r/2+1, (r+1)/2, \dots, (r+1-q)/2; t_i/2)}{g_{u_i}^o(r/2, (r+1)/2, \dots, (r+1-q)/2; t_i/2)}, \quad (3.3.4.3)$$

where γ_i is a weight dependent on the design matrix V_i^* . If the functions $g_i^o = g_{u_i}^o(r; t_i/2)$ are known, then the statistic proposed in (3.3.4.3) can be calculated. For certain cases, the functions g_i^o are known. See, for example, Gupta [1971] for the density of $U_{q,r,t}$ when $q = 2, 3, 4$, or 5 . Notice that if $q = 1$ or $r = 1$, $\mu_{q \times r}$ reduces to a vector and the likelihood statistic is distributed as a single beta-distributed variate which is noncentral under the alternative hypothesis. In this case the results derived earlier apply directly.

Assume again that the null hypothesis specifies the value of a vector rather than a matrix. Results similar to those in the previous sections and chapter can be derived if it is assumed that the statistics to be combined, X_1, \dots, X_s , have chi-squared distributions which are noncentral under the alternative hypothesis. If X_i has a noncentral chi-squared distribution with $2b_i$ degrees of freedom and noncentrality parameter θ_i , then the density of X_i is given by

$$\left[e^{-\theta_i/2} e^{-x_i/2} / 2^{b_i} \right] \sum_{j=0}^{\infty} \frac{x_i^{b_i+j-1} \theta_i^j}{x_i^{b_i+j-1} \theta_i^j / [\Gamma(b_i+j) 2^{2j} j!]}, \quad x > 0,$$

where $\Gamma(a) = (a-1)!$. It is easily shown that the type D and type F statistics for combining X_1, \dots, X_s are of the form

$$\sum_{i=1}^s \gamma_i x_i / (2b_i) = \sum_{i=1}^s \gamma_i \frac{\chi^2(2b_i+2)}{\chi^2(2b_i)} \quad (3.3.4.3)$$

where γ_i is a weight dependent on the design matrix V_i as before and $\chi^2(2b_i)$ denotes the density of a central chi-squared variate with $2b_i$ degrees of freedom.

However, the chi-squared-distributed statistics which arise in real problems are often statistics used in the testing of (3.3.4.1). In particular, Sugiura and Fujikoshi [1969] have shown that the non-null distribution of $-2 \log \Lambda$, where Λ is the likelihood ratio statistic mentioned above, can be approximated by a linear combination of non-central chi-squared-distributed variates, up to order N^{-2} where N is the sample size. For large samples, the distribution is approximately a chi-squared-distribution which is noncentral under the alternative. In that case the statistic (3.3.4.3) can be used to provide a combined test, with $X_i = -2 \log \Lambda_i$, $1 \leq i \leq s$. For smaller samples the result is more complicated. The overall statistic is similar to (3.3.4.2), with the central densities there replaced by the appropriate linear combinations of central chi-squared densities. Similarly, Siotani [1971] has given an asymptotic expansion of the nonnull distribution of the Hotelling-Lawley T_0^2 up to order N^{-2} . Again the expansion is a linear combination of noncentral chi-squared variates which can be approximated by a single noncentral chi-squared variate for large samples. The extensions here follow in the same fashion as those involving combinations of likelihood ratio statistics.

Finally, Pillai and Young [1971] have considered approximations to the density of T_0^2 for the central case. They have proposed that the central density of T_0^2 be approximated by a generalized type II beta

variate having density

$$u^a [k^{a+1} \beta(a+1, b-a-1) (1+u/k)^b]^{-1}, \quad 0 \leq u \leq 1.$$

If research extending this result shows that the noncentral density of T_0^2 could be approximated by a noncentral beta-type distribution, a procedure for combining s such statistics could easily be determined along the lines of the above extensions.

CHAPTER IV

APPROXIMATE CRITICAL VALUES FOR TYPE D AND TYPE F TESTS

In Chapters II and III the type D and type F critical regions for tests of hypotheses specifying the value of a vector were developed. Formulas which can be used to tabulate critical values for two-sample problems were found in Section 2.5, and some tables of critical values are presented in Chapter V. Although it is, perhaps, reasonable to seek exact solutions for three-sample and four-sample problems, as the number of samples to be combined increases, both the number of cases to be considered in performing the integration and the complexity of the integrated results increase. Furthermore, the computer costs of creating the tables of critical values increase rapidly as the calculations become longer.

It is possible, however, to use approximate rather than exact critical values in tests combining more than two samples. The type D and type F test statistics are both of the form $\sum_{i=1}^s f_i U_i$, where U_i is beta-distributed with parameters b and a_i , and $f_i = (a_i + b)\gamma_i/b$ with γ_i an appropriately defined constant. Since the s statistics U_i , $1 \leq i \leq s$, are bounded and $\sum_{i=1}^s \text{Var}(f_i U_i) \rightarrow \infty$ with s , the Central Limit Theorem applies (Fisz [1963, pp. 206-207]).

Therefore, for all real z

$$\Pr(Z \leq z) = \Pr \left\{ \frac{\sum_{i=1}^s [f_i U_i - E_{\circ}(f_i U_i)]}{[\sum_{i=1}^s \text{Var}_{\circ}(f_i U_i)]^{1/2}} \leq z \right\}_{s \rightarrow \infty} \rightarrow \Phi(z), \quad (4.1)$$

where $Z = \sum_{i=1}^s [f_i U_i - E_{\circ}(f_i U_i)] / [\sum_{i=1}^s \text{Var}_{\circ}(f_i U_i)]^{1/2}$ and where Φ is the cumulative distribution function of a standard normal random variable.

It would be useful to determine values of s which are sufficiently large to provide good approximations in (4.1) so that the standard normal tables can be used to provide approximate critical values. Computer simulations were used to find the minimum s necessary to provide good normal approximations for normalized type D and type F test statistics for several different testing problems. Different values of b , a_i , and γ_i are used to define different combination problems.

The algorithm for the simulation is as follows:

1. Define the testing problem by determining b , and the range of the a_i 's and γ_i 's.
2. Set s .
3. Generate s beta-distributed variates. (The beta variates were found by transforming two central chi-squared variates, say χ_1 and χ_2 , with $2b$ and $2a$ degrees of freedom respectively. When χ_1 and χ_2 are so distributed, $\beta = \chi_1 / (\chi_1 + \chi_2)$ has a central beta distribution with parameters b and a . The variates χ_1 and χ_2 were found by summing the squares of, respectively, $2b$ and $2a$ numbers which were generated from a standard normal population. These normally distributed numbers were generated by the PL-1 version of program VARGEN

as installed on the University of North Carolina Computation Center's IBM 360/75.)

4. Normalize the sum of the s beta-distributed variates in order to calculate Z in (4.1).
5. Repeat steps 3 and 4 until a total of 5000 values, say Z_j , $1 \leq j \leq 5000$, are generated.
6. Perform the Kolmogorov-Smirnov goodness of fit test at the .05 level of significance to measure the maximum distance D between the empirical distribution function of the $\{Z_i\}$ and the standard normal cumulative distribution function and to test whether or not the values $\{Z_i\}$ are consistent with a standard normal population. (The PL-1 Scientific Subroutine Package NDTR as installed at the U.N.C. Computation Center was used to determine values of the standard normal cumulative distribution function.) If $D \leq 1.36/\sqrt{5000} = 0.0193$, the hypothesis that the $\{Z_i\}$ were likely to have been drawn from a standard normal population is not rejected.
7. If $D > 0.0193$, increase s and go to step 3; otherwise, go to step 1 to define a new problem or else stop.

In Table 4.1 the maximum distance between the empirical distribution function of the normalized variates Z_i , $1 \leq i \leq 5000$, and the standard normal cumulative distribution function is given for a number of testing problems. It is assumed that $a_1 = \dots = a_s$ and $\gamma_1 = \dots = \gamma_s$. If $\gamma_i = \gamma$, $1 \leq i \leq s$, Z_i is not a function of the γ_i 's and hence there is no loss in assuming that $\gamma_i = \gamma = 1$, $1 \leq i \leq s$.

TABLE 4.1

VALUES OF THE KOLMOGOROV-SMIRNOV STATISTIC FOR CERTAIN
SIMULATION PROBLEMS, ASSUMING CERTAIN PARAMETERS
ARE CONSTANT FOR ALL SAMPLES

| b | γ | a | s | D |
|---|----------|----|----|--------|
| 1 | 1 | 1 | 3 | 0.0262 |
| 1 | 1 | 1 | 4 | 0.0154 |
| 1 | 1 | 2 | 4 | 0.0299 |
| 1 | 1 | 2 | 5 | 0.0152 |
| 1 | 1 | 8 | 7 | 0.0348 |
| 1 | 1 | 8 | 12 | 0.0207 |
| 1 | 1 | 8 | 20 | 0.0182 |
| 2 | 1 | 2 | 3 | 0.0178 |
| 2 | 1 | 8 | 7 | 0.0278 |
| 2 | 1 | 8 | 12 | 0.0145 |
| 3 | 1 | 8 | 8 | 0.0158 |
| 3 | 1 | 15 | 12 | 0.0217 |
| 3 | 1 | 15 | 16 | 0.0136 |

Notice that for fixed b , as a increases the s needed to attain values of D less than 0.0193 also increases. This is not surprising since the beta distribution is symmetric if $a = b$ (uniform if $a = b = 1$, bell-shaped if $a = b > 1$) but is skewed if $a \neq b$. The parameter a is more likely to be larger than b in practical applications, and so the table has been prepared in that fashion. However, if $a_i = a$ and $\gamma_i = \gamma$, $1 \leq i \leq s$, then the parameters a and b are algebraically interchangeable. For example, from the table it can be concluded that if $b = 2$ and $a = a_i = 1$, $1 \leq i \leq s$, $s = 5$ is the minimum s which is large enough to provide a satisfactory approximation to normality, according to the criteria previously established. It can also be concluded, considering

the relative skewness of the beta density functions involved, that since s must exceed 12 if $b = 3$ and $a = 15$ before approximate normality is achieved, s must be quite large indeed if $a = 15$ and $b < 3$.

Table 4.2 indicates results of simulations when it is not assumed that the γ_i and the a_i are necessarily constant. Let $\underline{\gamma} = (\gamma_1, \dots, \gamma_s)$ and $\underline{a} = (a_1, \dots, a_s)$. Other simulations indicated, as expected, that the approximate normality was very sensitive to the inclusion of large a_i when b was small. While this table is illustrative of some effects of having varying γ_i and a_i , the exact parameters of the γ_i and a_i would have to be known before a reasonable guess at the minimum s which is large enough to produce good approximations to normality could be made.

TABLE 4.2

VALUES OF THE KOLMOGOROV-SMIRNOV STATISTIC FOR CERTAIN
SIMULATION PROBLEMS WITHOUT RESTRICTION ON THE
CONSTANCY OF PARAMETERS

| b | $\underline{\gamma}$ | \underline{a} | s | D |
|-----|--|---|-----|--------|
| 2 | (1.0, 1.2, 1.4, 1.5, 1.6, 1.8, 2.0, 2.2, 2.4, 2.5, 2.6, 3.0) | $\underline{1}$ | 12 | 0.0137 |
| 2 | (1.0, 1.5, 1.75, 2.0, 2.25, 2.5, 2.75, 3.25) | (1, 4, 5, 7, 8, 10, 11, 14) | 8 | 0.0207 |
| 2 | (1.0, 1.2, 1.4, 1.5, 1.6, 1.8, 2.0, 2.2, 2.4, 2.6, 2.6, 2.8) | (1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12) | 12 | 0.0189 |
| 2 | $\underline{1}$ | (1, 2, 5, 7, 8, 10, 13, 15) | 8 | 0.0145 |

CHAPTER V

TABLES OF CRITICAL VALUES FOR TYPE D AND TYPE F TESTS

AND A NUMERICAL ILLUSTRATION

5.1. Tables of Critical Values for Type D and Type F Tests

Tables of critical values for type F and type D statistics have been prepared for certain parameter values. Assume that U_i , $i = 1, 2$, has a central beta distribution with parameters b and a_i and let $f = (a_1 + b)\gamma_1 / [(a_2 + b)\gamma_2]$ where γ_i has an appropriate definition (see Sections 2.4 and 3.2). Table 5.1.1 gives values of k which satisfy

$$\Pr(fU_1 + U_2 \geq k) = \alpha$$

for $\alpha = .01, .05$; $b = 1, 2$; $a_i = 1$ (1) 10 (2) 30, $i = 1, 2$; and $f = 1$ (.5) 2.5. Incomplete tables are also given for $b = 1.5$. By comparing, for example (2.5.3.2) and (2.5.3.3) for Case 3, it is clear that there are generally more terms in the series to evaluate if b is not an integer but a_1 and a_2 are integers, for the range of parameters chosen. Since there are more terms to evaluate, the tables for $b = 1.5$ are substantially more expensive to compute. By the time the Case 3 runs for $b = 1.5$ had been completed it was evident that sufficient computer funds to complete the table were not available.

