Multilevel dynamic substructure iteration method (iterative eigensolution of a multilevel substructure system)

R.Leo & M.Mazzoli
Ansaldo S.p.A., Genova, Italy

1 INTRODUCTION

The use of substructure techniques in linear dynamic analysis is attractive from two points of wiew:

- substructure analyses can be performed independently by different people or organizations.
- large structures (and/or integrated systems of interacting structures) can be subdivided into substructures (of size compatible with computer capacity and solution cost) whose number of degrees of freedom can be further on reduced "condensing" out part (or all) of the "internal" degrees of freedom.

In static analysis the "condensation" is an exact procedure. In dynamic analysis any "condensation" tecnique (which is always a particular Ritz analysis) is approximated. The actual accuracy achieved in dynamic condensation is not known and is highly dependent on the Ritz basis vectors chosen. Exact dynamic solutions can, therefore, be obtained only by iterative procedures (that iterate on the whole system of substructures).

In this paper a method is presented for the iterative calculation of the first eigenmodes of a structure subdivided into substructures organized in a multilevel tree.

The method, called MDSI (Multilevel Dynamic Substructure Iteration) is the application to a substructure tree of the subspace iteration method (Clough & Penzien 1975), (Bathe 1982).

It is an extension to multilevel substructure systems and a modification of the DSI method proposed by Dickens and Wilson (1980). The DSI method is a particular form of subspace iteration, while the present method is an application of the standard method.

The DSI method uses, to construct the structure Rayleigh-Ritz problem, the portions of the structure eigenvectors belonging to each single substructure, whereas the present method uses the complete eigenvectors.

Cast in the DSI method is the danger (particularly for integrated systems of structures) of construct non positive definite matrices (due to the fact that substructure portions of structure eigenvectors can be linearly dependent). This problem can be overcome by solution subroutines able to recognize and skip linearly dependent equations, but at the expense of a logic complication (and of a greater programming effort).

The problem does not exist for the proposed method.

2 SUBSTRUCTURE ORGANIZATION AND MATRIX ASSEMBLAGE

The structure is subdivided into a number N_0 of elementary substructures that constitute the minimum level (level $\ell=0$).

Intermediate levels ($0<\ell<L$) contain a number N_ℓ of substructures ($N_{\ell-1}\leqslant N_\ell \leqslant N_{\ell+1}$). The typical substructure s of level ℓ is the assembly of a certain number $N_\ell{}^s$ of substructures of level $\ell-1$, whose "internal" d.o.f.s have been previously "condensed" (that is transformed into a reduced number of Ritz coordinates). The maximum level ($\ell=L$) contains, as his unique substructure ($N_L=1$) the whole structure completely "condensed".

The stiffness and mass matrices of the typical substructure s of level ℓ are, therefore

$$(1) \qquad [K] = \sum [K^*] \qquad [M] = \sum [M^*]$$

where the symbol $\sum$ means assemblage (a true summation only occurs for the "attachment" degrees of freedom), the asterisk indicates "condensed" matrices, and the assemblage is executed only on the substructures of level $\ell-1$ that are the "components" of the present substructure. If we call $m_{\ell-1}$ the number of "attachment" d.o.f.s (variable with the level; it varies also with the substructure but, formally, we can suppose that all matrices are extended, using null elements, to all $m_{\ell-1}$ d.o.f.s) and $g_{\ell-1,s}$ the number of "internal" Ritz coordinates (different for each component substructure), eq.(1) becomes

$$(2) \qquad [K] = \begin{bmatrix} \sum K^*_{mm} & {}^1K^*_{mg} & {}^2K^*_{mg} & \dots & {}^NK^*_{mg} \\ {}^1K^*_{gm} & {}^1K^*_{gg} & 0 & \dots & 0 \\ {}^2K^*_{gm} & 0 & {}^2K^*_{gg} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ {}^NK^*_{gm} & 0 & 0 & \dots & {}^NK^*_{gg} \end{bmatrix} \qquad [M] = \text{(analogous)}$$

where N is the number of component substructures (of level $\ell-1$).

Eq.(1) and (2) hold also for the whole structure assembled in the level ℓ, if N is the total number of substructures in level $\ell-1$.

