

A NOTE ON THE NORMAL INTEGRAL

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Special report to the Office  
of Naval Research of work at  
Chapel Hill under Contract  
N7-onr-28402 for research in  
Probability and Statistics.

Institute of Statistics  
Mimeograph Series No. 114  
June, 1954

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Let

$$(1) \quad v = \int_0^x (2\pi)^{-1/2} e^{-t^2/2} dt, \quad x \geq 0.$$

G. Polya [1] and J. D. Williams [2] proved independently that

$$(2) \quad v \leq \frac{1}{2}(1 - e^{-2x^2/\pi})^{1/2}.$$

Two simple questions follow naturally. 1. Is it possible to replace the constant  $2/\pi$  in (2) by a smaller quantity without breaking the inequality? 2. Does there exist a lower bound, in a similar form, for  $v$ ? We find the following answer.

If for all  $x \geq 0$ , the integral  $v$  given by (1) satisfies

$$(3) \quad \frac{1}{2} (1 - e^{-ax^2})^{1/2} \leq v \leq \frac{1}{2} (1 - e^{-bx^2})^{1/2},$$

then it is necessary and sufficient that  $a \leq 1/2$  and  $b \geq 2/\pi$ .

The proof of this statement is simple. First

$$\lim_{x \rightarrow 0} \frac{v^2}{(1 - e^{-bx^2})} = (2\pi b)^{-1}.$$

Hence if (3) is true,  $b \geq 2/\pi$ . On the other hand,  $x^2 / [-\log(1 - 4v^2)] \leq 1/a$  for all real  $x$ . Since the limit of this ratio, as  $x \rightarrow \infty$ , is 2. We have

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$a \leq 1/2$ . Finally, it is easy to prove that

$$\frac{1}{2}(1 - e^{-x^2/2})^{1/2} \leq v.$$

Further, it follows from [1] and [2] that as  $x$  varies from 0 to  $\infty$ ,

$$2v/(1 - e^{-2x^2/\pi})^{1/2}$$

decreases steadily from 1 to its minimum value .9930 at  $x = 1.6$  (approximately), then increases steadily. Using a similar method used by G. Polya, it can be shown that

$$(4) \quad 2v/(1 - e^{-x^2/2})^{1/2}$$

is a steadily decreasing function of  $x$  for all  $x \geq 0$ . This is because the derivative of this ratio (4) has the same sign as

$$2(e^{x^2/2} - 1) - x e^{x^2/2} \int_0^x e^{-t^2/2} dt,$$

which is non-positive since

$$e^{x^2/2} \int_0^x e^{-t^2/2} dt = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{1 \cdot 3 \dots (2n-1)}.$$

As a consequence, we obtain that the ratio (4) has an upper bound  $(4/\pi)^{1/2}$ .

REFERENCES

[1] G. Polya, Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, Berkeley, 1949, pp. 63-78.

[2] J. D. Williams, "An approximation to the normal integral," Ann. Math. Stat. Vol. 17 (1946), pp. 363-365.