Solutions for values of k given a_1 , a_2 , b , and f were found iteratively for the specified value of α , except for the few instances in which direct solutions exist. Iteration started with an initial

TABLE 5.1.1 (Continued)
 b = 1.5 $\alpha = 0.05$

| | a ₁ | f | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 | 26 | 28 | 30 |
|-----|----------------|---|-------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
| 12 | 1.0 | | .962 | .837 | .741 | .667 | .610 | .566 | .530 | .501 | .477 | .441 | .414 | .395 | .379 | .367 | .357 | .349 | .343 | .337 | .332 | .332 |
| 1.5 | | | .922 | .830 | .761 | .709 | .668 | .636 | .611 | .590 | .570 | .558 | .558 | .518 | .505 | .495 | .487 | .480 | .474 | .469 | .465 | .465 |
| 2.0 | | | .933 | .869 | .821 | .784 | .755 | .732 | .713 | .685 | .666 | .656 | .654 | .649 | .637 | .627 | .620 | .613 | .608 | .601 | .599 | .599 |
| 2.5 | | | .986 | .941 | .907 | .881 | .859 | .842 | .835 | .815 | .797 | .782 | .781 | .782 | .774 | .762 | .754 | .748 | .743 | .739 | .735 | .735 |
| 14 | 1.0 | | .943 | .818 | .720 | .646 | .588 | .543 | .506 | .477 | .452 | .414 | .387 | .366 | .350 | .337 | .327 | .319 | .312 | .306 | .301 | .301 |
| 1.5 | | | .909 | .809 | .755 | .724 | .670 | .629 | .595 | .569 | .547 | .514 | .490 | .472 | .458 | .447 | .432 | .431 | .425 | .420 | .416 | .416 |
| 2.0 | | | .970 | .881 | .814 | .764 | .726 | .696 | .672 | .652 | .622 | .601 | .585 | .585 | .572 | .562 | .554 | .548 | .542 | .537 | .533 | .533 |
| 2.5 | | | .976 | .893 | .813 | .767 | .728 | .698 | .673 | .653 | .623 | .601 | .584 | .584 | .570 | .560 | .552 | .546 | .540 | .535 | .530 | .530 |
| 16 | 1.0 | | .929 | .803 | .705 | .630 | .572 | .526 | .489 | .459 | .433 | .395 | .366 | .345 | .328 | .315 | .304 | .295 | .288 | .282 | .276 | .276 |
| 1.5 | | | .907 | .804 | .769 | .737 | .692 | .659 | .635 | .613 | .585 | .549 | .526 | .506 | .488 | .471 | .458 | .454 | .449 | .444 | .439 | .437 |
| 2.0 | | | .953 | .842 | .773 | .722 | .682 | .651 | .626 | .603 | .574 | .552 | .535 | .535 | .522 | .512 | .503 | .497 | .491 | .486 | .482 | .482 |
| 2.5 | | | .922 | .858 | .810 | .773 | .744 | .721 | .702 | .673 | .654 | .624 | .602 | .586 | .574 | .564 | .556 | .550 | .544 | .539 | .535 | .535 |
| 18 | 1.0 | | .918 | .792 | .694 | .618 | .559 | .513 | .475 | .445 | .419 | .379 | .350 | .328 | .311 | .297 | .286 | .277 | .269 | .263 | .257 | .257 |
| 1.5 | | | .969 | .845 | .749 | .676 | .620 | .576 | .541 | .513 | .488 | .454 | .428 | .408 | .393 | .381 | .372 | .364 | .357 | .352 | .347 | .347 |
| 2.0 | | | .904 | .812 | .742 | .689 | .648 | .614 | .590 | .569 | .536 | .513 | .495 | .495 | .482 | .471 | .463 | .456 | .450 | .445 | .441 | .441 |
| 2.5 | | | .971 | .881 | .815 | .766 | .727 | .697 | .673 | .654 | .624 | .602 | .586 | .586 | .574 | .564 | .556 | .550 | .544 | .539 | .535 | .535 |
| 20 | 1.0 | | .909 | .783 | .684 | .608 | .549 | .502 | .465 | .433 | .408 | .367 | .337 | .315 | .297 | .283 | .271 | .262 | .254 | .247 | .242 | .242 |
| 1.5 | | | .954 | .830 | .733 | .659 | .603 | .558 | .522 | .493 | .469 | .433 | .406 | .386 | .370 | .358 | .348 | .340 | .333 | .327 | .322 | .322 |
| 2.0 | | | .942 | .828 | .741 | .673 | .621 | .588 | .561 | .539 | .506 | .482 | .464 | .464 | .450 | .439 | .430 | .423 | .417 | .411 | .407 | .407 |
| 2.5 | | | .940 | .849 | .782 | .731 | .691 | .650 | .626 | .605 | .575 | .544 | .522 | .506 | .494 | .483 | .475 | .468 | .462 | .457 | .453 | .453 |
| 22 | 1.0 | | .902 | .776 | .677 | .601 | .541 | .494 | .456 | .424 | .398 | .357 | .327 | .304 | .286 | .271 | .260 | .250 | .242 | .235 | .229 | .229 |
| 1.5 | | | .942 | .817 | .720 | .646 | .589 | .543 | .507 | .478 | .453 | .416 | .388 | .367 | .351 | .339 | .328 | .320 | .313 | .307 | .302 | .302 |
| 2.0 | | | .987 | .864 | .769 | .697 | .642 | .600 | .566 | .538 | .516 | .481 | .456 | .437 | .423 | .412 | .403 | .395 | .389 | .383 | .379 | .379 |
| 2.5 | | | .915 | .823 | .754 | .702 | .662 | .620 | .583 | .551 | .528 | .493 | .468 | .448 | .435 | .425 | .418 | .412 | .406 | .401 | .397 | .397 |
| 24 | 1.0 | | .895 | .769 | .670 | .594 | .535 | .487 | .449 | .417 | .391 | .349 | .319 | .295 | .277 | .262 | .250 | .240 | .231 | .224 | .218 | .218 |
| 1.5 | | | .933 | .807 | .710 | .635 | .577 | .531 | .495 | .465 | .440 | .401 | .373 | .352 | .336 | .323 | .312 | .303 | .296 | .290 | .285 | .285 |
| 2.0 | | | .972 | .849 | .754 | .681 | .625 | .582 | .547 | .519 | .496 | .456 | .430 | .416 | .401 | .389 | .380 | .372 | .365 | .358 | .353 | .353 |
| 2.5 | | | .945 | .855 | .802 | .732 | .679 | .637 | .605 | .578 | .555 | .524 | .501 | .483 | .469 | .459 | .450 | .443 | .437 | .432 | .427 | .427 |
| 26 | 1.0 | | .891 | .764 | .665 | .589 | .529 | .481 | .443 | .411 | .384 | .343 | .312 | .288 | .269 | .254 | .242 | .231 | .223 | .216 | .209 | .209 |
| 1.5 | | | .924 | .799 | .701 | .626 | .567 | .521 | .484 | .454 | .429 | .390 | .361 | .339 | .323 | .309 | .298 | .289 | .282 | .276 | .270 | .270 |
| 2.0 | | | .961 | .837 | .741 | .667 | .611 | .567 | .532 | .503 | .479 | .443 | .417 | .397 | .382 | .370 | .360 | .352 | .346 | .340 | .335 | .335 |
| 2.5 | | | 1.000 | .878 | .784 | .713 | .659 | .617 | .584 | .557 | .535 | .501 | .477 | .459 | .445 | .434 | .425 | .418 | .411 | .406 | .401 | .401 |
| 28 | 1.0 | | .887 | .760 | .660 | .584 | .524 | .476 | .438 | .406 | .379 | .337 | .306 | .282 | .263 | .247 | .235 | .224 | .216 | .208 | .202 | .202 |
| 1.5 | | | .917 | .792 | .693 | .618 | .559 | .513 | .475 | .445 | .419 | .380 | .351 | .329 | .311 | .298 | .287 | .277 | .270 | .263 | .258 | .258 |
| 2.0 | | | .951 | .826 | .730 | .656 | .599 | .554 | .519 | .489 | .465 | .429 | .402 | .382 | .366 | .354 | .344 | .338 | .332 | .326 | .321 | .321 |
| 2.5 | | | .985 | .864 | .769 | .698 | .643 | .600 | .566 | .539 | .516 | .482 | .456 | .438 | .424 | .412 | .403 | .396 | .390 | .384 | .378 | .378 |
| 30 | 1.0 | | .883 | .756 | .657 | .580 | .520 | .472 | .433 | .401 | .374 | .332 | .301 | .276 | .257 | .242 | .229 | .218 | .209 | .202 | .195 | .195 |
| 1.5 | | | .912 | .785 | .687 | .611 | .552 | .506 | .468 | .437 | .411 | .371 | .342 | .319 | .302 | .288 | .277 | .269 | .262 | .255 | .249 | .249 |
| 2.0 | | | .942 | .817 | .720 | .646 | .589 | .544 | .507 | .478 | .454 | .416 | .389 | .368 | .352 | .339 | .329 | .321 | .314 | .308 | .303 | .303 |
| 2.5 | | | .975 | .852 | .756 | .684 | .629 | .585 | .551 | .523 | .500 | .465 | .439 | .420 | .406 | .394 | .385 | .377 | .370 | .363 | .357 | .357 |

TABLE 5.1.1 (Continued)
b = 2.0 $\alpha = 0.01$

| a ₁ f | a ₂ | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|------------------|----------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|--|--|--|--|--|--|--|--|--|--|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 | 26 | 28 | 30 | | | | | | | | | | |
| 1 | 1.928 | 1.621 | 1.717 | 1.628 | 1.554 | 1.493 | 1.441 | 1.398 | 1.362 | 1.331 | 1.281 | 1.242 | 1.212 | 1.187 | 1.167 | 1.150 | 1.136 | 1.124 | 1.114 | 1.105 | | | | | | | | | | |
| 1.5 | 2.411 | 2.295 | 2.107 | 2.096 | 2.022 | 1.961 | 1.911 | 1.869 | 1.834 | 1.804 | 1.756 | 1.719 | 1.690 | 1.667 | 1.648 | 1.633 | 1.620 | 1.609 | 1.599 | 1.591 | | | | | | | | | | |
| 2.0 | 2.897 | 2.774 | 2.662 | 2.571 | 2.497 | 2.437 | 2.387 | 2.347 | 2.312 | 2.283 | 2.237 | 2.202 | 2.174 | 2.152 | 2.135 | 2.120 | 2.109 | 2.097 | 2.088 | 2.081 | | | | | | | | | | |
| 2.5 | 3.385 | 3.256 | 3.142 | 3.050 | 2.976 | 2.917 | 2.868 | 2.828 | 2.795 | 2.767 | 2.722 | 2.688 | 2.661 | 2.640 | 2.624 | 2.610 | 2.598 | 2.588 | 2.580 | 2.572 | | | | | | | | | | |
| 2 | 1.621 | 1.595 | 1.585 | 1.495 | 1.422 | 1.362 | 1.313 | 1.273 | 1.239 | 1.210 | 1.165 | 1.131 | 1.105 | 1.084 | 1.068 | 1.054 | 1.043 | 1.034 | 1.026 | 1.020 | | | | | | | | | | |
| 1.5 | 2.265 | 2.126 | 2.011 | 1.920 | 1.848 | 1.790 | 1.744 | 1.706 | 1.675 | 1.649 | 1.608 | 1.578 | 1.558 | 1.538 | 1.524 | 1.513 | 1.504 | 1.496 | 1.490 | 1.484 | | | | | | | | | | |
| 2.0 | 2.715 | 2.566 | 2.449 | 2.358 | 2.288 | 2.232 | 2.188 | 2.153 | 2.124 | 2.100 | 2.063 | 2.036 | 2.016 | 2.000 | 1.988 | 1.978 | 1.970 | 1.963 | 1.957 | 1.952 | | | | | | | | | | |
| 2.5 | 3.169 | 3.013 | 2.894 | 2.796 | 2.736 | 2.683 | 2.641 | 2.607 | 2.580 | 2.558 | 2.524 | 2.500 | 2.481 | 2.467 | 2.455 | 2.446 | 2.438 | 2.432 | 2.426 | 2.421 | | | | | | | | | | |
| 3 | 1.717 | 1.595 | 1.475 | 1.395 | 1.313 | 1.254 | 1.206 | 1.167 | 1.135 | 1.107 | 1.065 | 1.033 | 1.010 | .991 | .977 | .965 | .955 | .947 | .940 | .934 | | | | | | | | | | |
| 1.5 | 2.116 | 1.970 | 1.855 | 1.765 | 1.695 | 1.640 | 1.596 | 1.561 | 1.532 | 1.508 | 1.471 | 1.445 | 1.425 | 1.409 | 1.397 | 1.386 | 1.378 | 1.371 | 1.365 | 1.359 | | | | | | | | | | |
| 2.0 | 2.523 | 2.368 | 2.250 | 2.162 | 2.095 | 2.043 | 2.003 | 1.971 | 1.945 | 1.923 | 1.890 | 1.866 | 1.848 | 1.833 | 1.822 | 1.812 | 1.804 | 1.796 | 1.792 | 1.787 | | | | | | | | | | |
| 2.5 | 2.935 | 2.773 | 2.656 | 2.570 | 2.505 | 2.457 | 2.419 | 2.389 | 2.364 | 2.344 | 2.313 | 2.291 | 2.273 | 2.260 | 2.249 | 2.239 | 2.232 | 2.225 | 2.220 | 2.215 | | | | | | | | | | |
| 4 | 1.628 | 1.495 | 1.385 | 1.296 | 1.224 | 1.166 | 1.119 | 1.080 | 1.048 | 1.021 | .980 | .949 | .926 | .908 | .894 | .883 | .873 | .865 | .858 | .853 | | | | | | | | | | |
| 1.5 | 1.986 | 1.838 | 1.723 | 1.634 | 1.565 | 1.511 | 1.468 | 1.434 | 1.406 | 1.383 | 1.347 | 1.321 | 1.302 | 1.286 | 1.274 | 1.264 | 1.256 | 1.248 | 1.242 | 1.237 | | | | | | | | | | |
| 2.0 | 2.351 | 2.195 | 2.078 | 1.991 | 1.926 | 1.876 | 1.836 | 1.805 | 1.780 | 1.759 | 1.726 | 1.702 | 1.684 | 1.670 | 1.659 | 1.649 | 1.641 | 1.635 | 1.629 | 1.624 | | | | | | | | | | |
| 2.5 | 2.722 | 2.560 | 2.444 | 2.360 | 2.297 | 2.250 | 2.213 | 2.183 | 2.160 | 2.140 | 2.109 | 2.087 | 2.070 | 2.056 | 2.045 | 2.036 | 2.029 | 2.022 | 2.017 | 2.012 | | | | | | | | | | |
| 5 | 1.554 | 1.422 | 1.313 | 1.224 | 1.152 | 1.094 | 1.047 | 1.009 | .977 | .950 | .908 | .879 | .854 | .837 | .822 | .811 | .801 | .793 | .786 | .780 | | | | | | | | | | |
| 1.5 | 1.875 | 1.728 | 1.613 | 1.525 | 1.456 | 1.403 | 1.360 | 1.326 | 1.298 | 1.275 | 1.239 | 1.213 | 1.194 | 1.178 | 1.166 | 1.156 | 1.147 | 1.140 | 1.134 | 1.129 | | | | | | | | | | |
| 2.0 | 2.204 | 2.048 | 1.932 | 1.846 | 1.781 | 1.731 | 1.692 | 1.661 | 1.634 | 1.614 | 1.582 | 1.558 | 1.540 | 1.526 | 1.514 | 1.505 | 1.497 | 1.490 | 1.485 | 1.480 | | | | | | | | | | |
| 2.5 | 2.539 | 2.378 | 2.263 | 2.179 | 2.117 | 2.069 | 2.033 | 2.003 | 1.979 | 1.959 | 1.929 | 1.906 | 1.889 | 1.875 | 1.865 | 1.856 | 1.848 | 1.842 | 1.836 | 1.831 | | | | | | | | | | |
| 6 | 1.493 | 1.362 | 1.254 | 1.166 | 1.094 | 1.036 | .989 | .950 | .918 | .890 | .848 | .817 | .794 | .776 | .761 | .749 | .740 | .731 | .725 | .719 | | | | | | | | | | |
| 1.5 | 1.782 | 1.636 | 1.523 | 1.435 | 1.366 | 1.312 | 1.269 | 1.235 | 1.207 | 1.183 | 1.147 | 1.121 | 1.101 | 1.085 | 1.073 | 1.063 | 1.054 | 1.047 | 1.041 | 1.036 | | | | | | | | | | |
| 2.0 | 2.080 | 1.925 | 1.810 | 1.724 | 1.659 | 1.609 | 1.569 | 1.538 | 1.512 | 1.491 | 1.458 | 1.434 | 1.416 | 1.402 | 1.390 | 1.381 | 1.373 | 1.366 | 1.360 | 1.355 | | | | | | | | | | |
| 2.5 | 2.384 | 2.224 | 2.109 | 2.025 | 1.963 | 1.915 | 1.878 | 1.849 | 1.824 | 1.805 | 1.774 | 1.751 | 1.734 | 1.720 | 1.709 | 1.700 | 1.692 | 1.686 | 1.680 | 1.676 | | | | | | | | | | |
| 7 | 1.441 | 1.313 | 1.206 | 1.119 | 1.047 | .989 | .941 | .902 | .869 | .841 | .798 | .767 | .743 | .724 | .709 | .697 | .687 | .679 | .672 | .666 | | | | | | | | | | |
| 1.5 | 1.704 | 1.506 | 1.447 | 1.360 | 1.291 | 1.236 | 1.193 | 1.158 | 1.130 | 1.106 | 1.069 | 1.042 | 1.022 | 1.006 | .994 | .983 | .975 | .967 | .961 | .956 | | | | | | | | | | |
| 2.0 | 1.976 | 1.802 | 1.708 | 1.621 | 1.558 | 1.505 | 1.463 | 1.430 | 1.408 | 1.386 | 1.353 | 1.328 | 1.310 | 1.295 | 1.284 | 1.274 | 1.266 | 1.259 | 1.254 | 1.249 | | | | | | | | | | |
| 2.5 | 2.253 | 2.094 | 1.979 | 1.895 | 1.832 | 1.784 | 1.747 | 1.717 | 1.693 | 1.672 | 1.641 | 1.618 | 1.601 | 1.587 | 1.576 | 1.567 | 1.559 | 1.553 | 1.547 | 1.542 | | | | | | | | | | |
| 8 | 1.398 | 1.273 | 1.167 | 1.080 | 1.009 | .950 | .902 | .862 | .828 | .800 | .756 | .724 | .699 | .680 | .665 | .653 | .643 | .634 | .627 | .621 | | | | | | | | | | |
| 1.5 | 1.639 | 1.496 | 1.384 | 1.297 | 1.228 | 1.173 | 1.129 | 1.093 | 1.064 | 1.040 | 1.000 | .976 | .955 | .939 | .926 | .915 | .907 | .899 | .893 | .889 | | | | | | | | | | |
| 2.0 | 1.887 | 1.735 | 1.621 | 1.535 | 1.469 | 1.417 | 1.377 | 1.345 | 1.318 | 1.296 | 1.263 | 1.234 | 1.219 | 1.204 | 1.193 | 1.183 | 1.175 | 1.168 | 1.162 | 1.157 | | | | | | | | | | |
| 2.5 | 2.141 | 1.983 | 1.869 | 1.784 | 1.721 | 1.673 | 1.635 | 1.604 | 1.580 | 1.559 | 1.528 | 1.505 | 1.487 | 1.473 | 1.462 | 1.452 | 1.445 | 1.438 | 1.433 | 1.428 | | | | | | | | | | |
| 9 | 1.362 | 1.239 | 1.135 | 1.046 | .977 | .918 | .869 | .828 | .794 | .766 | .721 | .688 | .662 | .643 | .627 | .615 | .604 | .596 | .589 | .582 | | | | | | | | | | |
| 1.5 | 1.582 | 1.442 | 1.331 | 1.244 | 1.174 | 1.119 | 1.074 | 1.038 | 1.009 | .984 | .946 | .918 | .897 | .881 | .867 | .857 | .848 | .840 | .834 | .829 | | | | | | | | | | |
| 2.0 | 1.811 | 1.661 | 1.547 | 1.461 | 1.394 | 1.342 | 1.301 | 1.269 | 1.242 | 1.220 | 1.185 | 1.160 | 1.141 | 1.124 | 1.114 | 1.104 | 1.095 | 1.089 | 1.083 | 1.078 | | | | | | | | | | |
| 2.5 | 2.045 | 1.899 | 1.774 | 1.690 | 1.626 | 1.577 | 1.538 | 1.508 | 1.483 | 1.462 | 1.430 | 1.406 | 1.378 | 1.358 | 1.354 | 1.346 | 1.336 | 1.330 | 1.324 | 1.320 | | | | | | | | | | |
| 10 | 1.331 | 1.210 | 1.107 | 1.021 | .950 | .890 | .841 | .800 | .766 | .737 | .691 | .657 | .631 | .611 | .595 | .582 | .571 | .562 | .555 | .548 | | | | | | | | | | |
| 1.5 | 1.534 | 1.396 | 1.286 | 1.199 | 1.129 | 1.073 | 1.028 | .991 | .961 | .936 | .897 | .869 | .847 | .830 | .817 | .806 | .797 | .789 | .783 | .777 | | | | | | | | | | |
| 2.0 | 1.766 | 1.597 | 1.484 | 1.397 | 1.330 | 1.278 | 1.236 | 1.203 | 1.176 | 1.153 | 1.118 | 1.093 | 1.073 | 1.058 | 1.046 | 1.036 | 1.026 | 1.021 | 1.015 | 1.009 | | | | | | | | | | |
| 2.5 | 1.982 | 1.807 | 1.693 | 1.608 | 1.544 | 1.494 | 1.455 | 1.424 | 1.399 | 1.378 | 1.345 | 1.321 | 1.303 | 1.289 | 1.277 | 1.268 | 1.260 | 1.253 | 1.244 | 1.243 | | | | | | | | | | |

TABLE 5.1.1 (Continued)
 $b = 2.0 \quad \alpha = 0.01$

| | | a2 | | | | | | | | | | | | | | | | | | | | |
|-----|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|------|------|
| | | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 | 26 | 28 | 30 | |
| 12 | 1.0 | 1.261 | 1.165 | 1.065 | .980 | .908 | .848 | .798 | .756 | .721 | .691 | .663 | .638 | .616 | .597 | .580 | .565 | .551 | .538 | .526 | .515 | .504 |
| 1.5 | 1.0 | 1.456 | 1.323 | 1.214 | 1.127 | 1.057 | 1.000 | .954 | .916 | .885 | .858 | .833 | .811 | .792 | .775 | .759 | .744 | .730 | .717 | .705 | .693 | .683 |
| 2.0 | 1.0 | 1.639 | 1.494 | 1.382 | 1.295 | 1.227 | 1.174 | 1.131 | 1.096 | 1.068 | 1.045 | 1.023 | .999 | .974 | .949 | .924 | .900 | .876 | .852 | .829 | .807 | .785 |
| 2.5 | 1.0 | 1.827 | 1.675 | 1.560 | 1.473 | 1.406 | 1.350 | 1.313 | 1.282 | 1.254 | 1.230 | 1.207 | 1.181 | 1.154 | 1.128 | 1.102 | 1.076 | 1.050 | 1.024 | 1.000 | .974 | .949 |
| 14 | 1.0 | 1.242 | 1.151 | 1.053 | .969 | .898 | .837 | .787 | .747 | .712 | .682 | .654 | .629 | .605 | .583 | .562 | .542 | .523 | .505 | .488 | .472 | .456 |
| 1.5 | 1.0 | 1.395 | 1.266 | 1.160 | 1.073 | 1.003 | .943 | .893 | .854 | .820 | .790 | .762 | .737 | .713 | .690 | .669 | .649 | .629 | .610 | .593 | .576 | .560 |
| 2.0 | 1.0 | 1.553 | 1.415 | 1.304 | 1.217 | 1.149 | 1.094 | 1.050 | 1.014 | .985 | .961 | .937 | .913 | .889 | .865 | .842 | .819 | .797 | .775 | .754 | .733 | .712 |
| 2.5 | 1.0 | 1.722 | 1.575 | 1.460 | 1.374 | 1.307 | 1.255 | 1.214 | 1.181 | 1.154 | 1.132 | 1.097 | 1.072 | 1.053 | 1.037 | 1.025 | 1.016 | 1.007 | 1.000 | .994 | .989 | .984 |
| 16 | 1.0 | 1.212 | 1.105 | 1.010 | .926 | .854 | .794 | .743 | .699 | .662 | .631 | .603 | .578 | .554 | .532 | .513 | .494 | .475 | .457 | .440 | .424 | .417 |
| 1.5 | 1.0 | 1.342 | 1.223 | 1.118 | 1.032 | .961 | .903 | .855 | .815 | .781 | .753 | .727 | .702 | .678 | .654 | .633 | .612 | .592 | .573 | .554 | .537 | .521 |
| 2.0 | 1.0 | 1.490 | 1.353 | 1.244 | 1.157 | 1.087 | 1.032 | .987 | .950 | .920 | .896 | .871 | .846 | .822 | .800 | .779 | .758 | .737 | .716 | .695 | .674 | .654 |
| 2.5 | 1.0 | 1.638 | 1.492 | 1.380 | 1.293 | 1.226 | 1.173 | 1.131 | 1.097 | 1.069 | 1.046 | 1.019 | .994 | .971 | .949 | .926 | .904 | .882 | .860 | .838 | .816 | .795 |
| 18 | 1.0 | 1.187 | 1.084 | .991 | .908 | .837 | .776 | .724 | .680 | .643 | .611 | .582 | .556 | .532 | .509 | .489 | .469 | .449 | .430 | .411 | .393 | .375 |
| 1.5 | 1.0 | 1.309 | 1.194 | 1.086 | 1.000 | .929 | .870 | .821 | .780 | .746 | .717 | .692 | .668 | .644 | .621 | .599 | .577 | .555 | .533 | .511 | .490 | .470 |
| 2.0 | 1.0 | 1.437 | 1.303 | 1.195 | 1.109 | 1.039 | .982 | .936 | .899 | .867 | .841 | .816 | .791 | .767 | .743 | .720 | .700 | .679 | .657 | .635 | .613 | .592 |
| 2.5 | 1.0 | 1.570 | 1.427 | 1.316 | 1.229 | 1.161 | 1.107 | 1.064 | 1.029 | 1.000 | .976 | .952 | .928 | .904 | .881 | .858 | .835 | .812 | .789 | .766 | .743 | .721 |
| 20 | 1.0 | 1.157 | 1.054 | .965 | .883 | .811 | .749 | .697 | .653 | .615 | .582 | .553 | .528 | .504 | .482 | .461 | .440 | .420 | .400 | .381 | .362 | .343 |
| 1.5 | 1.0 | 1.277 | 1.159 | 1.050 | .967 | .903 | .844 | .794 | .752 | .717 | .688 | .661 | .636 | .611 | .587 | .564 | .542 | .520 | .498 | .476 | .454 | .433 |
| 2.0 | 1.0 | 1.393 | 1.263 | 1.156 | 1.070 | .999 | .942 | .895 | .857 | .825 | .798 | .776 | .754 | .732 | .710 | .688 | .666 | .644 | .622 | .600 | .578 | .556 |
| 2.5 | 1.0 | 1.515 | 1.374 | 1.264 | 1.177 | 1.108 | 1.053 | 1.009 | .973 | .944 | .919 | .891 | .863 | .835 | .807 | .780 | .752 | .724 | .696 | .668 | .640 | .613 |
| 22 | 1.0 | 1.150 | 1.054 | .965 | .883 | .811 | .749 | .697 | .653 | .615 | .582 | .553 | .528 | .504 | .482 | .461 | .440 | .420 | .400 | .381 | .362 | .343 |
| 1.5 | 1.0 | 1.250 | 1.136 | 1.028 | .944 | .882 | .822 | .772 | .730 | .694 | .664 | .638 | .613 | .588 | .563 | .538 | .513 | .488 | .463 | .438 | .413 | .388 |
| 2.0 | 1.0 | 1.356 | 1.229 | 1.124 | 1.038 | .968 | .910 | .862 | .823 | .780 | .762 | .735 | .708 | .681 | .654 | .627 | .600 | .573 | .546 | .519 | .492 | .465 |
| 2.5 | 1.0 | 1.466 | 1.329 | 1.220 | 1.134 | 1.064 | 1.009 | .964 | .927 | .897 | .871 | .845 | .818 | .791 | .764 | .737 | .710 | .683 | .656 | .629 | .602 | .575 |
| 24 | 1.0 | 1.136 | 1.043 | .955 | .873 | .801 | .740 | .687 | .643 | .604 | .571 | .542 | .517 | .492 | .467 | .443 | .419 | .395 | .371 | .347 | .323 | .300 |
| 1.5 | 1.0 | 1.228 | 1.117 | 1.020 | .937 | .865 | .805 | .754 | .712 | .675 | .645 | .619 | .593 | .567 | .541 | .515 | .489 | .463 | .437 | .411 | .385 | .359 |
| 2.0 | 1.0 | 1.325 | 1.201 | 1.098 | 1.012 | .941 | .883 | .834 | .794 | .761 | .733 | .708 | .682 | .656 | .630 | .604 | .578 | .552 | .526 | .500 | .474 | .448 |
| 2.5 | 1.0 | 1.426 | 1.292 | 1.184 | 1.098 | 1.028 | .972 | .926 | .888 | .857 | .831 | .806 | .780 | .754 | .728 | .702 | .676 | .650 | .624 | .598 | .572 | .546 |
| 26 | 1.0 | 1.124 | 1.034 | .947 | .865 | .793 | .731 | .679 | .634 | .596 | .562 | .533 | .508 | .483 | .458 | .433 | .408 | .383 | .358 | .333 | .308 | .283 |
| 1.5 | 1.0 | 1.209 | 1.101 | 1.006 | .923 | .851 | .790 | .739 | .696 | .660 | .628 | .599 | .574 | .549 | .524 | .499 | .474 | .449 | .424 | .400 | .375 | .350 |
| 2.0 | 1.0 | 1.298 | 1.171 | 1.075 | .990 | .919 | .860 | .811 | .771 | .736 | .708 | .682 | .656 | .630 | .604 | .578 | .552 | .526 | .500 | .474 | .448 | .422 |
| 2.5 | 1.0 | 1.391 | 1.260 | 1.154 | 1.068 | .998 | .940 | .894 | .856 | .824 | .797 | .771 | .745 | .719 | .693 | .667 | .641 | .615 | .589 | .563 | .537 | .511 |
| 28 | 1.0 | 1.114 | 1.026 | .940 | .858 | .786 | .725 | .672 | .627 | .588 | .555 | .526 | .500 | .475 | .450 | .425 | .400 | .375 | .350 | .325 | .300 | .275 |
| 1.5 | 1.0 | 1.192 | 1.087 | .993 | .911 | .839 | .778 | .727 | .683 | .646 | .615 | .589 | .564 | .539 | .514 | .489 | .464 | .439 | .414 | .389 | .364 | .339 |
| 2.0 | 1.0 | 1.275 | 1.157 | 1.056 | .972 | .901 | .841 | .792 | .751 | .716 | .686 | .660 | .635 | .610 | .585 | .560 | .535 | .510 | .485 | .460 | .435 | .410 |
| 2.5 | 1.0 | 1.362 | 1.233 | 1.128 | 1.042 | .972 | .914 | .867 | .824 | .795 | .768 | .742 | .716 | .690 | .664 | .638 | .612 | .586 | .560 | .534 | .508 | .482 |
| 30 | 1.0 | 1.105 | 1.020 | .934 | .853 | .780 | .719 | .666 | .621 | .584 | .551 | .522 | .496 | .470 | .444 | .418 | .392 | .366 | .340 | .314 | .288 | .262 |
| 1.5 | 1.0 | 1.177 | 1.075 | .983 | .900 | .829 | .768 | .716 | .672 | .635 | .603 | .575 | .549 | .523 | .497 | .471 | .445 | .419 | .393 | .367 | .341 | .315 |
| 2.0 | 1.0 | 1.255 | 1.139 | 1.040 | .956 | .885 | .825 | .775 | .733 | .698 | .668 | .641 | .615 | .589 | .563 | .537 | .511 | .485 | .459 | .433 | .407 | .381 |
| 2.5 | 1.0 | 1.335 | 1.210 | 1.106 | 1.020 | .949 | .891 | .843 | .804 | .771 | .743 | .716 | .690 | .664 | .638 | .612 | .586 | .560 | .534 | .508 | .482 | .456 |

TABLE 5.1.1 (Continued)
b = 2.0 α = 0.05

| a1 f | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 | 26 | 28 | 30 |
|--------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 1 1.0 | 1.632 | 1.681 | 1.560 | 1.467 | 1.396 | 1.340 | 1.296 | 1.259 | 1.230 | 1.250 | 1.166 | 1.137 | 1.115 | 1.098 | 1.084 | 1.073 | 1.064 | 1.056 | 1.049 | 1.044 |
| 1 1.5 | 2.295 | 2.334 | 2.011 | 1.920 | 1.850 | 1.797 | 1.753 | 1.721 | 1.695 | 1.670 | 1.635 | 1.609 | 1.589 | 1.571 | 1.561 | 1.551 | 1.543 | 1.536 | 1.531 | 1.526 |
| 2 0 | 2.762 | 2.895 | 2.472 | 2.382 | 2.315 | 2.263 | 2.223 | 2.191 | 2.165 | 2.143 | 2.110 | 2.086 | 2.068 | 2.054 | 2.043 | 2.034 | 2.027 | 2.021 | 2.016 | 2.012 |
| 2 5 | 3.233 | 3.562 | 2.939 | 2.850 | 2.785 | 2.735 | 2.696 | 2.666 | 2.641 | 2.621 | 2.590 | 2.568 | 2.551 | 2.538 | 2.528 | 2.520 | 2.513 | 2.508 | 2.503 | 2.500 |
| 3 1.0 | 1.681 | 1.524 | 1.404 | 1.314 | 1.245 | 1.193 | 1.151 | 1.116 | 1.091 | 1.069 | 1.035 | 1.011 | .992 | .978 | .967 | .957 | .950 | .943 | .937 | .932 |
| 1 5 | 2.682 | 1.515 | 1.795 | 1.708 | 1.644 | 1.596 | 1.559 | 1.529 | 1.506 | 1.487 | 1.457 | 1.435 | 1.419 | 1.406 | 1.395 | 1.387 | 1.379 | 1.373 | 1.366 | 1.363 |
| 2 0 | 2.491 | 2.330 | 2.201 | 2.118 | 2.058 | 2.014 | 1.979 | 1.952 | 1.931 | 1.913 | 1.885 | 1.864 | 1.848 | 1.836 | 1.826 | 1.817 | 1.810 | 1.804 | 1.799 | 1.795 |
| 2 5 | 2.906 | 2.733 | 2.617 | 2.537 | 2.481 | 2.438 | 2.406 | 2.380 | 2.359 | 2.342 | 2.315 | 2.295 | 2.279 | 2.267 | 2.257 | 2.249 | 2.242 | 2.236 | 2.231 | 2.226 |
| 3 1.0 | 1.557 | 1.404 | 1.285 | 1.196 | 1.129 | 1.077 | 1.036 | 1.004 | .978 | .956 | .923 | .899 | .881 | .866 | .855 | .845 | .837 | .831 | .826 | .820 |
| 1 5 | 1.903 | 1.738 | 1.620 | 1.535 | 1.473 | 1.426 | 1.390 | 1.361 | 1.338 | 1.319 | 1.289 | 1.267 | 1.250 | 1.237 | 1.226 | 1.217 | 1.210 | 1.204 | 1.198 | 1.194 |
| 2 0 | 2.257 | 2.088 | 1.972 | 1.891 | 1.833 | 1.789 | 1.755 | 1.728 | 1.706 | 1.688 | 1.660 | 1.639 | 1.623 | 1.610 | 1.600 | 1.591 | 1.584 | 1.578 | 1.573 | 1.568 |
| 2 5 | 2.617 | 2.446 | 2.334 | 2.256 | 2.199 | 2.157 | 2.125 | 2.098 | 2.077 | 2.060 | 2.033 | 2.012 | 1.997 | 1.984 | 1.974 | 1.966 | 1.959 | 1.953 | 1.948 | 1.943 |
| 4 1.0 | 1.467 | 1.314 | 1.196 | 1.108 | 1.040 | .988 | .947 | .914 | .888 | .866 | .832 | .807 | .789 | .774 | .762 | .752 | .744 | .738 | .732 | .727 |
| 1 5 | 1.764 | 1.600 | 1.483 | 1.399 | 1.337 | 1.289 | 1.253 | 1.223 | 1.200 | 1.180 | 1.150 | 1.127 | 1.110 | 1.097 | 1.086 | 1.077 | 1.070 | 1.063 | 1.058 | 1.053 |
| 2 0 | 2.070 | 1.903 | 1.788 | 1.708 | 1.649 | 1.604 | 1.570 | 1.542 | 1.520 | 1.502 | 1.473 | 1.452 | 1.436 | 1.423 | 1.412 | 1.404 | 1.397 | 1.391 | 1.385 | 1.381 |
| 2 5 | 2.384 | 2.215 | 2.102 | 2.024 | 1.967 | 1.925 | 1.892 | 1.865 | 1.844 | 1.826 | 1.799 | 1.778 | 1.762 | 1.750 | 1.740 | 1.731 | 1.724 | 1.718 | 1.713 | 1.709 |
| 5 1.0 | 1.396 | 1.245 | 1.123 | 1.040 | .972 | .919 | .878 | .844 | .817 | .795 | .760 | .735 | .715 | .700 | .688 | .678 | .670 | .663 | .657 | .652 |
| 1 5 | 1.654 | 1.493 | 1.377 | 1.292 | 1.229 | 1.181 | 1.143 | 1.114 | 1.089 | 1.069 | 1.037 | 1.016 | .998 | .985 | .974 | .965 | .957 | .950 | .945 | .940 |
| 2 0 | 1.923 | 1.758 | 1.643 | 1.562 | 1.502 | 1.457 | 1.422 | 1.394 | 1.371 | 1.353 | 1.323 | 1.302 | 1.285 | 1.272 | 1.262 | 1.253 | 1.246 | 1.240 | 1.234 | 1.230 |
| 2 5 | 2.199 | 2.031 | 1.918 | 1.839 | 1.782 | 1.739 | 1.705 | 1.679 | 1.657 | 1.639 | 1.611 | 1.590 | 1.574 | 1.562 | 1.551 | 1.543 | 1.536 | 1.530 | 1.524 | 1.520 |
| 6 1.0 | 1.340 | 1.193 | 1.077 | .988 | .919 | .865 | .823 | .789 | .761 | .738 | .702 | .676 | .656 | .641 | .629 | .618 | .610 | .603 | .597 | .591 |
| 1 5 | 1.568 | 1.409 | 1.293 | 1.207 | 1.144 | 1.095 | 1.056 | 1.026 | 1.001 | .981 | .949 | .926 | .908 | .894 | .883 | .874 | .866 | .859 | .854 | .849 |
| 2 0 | 1.806 | 1.642 | 1.527 | 1.445 | 1.385 | 1.339 | 1.303 | 1.275 | 1.252 | 1.233 | 1.203 | 1.181 | 1.164 | 1.151 | 1.140 | 1.132 | 1.124 | 1.118 | 1.113 | 1.108 |
| 2 5 | 2.051 | 1.884 | 1.771 | 1.691 | 1.633 | 1.589 | 1.555 | 1.528 | 1.506 | 1.488 | 1.459 | 1.438 | 1.422 | 1.409 | 1.399 | 1.391 | 1.383 | 1.377 | 1.372 | 1.367 |
| 7 1.0 | 1.295 | 1.151 | 1.036 | .947 | .878 | .823 | .780 | .745 | .716 | .693 | .656 | .629 | .609 | .593 | .580 | .570 | .561 | .554 | .548 | .542 |
| 1 5 | 1.498 | 1.342 | 1.226 | 1.140 | 1.075 | 1.025 | .986 | .955 | .930 | .909 | .877 | .853 | .835 | .820 | .809 | .800 | .792 | .785 | .779 | .774 |
| 2 0 | 1.711 | 1.549 | 1.434 | 1.351 | 1.290 | 1.243 | 1.207 | 1.178 | 1.154 | 1.135 | 1.105 | 1.082 | 1.065 | 1.052 | 1.041 | 1.032 | 1.025 | 1.018 | 1.013 | 1.008 |
| 2 5 | 1.930 | 1.765 | 1.651 | 1.571 | 1.512 | 1.468 | 1.433 | 1.405 | 1.383 | 1.364 | 1.336 | 1.314 | 1.298 | 1.285 | 1.275 | 1.266 | 1.259 | 1.253 | 1.247 | 1.243 |
| 8 1.0 | 1.259 | 1.118 | 1.004 | .914 | .844 | .789 | .745 | .709 | .680 | .656 | .618 | .591 | .570 | .553 | .540 | .530 | .521 | .513 | .507 | .501 |
| 1 5 | 1.441 | 1.287 | 1.172 | 1.085 | 1.019 | .969 | .929 | .897 | .871 | .850 | .817 | .792 | .774 | .759 | .748 | .738 | .730 | .723 | .718 | .713 |
| 2 0 | 1.633 | 1.472 | 1.357 | 1.274 | 1.212 | 1.164 | 1.127 | 1.098 | 1.074 | 1.054 | 1.023 | 1.001 | .983 | .970 | .959 | .950 | .942 | .936 | .930 | .925 |
| 2 5 | 1.831 | 1.667 | 1.553 | 1.472 | 1.412 | 1.367 | 1.332 | 1.304 | 1.281 | 1.262 | 1.233 | 1.212 | 1.195 | 1.182 | 1.171 | 1.163 | 1.155 | 1.149 | 1.144 | 1.139 |
| 9 1.0 | 1.230 | 1.091 | .978 | .888 | .817 | .761 | .716 | .680 | .650 | .626 | .587 | .559 | .537 | .521 | .507 | .496 | .487 | .479 | .473 | .467 |
| 1 5 | 1.394 | 1.242 | 1.127 | 1.040 | .974 | .922 | .882 | .849 | .823 | .801 | .767 | .742 | .723 | .708 | .696 | .687 | .678 | .672 | .666 | .661 |
| 2 0 | 1.568 | 1.409 | 1.294 | 1.210 | 1.147 | 1.099 | 1.061 | 1.031 | 1.007 | .987 | .955 | .932 | .915 | .901 | .890 | .880 | .873 | .866 | .861 | .856 |
| 2 5 | 1.749 | 1.585 | 1.471 | 1.390 | 1.329 | 1.283 | 1.248 | 1.219 | 1.196 | 1.177 | 1.147 | 1.125 | 1.108 | 1.095 | 1.085 | 1.076 | 1.068 | 1.062 | 1.057 | 1.052 |
| 10 1.0 | 1.250 | 1.069 | .956 | .866 | .795 | .738 | .693 | .656 | .626 | .600 | .561 | .532 | .510 | .493 | .479 | .468 | .459 | .451 | .444 | .438 |
| 1 5 | 1.355 | 1.205 | 1.090 | 1.003 | .936 | .883 | .842 | .809 | .782 | .759 | .725 | .699 | .680 | .665 | .653 | .643 | .635 | .628 | .622 | .616 |
| 2 0 | 1.514 | 1.356 | 1.241 | 1.156 | 1.093 | 1.044 | 1.006 | .975 | .950 | .929 | .898 | .874 | .856 | .842 | .831 | .822 | .814 | .807 | .802 | .797 |
| 2 5 | 1.679 | 1.517 | 1.403 | 1.320 | 1.259 | 1.213 | 1.176 | 1.147 | 1.124 | 1.104 | 1.074 | 1.052 | 1.035 | 1.021 | 1.011 | 1.002 | .994 | .986 | .982 | .976 |

a2

TABLE 5.1.1 (Continued)
 $b = 2.0 \quad \alpha = 0.05$

| $a_1 f$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 12 | 14 | 16 | 18 | 20 | 22 | 24 | 26 | 28 | 30 |
|---------|-------|-------|-------|-------|-------|-------|-------|-------|-------|------|------|------|------|------|------|------|------|------|------|------|
| 12 1.0 | 1.166 | 1.035 | .923 | .832 | .760 | .702 | .656 | .618 | .587 | .561 | .520 | .490 | .467 | .449 | .435 | .423 | .413 | .405 | .395 | .392 |
| 1.5 | 1.292 | 1.147 | 1.033 | .945 | .876 | .823 | .780 | .746 | .718 | .694 | .658 | .632 | .612 | .596 | .583 | .573 | .564 | .557 | .551 | .546 |
| 2.0 | 1.427 | 1.273 | 1.158 | 1.072 | 1.007 | .957 | .917 | .886 | .860 | .839 | .807 | .781 | .763 | .748 | .737 | .727 | .719 | .712 | .706 | .701 |
| 2.5 | 1.568 | 1.409 | 1.284 | 1.210 | 1.148 | 1.100 | 1.063 | 1.033 | 1.008 | .908 | .857 | .834 | .817 | .803 | .792 | .783 | .775 | .769 | .763 | .758 |
| 14 1.0 | 1.137 | 1.011 | .899 | .807 | .735 | .676 | .629 | .591 | .559 | .532 | .490 | .459 | .435 | .416 | .401 | .389 | .379 | .371 | .363 | .357 |
| 1.5 | 1.246 | 1.105 | .992 | .903 | .833 | .778 | .734 | .699 | .670 | .646 | .608 | .581 | .560 | .543 | .530 | .519 | .511 | .503 | .497 | .491 |
| 2.0 | 1.362 | 1.212 | 1.097 | 1.010 | .944 | .892 | .851 | .819 | .792 | .770 | .736 | .710 | .691 | .676 | .664 | .653 | .646 | .639 | .633 | .628 |
| 2.5 | 1.484 | 1.327 | 1.213 | 1.128 | 1.064 | 1.015 | .977 | .946 | .921 | .900 | .868 | .844 | .826 | .812 | .801 | .792 | .784 | .777 | .771 | .767 |
| 16 1.0 | 1.115 | .992 | .881 | .799 | .715 | .656 | .609 | .570 | .537 | .510 | .467 | .435 | .411 | .391 | .376 | .363 | .353 | .344 | .336 | .330 |
| 1.5 | 1.210 | 1.073 | .960 | .870 | .800 | .744 | .699 | .663 | .633 | .608 | .569 | .541 | .519 | .502 | .489 | .478 | .469 | .461 | .454 | .449 |
| 2.0 | 1.312 | 1.165 | 1.051 | .963 | .895 | .842 | .801 | .767 | .739 | .717 | .681 | .655 | .635 | .620 | .608 | .598 | .590 | .582 | .576 | .571 |
| 2.5 | 1.419 | 1.265 | 1.150 | 1.065 | 1.000 | .949 | .910 | .878 | .852 | .831 | .794 | .774 | .755 | .741 | .729 | .720 | .712 | .705 | .699 | .694 |
| 18 1.0 | 1.098 | .979 | .866 | .774 | .700 | .641 | .593 | .553 | .521 | .493 | .449 | .416 | .391 | .371 | .355 | .342 | .331 | .322 | .314 | .308 |
| 1.5 | 1.191 | 1.048 | .936 | .846 | .774 | .717 | .672 | .635 | .604 | .579 | .539 | .510 | .487 | .470 | .456 | .444 | .435 | .427 | .420 | .414 |
| 2.0 | 1.271 | 1.128 | 1.014 | .926 | .857 | .804 | .761 | .726 | .698 | .674 | .638 | .611 | .591 | .575 | .562 | .552 | .543 | .536 | .529 | .524 |
| 2.5 | 1.367 | 1.215 | 1.101 | 1.015 | .949 | .897 | .857 | .824 | .798 | .776 | .742 | .717 | .698 | .683 | .671 | .661 | .653 | .646 | .640 | .635 |
| 20 1.0 | 1.084 | .967 | .855 | .762 | .689 | .629 | .580 | .540 | .507 | .479 | .435 | .401 | .376 | .355 | .339 | .326 | .314 | .305 | .297 | .290 |
| 1.5 | 1.159 | 1.028 | .917 | .826 | .754 | .696 | .650 | .612 | .581 | .555 | .514 | .484 | .461 | .443 | .429 | .417 | .407 | .399 | .392 | .386 |
| 2.0 | 1.235 | 1.098 | .986 | .897 | .827 | .772 | .729 | .693 | .664 | .640 | .603 | .575 | .554 | .538 | .525 | .514 | .505 | .497 | .491 | .484 |
| 2.5 | 1.324 | 1.176 | 1.062 | .975 | .908 | .856 | .814 | .781 | .754 | .731 | .696 | .670 | .651 | .636 | .623 | .613 | .605 | .598 | .592 | .587 |
| 22 1.0 | 1.073 | .957 | .845 | .752 | .678 | .618 | .570 | .530 | .496 | .468 | .423 | .389 | .363 | .342 | .326 | .312 | .300 | .291 | .282 | .275 |
| 1.5 | 1.139 | 1.012 | .901 | .809 | .737 | .679 | .632 | .594 | .562 | .536 | .494 | .463 | .440 | .421 | .406 | .394 | .384 | .376 | .369 | .362 |
| 2.0 | 1.213 | 1.074 | .962 | .873 | .802 | .747 | .702 | .665 | .637 | .612 | .574 | .545 | .524 | .507 | .493 | .482 | .473 | .466 | .459 | .453 |
| 2.5 | 1.289 | 1.144 | 1.030 | .943 | .875 | .821 | .779 | .745 | .717 | .694 | .658 | .632 | .612 | .596 | .584 | .573 | .565 | .558 | .552 | .546 |
| 24 1.0 | 1.064 | .950 | .837 | .744 | .670 | .610 | .561 | .521 | .487 | .459 | .413 | .379 | .353 | .331 | .314 | .300 | .289 | .279 | .270 | .263 |
| 1.5 | 1.124 | .999 | .888 | .796 | .723 | .665 | .617 | .579 | .547 | .520 | .477 | .446 | .422 | .403 | .389 | .375 | .365 | .356 | .349 | .343 |
| 2.0 | 1.190 | 1.055 | .943 | .853 | .782 | .726 | .681 | .644 | .614 | .589 | .549 | .520 | .498 | .481 | .467 | .456 | .447 | .439 | .432 | .426 |
| 2.5 | 1.260 | 1.117 | 1.004 | .916 | .847 | .793 | .750 | .715 | .687 | .663 | .626 | .599 | .579 | .563 | .550 | .539 | .531 | .523 | .517 | .512 |
| 26 1.0 | 1.055 | .943 | .831 | .738 | .663 | .603 | .554 | .513 | .479 | .451 | .405 | .371 | .344 | .322 | .305 | .291 | .279 | .269 | .260 | .253 |
| 1.5 | 1.110 | .988 | .877 | .785 | .711 | .652 | .605 | .566 | .534 | .506 | .463 | .431 | .407 | .387 | .372 | .359 | .349 | .340 | .332 | .326 |
| 2.0 | 1.171 | 1.038 | .927 | .837 | .765 | .708 | .662 | .625 | .594 | .569 | .529 | .499 | .477 | .459 | .445 | .433 | .424 | .416 | .409 | .403 |
| 2.5 | 1.235 | 1.094 | .982 | .893 | .824 | .769 | .725 | .690 | .661 | .637 | .599 | .572 | .551 | .534 | .521 | .510 | .501 | .494 | .488 | .482 |
| 28 1.0 | 1.049 | .937 | .825 | .732 | .657 | .597 | .548 | .507 | .473 | .444 | .398 | .363 | .336 | .315 | .299 | .282 | .270 | .260 | .251 | .244 |
| 1.5 | 1.099 | .979 | .867 | .775 | .701 | .642 | .594 | .555 | .522 | .495 | .451 | .419 | .394 | .374 | .358 | .345 | .334 | .325 | .318 | .311 |
| 2.0 | 1.154 | 1.025 | .913 | .822 | .750 | .693 | .647 | .609 | .578 | .552 | .511 | .481 | .458 | .440 | .425 | .414 | .404 | .396 | .389 | .383 |
| 2.5 | 1.213 | 1.075 | .964 | .874 | .804 | .749 | .704 | .668 | .639 | .614 | .576 | .549 | .526 | .510 | .495 | .485 | .475 | .468 | .462 | .456 |
| 30 1.0 | 1.044 | .933 | .820 | .727 | .652 | .591 | .542 | .501 | .467 | .438 | .392 | .357 | .330 | .308 | .290 | .275 | .263 | .253 | .244 | .236 |
| 1.5 | 1.089 | .971 | .859 | .767 | .693 | .633 | .585 | .546 | .513 | .485 | .441 | .409 | .383 | .363 | .347 | .333 | .322 | .313 | .305 | .298 |
| 2.0 | 1.140 | 1.013 | .902 | .810 | .738 | .680 | .634 | .595 | .564 | .538 | .496 | .464 | .442 | .424 | .409 | .397 | .387 | .378 | .371 | .365 |
| 2.5 | 1.195 | 1.059 | .948 | .858 | .787 | .731 | .686 | .650 | .620 | .595 | .556 | .527 | .505 | .488 | .474 | .463 | .454 | .446 | .440 | .434 |

value k_0 and $\alpha(k_0) = \Pr(fU_1 + U_2 \geq k_0)$ was evaluated. Iteration proceeded, producing successive values k_1, \dots, k_n , until $|k_n - k_{n-1}| < .00005$. Then k_n was truncated to three decimals--call the truncated value k_n^- . The values $\alpha(k_n^-)$ and $\alpha(k_n^+) = \alpha(k_n^- + 0.001)$ were evaluated. If $\alpha(k_n^-)$ was closer than $\alpha(k_n^+)$ to the desired α -level, k_n^- was chosen as the tabular value, k ; otherwise k_n^+ was chosen. The rounding of k to three decimals produces an error of $|\alpha(k) - \alpha|$, which does not exceed 0.00020 if $\alpha = .01$ is the desired level of the test and 0.00060 if $\alpha = .05$ is the desired level. For most values the errors are substantially smaller, usually less than one-half the stated maximum for the significance level.

For values not included in the tables, interpolation is sometimes useful. Interpolation over values of b does not lead to good approximations, but interpolations over the parameters a_1 , a_2 , and f does result in quite accurate values for k . Extrapolation for some larger values of f is also feasible. Extrapolation for larger values of a_1 and a_2 was not done since it seems clear from the values in Table 5.1.1 that extrapolation to values of a_1 and a_2 which are reasonably close to the maximum tabular value of 30 should yield good approximations to the true value of k . Table 5.1.2 presents values of parameters for which interpolation or extrapolation was performed. (Since the parameters a_1 and a_2 are interchangeable if $f = 1$, the results for certain examples of interpolation over a_1 apply to corresponding examples of interpolation over a_2 .) The asterisk indicates the parameter on which interpolation or extrapolation was performed. The value of k , say \hat{k} , which was found by interpolation or extrapolation is also given along with the corresponding error, $|\alpha(\hat{k}) - \alpha|$.

TABLE 5.1.2

ESTIMATED CRITICAL VALUES FOR TYPE D AND TYPE F TESTS
FOR CERTAIN INTERPOLATED AND EXTRAPOLATED VALUES
OF THE PARAMETERS a_1 , a_2 , AND f

| α | b | a_1 | a_2 | f | \hat{k} | $ \alpha(\hat{k}) - \alpha $ |
|----------|-----|-------|-------|-------|-----------|------------------------------|
| 0.01 | 1.0 | 1.5* | 1.0 | 1.00 | 1.770 | 0.00015 |
| 0.01 | 2.0 | 1.5* | 1.0 | 1.00 | 1.875 | 0.00008 |
| 0.01 | 1.0 | 1.5* | 1.0 | 2.50 | 3.102 | 0.00011 |
| 0.01 | 2.0 | 1.5* | 30.0 | 1.00 | 1.063 | 0.00027 |
| 0.01 | 1.0 | 1.5* | 30.0 | 2.50 | 2.414 | 0.00054 |
| 0.01 | 1.0 | 29.0* | 1.0 | 1.00 | 1.040 | 0.00020 |
| 0.01 | 2.0 | 29.0* | 1.0 | 1.00 | 1.109 | 0.00009 |
| 0.01 | 1.0 | 29.0* | 1.0 | 2.50 | 1.171 | 0.00005 |
| 0.01 | 2.0 | 29.0* | 1.0 | 2.50 | 1.349 | 0.00012 |
| 0.01 | 1.0 | 29.0* | 30.0 | 1.00 | 0.209 | 0.00006 |
| 0.01 | 2.0 | 29.0* | 30.0 | 1.00 | 0.304 | 0.00003 |
| 0.01 | 1.0 | 29.0* | 30.0 | 2.50 | 0.407 | 0.00010 |
| 0.05 | 2.0 | 30.0 | 1.5* | 2.50 | 1.124 | 0.00005 |
| 0.05 | 1.0 | 30.0 | 29.0* | 2.50 | 0.279 | 0.00041 |
| 0.05 | 2.0 | 30.0 | 29.0* | 2.50 | 0.436 | 0.00036 |
| 0.01 | 1.0 | 1.0 | 1.0 | 1.25* | 2.092 | 0.00001 |
| 0.01 | 1.0 | 1.0 | 30.0 | 1.25* | 1.280 | 0.00004 |
| 0.01 | 1.0 | 30.0 | 1.0 | 1.25* | 1.055 | 0.00001 |
| 0.01 | 1.0 | 30.0 | 30.0 | 1.25* | 0.234 | 0.00016 |
| 0.05 | 2.0 | 30.0 | 30.0 | 1.25* | 0.266 | 0.00042 |
| 0.05 | 2.0 | 1.0 | 1.0 | 2.25* | 2.997 | 0.00010 |
| 0.05 | 2.0 | 1.0 | 30.0 | 2.25* | 2.256 | 0.00073 |
| 0.01 | 2.0 | 30.0 | 30.0 | 2.25* | 0.518 | 0.00000 |
| 0.05 | 1.0 | 30.0 | 30.0 | 2.25* | 0.255 | 0.00025 |
| 0.05 | 1.0 | 1.0 | 1.0 | 3.00* | 3.450 | 0.00042 |
| 0.05 | 2.0 | 1.0 | 1.0 | 3.00* | 3.708 | 0.00040 |
| 0.01 | 1.0 | 1.0 | 30.0 | 3.00* | 3.001 | 0.00042 |
| 0.01 | 1.0 | 30.0 | 1.0 | 3.00* | 1.209 | 0.00032 |
| 0.01 | 1.0 | 30.0 | 30.0 | 3.00* | 0.464 | 0.00024 |

*The asterisk signifies the parameter for which the interpolation or extrapolation was performed.

Interpolation was done using Newton's forward-difference and backward-difference formulas (Conte [1965]). Extrapolation was also done using difference formulas. For interpolation on a_1 and a_2 , five known values of k were used as interpolating points; for interpolation and extrapolation on f , all four known values of k for specified a_1 , a_2 , b and α were used.

Programming for the critical values in Table 5.1.1 was done in PL-1 and the PL-1 Scientific Subroutine Package program BDTR was used in the computations. BDTR returns the incomplete beta function ratio $I_x(b,a)$ for given x , b , and a . As installed on the University of North Carolina Computation Center's IBM 360/75, BDTR returns incorrect values if $b = 1$, $a \neq 1$, and $x \neq 1$. It was possible to avoid using this subroutine in any such case by substituting the equivalent value of $I_{1-x}(a,b)$ for $I_x(b,a)$ as necessary.

5.2. Numerical Illustration

In this section the use of the type D and type F tests in analysis will be demonstrated in an application to data supplied by a pharmaceutical company. Data on the effects of six parallel treatments on diastolic blood pressure measured in sitting position in two hypertensive studies are used. The cell frequencies of the data are as follows:

TABLE 5.2.1
DISTRIBUTION OF PATIENTS AMONG SIX HYPERTENSION
TREATMENTS IN TWO CLINICS

| Clinic | Treatment | | | | | | N_i |
|--------|-----------|----|---|---|----|----|-------|
| | 1 | 2 | 3 | 4 | 5 | 6 | |
| 1 | 10 | 9 | 7 | 9 | 8 | 9 | 52 |
| 2 | 11 | 11 | 9 | 9 | 10 | 12 | 62 |

The observation x_{ijk} was obtained as the difference of the pre-treatment value (y_{ijk}) and the average of post-treatment values collected during four visits (z_{ijk}). Therefore, the average decrease in diastolic blood pressure of the j^{th} patient at clinic i receiving treatment k is given by

$$x_{ijk} = y_{ijk} - z_{ijk} \quad (i = 1, 2; j = 1, \dots, N_i; k = 1, \dots, 6).$$

Consider the model

$$x_{ijk} = \mu_i + \xi_{ik} + \varepsilon_{ijk}, \quad (5.2.1)$$

where μ_i is the overall mean at clinic i , ξ_{ik} is the k^{th} treatment effect at clinic i (after reparameterization) and ε_{ijk} is the error. Let \tilde{X}_i be a $N_i \times 1$ vector of the observations x_{ijk} for clinic i and let $\tilde{\xi}_i = (\mu_i, \xi_{i1}, \dots, \xi_{i5})'$. Then (5.2.1) can be rewritten as

$$\tilde{X}_i = A_i \tilde{\xi}_i + \tilde{\varepsilon}_i$$

where the m^{th} row of A_i corresponds to the m^{th} row (patient) of \tilde{X}_i (clinic i) and is

$$\left\{ \begin{array}{ll} 1 & 1 & 0 & 0 & 0 & 0 & \text{if patient } m \text{ received treatment 1} \\ 1 & 0 & 1 & 0 & 0 & 0 & \text{if patient } m \text{ received treatment 2} \\ 1 & 0 & 0 & 1 & 0 & 0 & \text{if patient } m \text{ received treatment 3} \\ 1 & 0 & 0 & 0 & 1 & 0 & \text{if patient } m \text{ received treatment 4} \\ 1 & 0 & 0 & 0 & 0 & 1 & \text{if patient } m \text{ received treatment 5} \\ 1 & -1 & -1 & -1 & -1 & -1 & \text{if patient } m \text{ received treatment 6} \end{array} \right.$$

and $\tilde{\varepsilon}_i$ is the vector of errors for clinic i .

In each clinic the hypothesis tested is

$$H_0: \underset{\sim}{C} \underset{\sim}{\xi}_i \underset{\sim}{M} = \underset{\sim}{0} \text{ versus } H_1: \underset{\sim}{C} \underset{\sim}{\xi}_i \underset{\sim}{M} \neq \underset{\sim}{0}$$

where $\underset{\sim}{C} = [0_{5 \times 1} \quad I_{5 \times 5}]$ and $\underset{\sim}{M} = 1$. Assuming that each error vector has an N_i -variate normal distribution with mean zero and $N_i \times N_i$

variance-covariance matrix $\sigma^2 \tilde{I}$, under H_0 the statistic $F_i = (N_i - 6) \frac{H_i E_i^{-1}}{\tilde{N}_i \tilde{N}_i} / 5$, $i = 1, 2$, has the central F distribution with 5 and $(N_i - 6)$ degrees of freedom, where \tilde{H}_i and \tilde{E}_i , defined as in Section 1.1, are scalars in this problem. In this example

$$F_1(5, 46) = 2.3964, P_1 = 0.05168;$$

and

$$F_2(5, 56) = 2.7523, P_2 = 0.02716;$$

the numbers in parentheses indicate the degrees of freedom. The statistics were calculated using the MGLM program, installed at the Triangle Universities Computation Center, and the P-values were calculated by the TSO program CFTDIS, also at TUCC.

Since $\tilde{M} = 1$, the null hypothesis to be tested is, for each clinic, $H_0: \lambda_i = C \tilde{\xi}_i = 0$, $i = 1, 2$. For this problem the noncentrality parameters are

$$\theta_i = (C \tilde{\xi}_i)' [C(A_i' A_i)^{-1} C']^{-1} C \tilde{\xi}_i / \sigma_i^2$$

so that

$$V_i = [C(A_i' A_i)^{-1} C']^{-1} / \sigma_i^2. \quad (5.2.2)$$

For each clinic the maximum likelihood estimate of σ_i^2 ,

$$\hat{\sigma}_i^2 = X_i' [I - A_i (A_i' A_i)^{-1} A_i'] X_i / N_i,$$

was substituted for σ_i^2 . It is not necessary to assume that $\sigma_1^2 = \sigma_2^2$, as it would be if a single model were fitted to the data. For the problem in this example, the test of $H_0: \sigma_1^2 = \sigma_2^2$ versus $H_1: \sigma_1^2 \neq \sigma_2^2$ leads to rejection of H_0 at $\alpha = .10$.

In order to calculate the statistic $f_1 U_1 + f_2 U_2$ for the type F test it is necessary to find the values of U_1 , U_2 , f_1 , and f_2 . Each U_i

is a simple transformation of the F-distributed statistics F_i . For this example, since $b = 5/2 = 2.5$, $a_1 = 46/2 = 23.0$, and $a_2 = 56/2 = 28.0$,

$$U_1 = \frac{bF_1/a_1}{1+bF_1/a_1} = \frac{2.5(2.3964)/23}{1 + 2.5(2.3964)/23} = 0.2067$$

and

$$U_2 = \frac{bF_2/a_2}{1+bF_2/a_2} = \frac{2.5(2.7523)/28}{1 + 2.5(2.7523)/28} = 0.1973.$$

In order to find the f_i 's it is necessary to calculate $\gamma_i = \text{tr}(\tilde{V}_i)$, $i = 1, 2$, where the \tilde{V}_i 's are of the form given in (5.2.2). To determine the matrices $[\tilde{C}(\tilde{A}'_i\tilde{A}_i)^{-1}\tilde{C}']^{-1}$, $i = 1, 2$, the 6×6 matrices $(\tilde{A}'_i\tilde{A}_i)^{-1}$ which are provided by MGLM can be used. Since $\tilde{C} = [0_{5 \times 1} : I_{5 \times 5}]$, $[\tilde{C}(\tilde{A}'_i\tilde{A}_i)^{-1}\tilde{C}']^{-1}$ can be found by deleting the first row and column of $(\tilde{A}'_i\tilde{A}_i)^{-1}$. The resulting 5×5 matrices were inverted to find

$$[\tilde{C}(\tilde{A}'_1\tilde{A}_1)^{-1}\tilde{C}']^{-1} = \begin{bmatrix} 18.981 & 9.000 & 9.038 & 9.000 & 9.019 \\ & 18.000 & 9.000 & 9.000 & 9.000 \\ & & 15.922 & 9.000 & 8.961 \\ & & & 18.000 & 9.000 \\ & & & & 16.981 \end{bmatrix}$$

and

$$[\tilde{C}(\tilde{A}'_2\tilde{A}_2)^{-1}\tilde{C}']^{-1} = \begin{bmatrix} 22.983 & 11.983 & 11.951 & 11.951 & 11.967 \\ & 22.983 & 11.951 & 11.951 & 11.967 \\ & & 20.854 & 11.854 & 11.902 \\ & & & 20.854 & 11.902 \\ & & & & 21.935 \end{bmatrix}.$$

MGLM also provides $N_i \hat{\sigma}_i^2$, so that $\hat{\sigma}_1^2$ and $\hat{\sigma}_2^2$ can be easily found:

$$N_1 \hat{\sigma}_1^2 = 2405.379 \Rightarrow \hat{\sigma}_1^2 = 2405.379/52 = 46.257;$$

$$N_2 \hat{\sigma}_2^2 = 2057.861 \Rightarrow \hat{\sigma}_2^2 = 2057.861/62 = 33.191.$$

It follows, therefore, that

$$\begin{aligned}\gamma_1 &= \text{tr}(\underline{V}_1) \doteq \text{tr}\{[\underline{C}(\underline{A}'_1\underline{A}_1)^{-1}\underline{C}']^{-1}/\hat{\sigma}_1^2\} \\ &= 87.884/46.257 \\ &= 1.900\end{aligned}$$

and, similarly,

$$\gamma_2 = \text{tr}(\underline{V}_2) \doteq 3.302.$$

The parameters γ_1 and γ_2 having been found, it is possible to calculate f_1 and f_2 :

$$f_1 = (a_1 + b) \gamma_1/b = (23.0 + 2.5)(1.900)/2.5 = 19.380;$$

$$f_2 = (a_2 + b) \gamma_2/b = (28.0 + 2.5)(3.302)/2.5 = 40.284.$$

Finally it is possible to calculate the statistic for the type F test. Since $f_2 > f_1$, the form $f_2 U_2/f_1 + U_1 = k$ is used. It follows from the above that

$$\begin{aligned}k &= 40.284 (0.1973)/19.380 + 0.2067 \\ &= 2.079 (0.1973) + 0.2067 = 0.617.\end{aligned}$$

Since $k \leq 1.0$ and $kf_1/f_2 \leq 1.0$, a program using the algorithm to provide significance levels for Case 3 situations can be used to determine that, for this problem,

$$\text{Pr}(2.079 U_2 + U_1 \geq 0.617) = .00869.$$

Therefore, the overall test is significant at $\alpha = .01$ whereas neither of the individual F statistics, F_1 and F_2 , is significant at that level.

By contrast, Fisher's statistic for this problem is $-2 \log (P_1 P_2) = -2 \log (0.05168 \times 0.02716) = 13.1374$. This statistic is distributed as a chi-squared with four degrees of freedom under H_0 . By

use of CFDIS it can be found that

$$\Pr(\chi_4^2 \geq 13.1374) = .01062.$$

For this problem, the type F test is found to be more sensitive than Fisher's test to deviations from the null hypothesis, even though there are only two statistics combined. This result is of particular interest in light of the results of Section 3.3, where it was shown that the type D and type F tests are more efficient assuming that there are a large number of statistics to be combined.

The above application illustrates the method of calculating a statistic for the type F test. Since the type D test coincides with the type F test if indeed the \underline{V}_i are proportional, the above method of calculation, with $\gamma_i = \text{tr}(\underline{V}_i)$, can be used for type D tests also.

CHAPTER VI

SUMMARY AND SOME SUGGESTIONS FOR FURTHER RESEARCH

In this dissertation two locally most powerful procedures were found for combining independent statistics which arise in the analysis of certain multiparametric and multivariate problems. The type D test for combining beta-distributed statistics was found for the case of a balanced design; the type F test was introduced for use in the case of an unbalanced design. The overall statistic for type D and type F tests is, in each case, a linear combination of the beta-distributed statistics to be combined. The exact distribution of the statistic was found in the two-sample case; the accuracy of using normal approximations for the general s -sample case was also considered. It was shown that for local alternatives, the type D test is more efficient than Fisher's test for combining statistics, according to Bahadur asymptotic relative efficiency, if there are a large number of small-sample statistics to be combined. Under the same conditions, it was shown that the type F test is more efficient than Fisher's test according to an average measure of Bahadur asymptotic relative efficiency. In a numerical example, the theoretical developments were applied to some pharmaceutical data.

Several areas of future research may be suggested. It would be useful to work out the details of the exact distribution of the type D and type F statistics for larger values of s . As noted

earlier, however, this task would be tedious and the calculations necessary to find critical values would be expensive.

It would also be useful to develop the details of the distributions of the type D and type F statistic if the statistics to be combined have chi-squared distributions. If $s = 2$, the exact distribution should be reasonably simple to calculate, with more complicated results for larger values of s .

Another area of further research is consideration of the extensions mentioned in Section 3.4. The exact distributions of the combined test statistics mentioned there may be sufficiently complicated to warrant study of approximations.

As mentioned earlier, Littell and Folks have shown that if there are a finite number of large-sample statistics to be combined, Fisher's method is more efficient than other methods according to Bahadur asymptotic relative efficiency; the locally most powerful combination procedures were shown to be more efficient if there are a large number of finite-sample statistics to be combined, for local alternatives. Research into the relative efficiencies of Fisher's test and the locally most powerful tests for combining an intermediate number of medium-sized sample statistics for local and non-local alternatives would be illuminating.

APPENDIX A

USEFUL MATHEMATICAL RESULTS

Lemma A.1. If U_i has a beta distribution with density $\beta_{u_i}^{\theta_i}$ specified in (1.1.2), then (2.3.4) and (2.3.5) are equivalent.

Proof. In (2.3.4) $\prod_{h=1}^s \beta_{u_h}^{\theta_h}$ is the product of s infinite series of the form of (1.1.2). Since each of these series is a positive term convergent series, each is absolutely convergent and (Taylor and Mann [1972, p. 635]) the order of summation can be interchanged. It remains to be shown that each of the infinite sums can be taken outside the integral. If $b \geq 1$ and $a_j \geq 1$, then the density of each U_j is uniformly bounded and the interchange is immediate. If $b = 1/2$ then for every j , the density of U_j goes to infinity as $U_j \downarrow 0$. If $a_j = 1/2$ for some j , the density goes to infinity as $U_j \uparrow 1$. However, since for all j $\int_0^1 \beta_{u_j}^{\theta_j} du_j = 1$, the dominated convergence theorem (Cramér [1945, p. 45]) can be applied. Therefore, each summation can be taken outside the integral and the lemma holds.

Lemma A.2. If $W = \{(u_1, u_2): 0 \leq u_i \leq 1, i = 1, 2\}$ and $\beta_{u_1}^{\theta_1}$ and $\beta_{u_2}^{\theta_2}$ are as defined in (1.1.3), then

$$\begin{aligned} \int_w \beta_{u_1}^{\circ} \beta_{u_2}^{\circ} du_1 du_2 &= \int_{u_2^{(1)}}^{u_2^{(2)}} \int_{U_1^{(1)}(u_2)}^{U_1^{(2)}(u_2)} \beta_{u_1}^{\circ} \beta_{u_2}^{\circ} du_1 du_2 \\ &= \int_{u_1^{(1)}}^{u_1^{(2)}} \int_{U_2^{(1)}(u_1)}^{U_2^{(2)}(u_1)} \beta_{u_1}^{\circ} \beta_{u_2}^{\circ} du_2 du_1, \end{aligned}$$

where the limits of integration depend on the boundary $w \subset W$.

Proof. As long as the function $\beta_{u_1}^{\circ} \beta_{u_2}^{\circ}$ is integrable in w , the result holds (Taylor and Mann [1972, p. 387]). The beta densities, $\beta_{u_1}^{\circ}$ and $\beta_{u_2}^{\circ}$, and their product, are integrable when $w \subset W$.

Lemma A.3. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R > 0$. The function f is continuous in the open interval of convergence of the series. If p and q are points in $[0, R)$, $p < q$, then

$$\int_p^q f(x) dx = \sum_{n=0}^{\infty} a_n \frac{q^{n+1} - p^{n+1}}{n+1}.$$

If $q = R$, the result holds if the right hand side is convergent.

Reference. Taylor and Mann [1972, p. 665].

Discussion. In all the instances in which this lemma is used in the dissertation, the series is a Taylor series in which

$$a_n = f^{(n)}(0)/n!,$$

where $f^{(n)}$ denotes the n^{th} derivative of f .

Lemma A.4. If $0 \leq x \leq c/d$ and if $n > 0$, then

$$(c-dx)^n = \sum_{i=0}^{n^*} \binom{n}{i} (-1)^i c^{n-i} d^i x^i$$

where

$$n^* = \begin{cases} n, & \text{if } n \text{ is an integer;} \\ \infty, & \text{otherwise.} \end{cases}$$

Reference. Taylor and Mann [1972, p. 632].

Lemma A.5. Let $0 \leq x \leq 1$, $b > 0$, and $a > 0$. If

$$\beta_x(b,a) = \int_0^x u^{b-1} (1-u)^{a-1} du$$

is the incomplete beta function and

$$I_x(b,a) = \beta_x(b,a)/\beta(b,a)$$

is the incomplete beta function ratio, then

$$\beta_x(b,a) = \sum_{i=0}^{(a-1)^*} \binom{a-1}{i} \frac{(-1)^i}{b+i} x^{b+i}, \quad (\text{A.5.1})$$

$$\beta(b,a) = \sum_{i=0}^{(a-1)^*} \binom{a-1}{i} \frac{(-1)^i}{b+i}, \quad \text{and} \quad (\text{A.5.2})$$

$$I_x(b,a) = \frac{1}{\beta(b,a)} \sum_{i=0}^{(a-1)^*} \binom{a-1}{i} \frac{(-1)^i}{b+i} x^{b+i}, \quad (\text{A.5.3})$$

where $(a-1)^*$ is defined in Lemma A.4.

Proof. By Lemma A.3 and A.4,

$$\beta_x(b,a) = \int_0^x u^{b-1} (1-u)^{a-1} du = \sum_{i=0}^{(a-1)^*} \binom{a-1}{i} (-1)^i \int_0^x u^{b+i-1} du$$

$$= \sum_{i=0}^{(a-1)^*} \binom{a-1}{i} \frac{(-1)^i}{b+i} x^{b+i}.$$

This verifies the first claim. By definition, $\beta_1(b,a) = \beta(b,a)$, so the second claim follows by setting $x = 1$ in the first result. The third claim follows by definition of $I_x(b,a)$. This completes the proof.

Lemma A.6. If $p, q, a,$ and b are real numbers such that $q > p, b > 0,$ and $a > 0,$ then

$$\int_p^q (u-p)^{b-1} (q-u)^{a-1} du = (q-p)^{a+b-1} \beta(b,a).$$

Discussion. Although the standard beta distribution specifies $p = 0$ and $q = 1,$ a family of beta distributions is given by density functions of the form

$$(u-p)^{b-1} (q-u)^{a-1} / [\beta(b,a) (q-p)^{a+b-1}]$$

(Johnson and Kotz, Vol. 2 [1970, p. 37]). If $V = (U-p)/(q-p),$ V has a standard beta distribution.

Lemma A.7. If $x \geq 1, b > 0, a > 0,$ and $c > 1,$ then

$$\sum_{i=0}^{(a-1)^*} \binom{a-1}{i} (-1)^i x^{b+i} \beta_{1/x}(b+i,c) \quad (\text{A.7.1})$$

converges.

Proof. If a is an integer the series is finite and, therefore, convergent. If a is not an integer, then notice that $\binom{a-1}{i} (-1)^i$ is either positive or negative for all $i > (a-1)$. If $[a]$ is even, the terms are positive; if $[a]$ is odd, the terms are negative, where $[a]$ is the greatest integer in a . If $[a]$ is odd, multiply the series by $(-1),$

which will not affect the convergence or divergence of the series. Therefore, without loss of generality, assume that the series is a positive term series in the tail.

It will be shown that (A.7.1) converges for non-integral a by showing that

$$\left| \sum_{i=0}^{\infty} \binom{a-1}{i} (-1)^i x^{b+i} \beta_{1/x}(b+i, c) \right| < M_1 + M_2 \left| \sum_{i=[a]}^{\infty} \binom{a-1}{i} \frac{(-1)^i}{b+i} \right| \quad (\text{A.7.2})$$

where M_1 and M_2 are positive constants. By Lemma A.5,

$\sum_{i=0}^{\infty} \binom{a-1}{i} \frac{(-1)^i}{b+i} = \beta(b, a)$ and, therefore, the tail of that series converges. It follows that (A.7.2) converges if M_1 and M_2 are finite.

Note that

$$\begin{aligned} \left| \sum_{i=0}^{\infty} \binom{a-1}{i} (-1)^i x^{b+i} \beta_{1/x}(b+i, c) \right| &\leq \left| \sum_{i=0}^{[a-1]} \binom{a-1}{i} (-1)^i x^{b+i} \beta_{1/x}(b+i, c) \right| \\ &+ \left| \sum_{i=[a]}^{\infty} \binom{a-1}{i} (-1)^i x^{b+i} \beta_{1/x}(b+i, c) \right|. \end{aligned} \quad (\text{A.7.3})$$

Next, by Lemma A.5,

$$\begin{aligned} x^{b+i} \beta_{1/x}(b+i, c) &= x^{b+i} \left[\sum_{j=0}^{[c-1]} \binom{c-1}{j} \frac{(-1)^j}{b+i+j} \left(\frac{1}{x}\right)^{b+i+j} \right. \\ &\quad \left. + \sum_{j=[c]}^{(c-1)^*} \binom{c-1}{j} \frac{(-1)^j}{b+i+j} \left(\frac{1}{x}\right)^{b+i+j} \right] \\ &= \frac{1}{b+i} \left[\sum_{j=0}^{[c-1]} \binom{c-1}{j} (-1)^j \frac{b+i}{b+i+j} \left(\frac{1}{x}\right)^j \right. \\ &\quad \left. + \sum_{j=[c]}^{(c-1)^*} \binom{c-1}{j} (-1)^j \frac{b+i}{b+i+j} \left(\frac{1}{x}\right)^j \right]. \end{aligned} \quad (\text{A.7.4})$$

The first series in (A.7.4) is finite and bounded by $M_3 = \sum_{j=0}^{[c-1]} \binom{c-1}{j} \left(\frac{1}{x}\right)^j$,

which is finite. If c is an integer, the second series in (A.7.4) is 0.

Otherwise, recall that $\sum_{j=0}^{\infty} \binom{c-1}{j} (-1)^j$ converges if $c-1 > 0$ (Taylor and Mann [1972, p. 632]). Therefore, $\sum_{j=[c]}^{\infty} \binom{c-1}{j} (-1)^j$ converges. Since

all terms in this series are of the same sign,

$$\left| \sum_{j=[c]}^{\infty} \binom{c-1}{j} (-1)^j \frac{b+i}{b+i+j} \left(\frac{1}{x}\right)^j \right| < \left| \sum_{j=[c]}^{\infty} \binom{c-1}{j} (-1)^j \right| = M_4 < \infty .$$

(A.7.5)

Let $M_2 = M_3 + M_4$, which is finite. Then

$$x^{b+i} \beta_{1/x}(b+i, c) < M_2 / (b+i). \quad (\text{A.7.6})$$

Let M_1 be the first term on the right hand side of (A.7.3). Clearly $M_1 < \infty$. Then (A.7.2) follows from (A.7.3) in light of (A.7.6). Since M_1 , M_2 and $\sum_{i=[a]}^{\infty} \binom{a-1}{i} \frac{(-1)^i}{b+i}$ are all finite, the lemma holds as long as $c > 1$. (Note that in all instances when the lemma is used in the text, $c > 1$.)

Lemma A.8. If $a > 0$, $b > 0$, $c > 0$ and $x \geq 1$,

$$\sum_{i=0}^{(a-1)^*} \binom{a-1}{i} \frac{(-1)^i}{b+i} x^{b+i} \beta_{1/x}(b+i+1, c)$$

converges.

Proof. The proof proceeds on the principles of the proof of the previous lemma. However, (A.7.4) is replaced by

$$\begin{aligned}
x^{b+i} \beta_{1/x}(b+i+1, c) &< \left| \sum_{j=0}^{[c-1]} \binom{c-1}{j} \frac{(-1)^j}{b+i+1+j} \left(\frac{1}{x}\right)^{j+1} \right| \\
&+ \left| \sum_{j=[c]}^{(c-1)^*} \binom{c-1}{j} \frac{(-1)^j}{b+i+1+j} \right|. \quad (\text{A.8.1})
\end{aligned}$$

If c is not an integer, the second term on the right-hand side is the tail of the series $\sum_{j=0}^{\infty} \binom{c-1}{j} \frac{(-1)^j}{b+i+1+j} = \beta(b+i+1, c)$, which is a decreasing function of i and, therefore, is bounded. If the right-hand side of (A.8.1) is set equal to $M_2 < \infty$, it is then clear that

$$\left| \sum_{i=0}^{\infty} \binom{a-1}{i} (-1)^i \frac{x^{b+i}}{b+i} \beta_{1/x}(b+i+1, c) \right| < M_1 + M_2 \left| \sum_{i=[a]}^{\infty} \binom{a-1}{i} \frac{(-1)^i}{b+i} \right|, \quad (\text{A.8.2})$$

where M_1 is the absolute value of the first a terms of the series on the left-hand side. Since the series on the right-hand side is the tail of the convergent series (A.5.2), (A.8.2) is finite and the lemma holds.

Lemma A.9. If $I_x(b, a)$ is an incomplete beta function ratio defined in Lemma A.5, then

$$I_x(b, a) = 1 - I_{1-x}(a, b).$$

Reference. K. Pearson [1968, p. vii].

BIBLIOGRAPHY

- Anderson, T. W. [1958]. *An Introduction to Multivariate Statistical Analysis*. Wiley, New York.
- Bahadur, R. R. [1960]. Stochastic comparison of tests. *Ann. Math. Statist.* 31, 276-295.
- Bahadur, R. R. [1967]. Rates of convergence of estimates and test statistics. *Ann. Math. Statist.* 38, 303-324.
- Bhattacharya, N. [1961]. Sampling experiments on the combination of independent χ^2 -tests. *Sankhyā A* 23, 191-196.
- Birnbaum, A. [1954]. Combining independent tests of significance. *J. Amer. Statist. Ass.* 49, 559-574.
- Birnbaum, A. [1955]. Characterizations of complete classes of tests of some multiparametric hypotheses, with applications to likelihood ratio tests. *Ann. Math. Statist.* 26, 21-36.
- Conte, S. D. [1965]. *Elementary Numerical Analysis*. McGraw-Hill, New York.
- Cramér, H. [1946]. *Mathematical Methods of Statistics*. Princeton University Press, Princeton.
- Davies, R. B. The weighted combination of independent normal test statistics. (Unpublished).
- Fisher, R. A. [1932]. *Statistical Methods for Research Workers* (4th and Later Editions). Oliver and Boyd, London.
- Fisz, M. [1963]. *Probability Theory and Mathematical Statistics*. Wiley, New York.
- Good, I. J. [1955]. On the weighted combination of significance tests. *J. R. Statist. Soc. B* 17, 264-265.
- Graybill, F. A. [1969]. *Introduction to Matrices with Applications in Statistics*. Wadsworth, Belmont, California.
- Gupta, A. K. [1971]. Noncentral distribution of Wilks' statistic in MANOVA. *Ann. Math. Statist.* 42, 1254-1261.

- Hill, A. B. [1971]. *Principles of Medical Statistics* (9th Edn.). Oxford University Press, New York.
- Isaacson, S. L. [1951]. On the theory of unbiased tests of simple statistical hypotheses specifying two or more parameters. *Ann. Math. Statist.* 22, 217-234.
- Johnson, N. L. and Kotz, S. [1970]. *Continuous Univariate Distributions*, Vols. 1 and 2. Houghton Mifflin, Boston.
- Kincaid, W. M. [1962]. The combination of test statistics based on discrete distributions. *J. Amer. Statist. Ass.* 57, 10-19.
- Lancaster, H. O. [1949]. The combination of probabilities arising from data in discrete distributions. *Biometrika* 36, 370-382.
- Lancaster, H. O. [1961]. The combination of probabilities: an application of orthonormal functions. *Austral. J. Statist.* 3, 20-33.
- Lipták, T. [1958]. On the combination of independent tests. *Magyar Tud. Akad. Mat. Kutató Int. Közl.* 3, 171-197.
- Littell, R. C. and Folks, J. L. [1971]. Asymptotic optimality of Fisher's method of combining independent tests. *J. Amer. Statist. Ass.* 66, 802-806.
- Littell, R. C. and Folks, J. L. [1973]. Asymptotic optimality of Fisher's method of combining independent tests II. *J. Amer. Statist. Ass.* 68, 193-194.
- Mantel, N. [1963]. Chi-squared tests with one degree of freedom; extensions of the Mantel-Haenszel procedures. *J. Amer. Statist. Ass.* 58, 690-700.
- Morrison, D. F. [1967]. *Multivariate Statistical Methods*, McGraw-Hill, New York.
- Naik, U. D. [1969]. The equal probability test and its applications to some simultaneous inference problems. *J. Amer. Statist. Ass.* 64, 986-998.
- Neyman, J. and Pearson, E. S. [1936]. Contributions to the theory of testing statistical hypotheses, part I. In *Statistical Research Memoirs*, Vol. 1. Issued by the Department of Statistics, University of London, University College, pp. 1-37.
- Neyman, J. and Pearson, E. S. [1938]. Contributions to the theory of testing statistical hypotheses, parts II and III. In *Statistical Research Memoirs*, Vol. 2. Issued by the Department of Statistics, University of London, University College, pp. 25-57.

- Oosterhoff, J. [1969]. *Combination of One-Sided Statistical Tests*. Tract No. 28, Amsterdam Mathematical Centre, Amsterdam.
- Pearson, E. S. [1938]. The probability integral transformation for testing goodness of fit and combining independent tests of significance. *Biometrika* 30, 134-148.
- Pearson, E. S. [1950]. On questions raised by the combination of tests based on discontinuous distributions. *Biometrika* 37, 383-398.
- Pearson, K. [1933]. On a method of determining whether a sample of given size n supposed to have been drawn from a parent population having a known probability integral has probably been drawn at random. *Biometrika* 25, 379-410.
- Pearson, K. (Ed.) [1968]. *Tables of the Incomplete Beta Function* (2nd Edn.). Cambridge University Press, London. (1st Edn., 1934).
- Pillai, K. C. S. and Young, D. L. [1971]. On the exact distribution of Hotelling's generalized T_0^2 . *J. Multivariate Anal.* 1, 90-107.
- Roy, S. N., Gnanadesikan, R. and Srivastava, J. N. [1971]. *Analysis and Design of Certain Quantitative Multiresponse Experiments*. Pergamon Press, New York.
- Schaafsma, W. [1968]. A comparison of the most stringent and the most stringent somewhere most powerful test for certain problems with restrictive alternative. *Ann. Math. Statist.* 39, 531-546.
- Sen, P. K. [1968]. Robustness of some nonparametric procedures in linear models. *Ann. Math. Statist.* 39, 1913-1922.
- Siotani, M. [1971]. An asymptotic expansion of the non-null distribution of Hotelling's generalized T_0^2 -statistic. *Ann. Math. Statist.* 42, 560-571.
- Sugiura, N. and Fujikoshi, Y. [1969]. Asymptotic expansions of the non-null distributions of the likelihood ratio criteria for multivariate linear hypothesis and independence. *Ann. Math. Statist.* 40, 942-952.
- Taylor, A. and Mann, W. R. [1972]. *Advanced Calculus*. Xerox College Publishing, Lexington, Massachusetts.
- Tippett, L. H. C. [1931]. *The Methods of Statistics*. Williams and Norgate, London.

- Van Elteren, Ph. [1958]. On the combination of independent two sample tests of Wilcoxon. *Bull. Intern. Statist. Inst.* 37, 351-361.
- Van Zwet, W. R. and Oosterhoff, J. [1967]. On the combination of independent test statistics. *Ann. Math. Statist.* 38, 659-680.
- Wald, A. [1943]. Tests of statistical hypotheses concerning several parameters when the number of observations is large. *Trans. Amer. Math. Soc.* 54, 426-482.
- Wallis, W. A. [1942]. Compounding probabilities from independent significance tests. *Econometrica* 10, 229-249.
- Weyl, H. [1939]. On the volume of tubes. *Amer. J. of Math.* 61, 461-472.
- Wilk, M. B. and Shapiro, S. S. [1968]. The joint assessment of normality of several independent samples. *Technometrics* 10, 825-839.
- Wilkinson, B. [1951]. A statistical consideration in psychological research. *Psych. Bull.* 48, 156-158.
- Zelen, M. and Joel, L. S. [1959]. The weighted compounding of two independent significance tests. *Ann. Math. Statist.* 30, 885-895.