Eq.(2) can be written as

$$(3) \qquad [K] = \begin{bmatrix} K_{mm} & K_{mn} \\ K_{nm} & K_{nn} \end{bmatrix} \qquad [M] = \begin{bmatrix} M_{mm} & M_{mn} \\ M_{nm} & M_{nn} \end{bmatrix}$$

where, now, m is the number m_ℓ of attachment d.o.f.s in the level ℓ ($m_\ell \leqslant m_{\ell-1}$) (that is a new partition of matrices has been performed). Eq.(3) holds also for the whole structure. The following relation holds for n in eq.(3)

$$(4) \qquad n = a + b \qquad a = m_{\ell-1} - m_\ell \qquad b = \sum g$$

If we think to the whole structure as the assemblage of the substructures of level ℓ, his matrices are

$$(5)\qquad [K] = \begin{bmatrix} \sum K_{mm} & {}^{1}K_{mn} & {}^{2}K_{mn} & \dots & {}^{N}K_{mn} \\ {}^{1}K_{nm} & {}^{1}K_{nn} & 0 & \dots & 0 \\ {}^{2}K_{nm} & 0 & {}^{2}K_{nn} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ {}^{N}K_{nm} & 0 & 0 & \dots & {}^{N}K_{nn} \end{bmatrix} \qquad [M] = \text{(analogous)}$$

where N is the total number of substructures in the level ℓ.

The assemblage of the whole structure will never been actually performed (except in the level L), as we will see later, but it is very useful to present the theory in a concise form.

3 MULTILEVEL SUBSPACE ITERATION

The exact structure eigenproblem is

$$(6)\qquad [{}_0K]\ [{}_0\Phi] = [{}_0M]\ [{}_0\Phi]\ [\Omega]$$

with the orthogonality conditions

$$(7)\qquad [{}_0\Phi]^T\ [{}_0K]\ [{}_0\Phi] = [\Omega] \qquad [{}_0\Phi]^T\ [{}_0M]\ [{}_0\Phi] = [I]$$

where $[{}_0K]$ and $[{}_0M]$ are the exact stiffness and mass matrices (assembled from level zero substructures), $[{}_0\Phi]$ is the exact eigenvector matrix and and $[\Omega]$ is the diagonal matrix of exact eigenvalues.

The eigenproblem (6) can contain rigid body modes (that can be eliminated imposing a sufficient number of constraint conditions, if desired). Rigid body modes in substructures are always eliminated by classifying as attachment d.o.f.s all those d.o.f.s that would make positive semidefinite the internal stiffness matrix.

Moreover, the eigenproblem (6) can contain infinite eigenvalues because mass matrix can be positive semidefinite (this fact is allowed also for substructure internal mass matrices). Infinite eigenvalues are never a problem because they can be easily detected by eigensolution subroutines (moreover, the use of appropriate cut-off frequencies usually eliminates the need of their calculation).

Say: $f = m_0 + n_0$ the order of matrices $[{}_0K]$ and $[{}_0M]$
$e \leqslant f$ the number of finite eigenvalues of problem (6)

The eigenvectors $[{}_0\Phi]$ form a mass-orthonormal basis of the e-dimensional space Ee in which the matrices $[{}_0K]$ and $[{}_0M]$ are defined (or the corresponding matrices with massless d.o.f.s condensed out, if $[{}_0M]$ is semidefinite positive).

We aim to determine the first q ($q \leqslant e$) solutions of eigenproblem (6). The first q eigenvectors form a mass-orthonormal basis of the q-dimensional least-dominant subspace of the operators $[{}_0K]$ and $[{}_0M]$, called E_q^{∞}.

Our purpose is to calculate the first q eigensolutions solving, iteratively, the Rayleigh-Ritz problem constituted by the eigenproblem of the structure condensed in the level L.

Define the approximate transformation

$$(8)\qquad [{}_0\Phi^k] = [X^k]\ [{}_L\Phi^k]$$

where $[_0\Phi^k]$ is the eigenvector approximation in iteration k, $[X^k]$ is the matrix of Ritz basis vectors, $[_L\Phi^k]$ is the matrix of Ritz coordinates.

The Ritz basis vectors (that must be linearly independent) span a q-dimensional subspace E_q^k. The eigenvectors $[_0\Phi^k]$, being a linear combination of the Ritz basis vectors, lie in the subspace E^k. The subspace E_q^k is contained in the space E_e in which the matrices $[_0K]$ and $[_0M]$ are defined.

Evaluating the projections of the operators $[_0K]$ and $[_0M]$ onto the subspace E_q^k we obtain the following Rayleigh-Ritz problem

$$(9) \qquad [_LK^k]\ [_L\Phi^k] = [_LM^k]\ [_L\Phi^k]\ [\Omega^k]$$

where

$$(10) \qquad [_LK^k] = [X^k]^T\ [_0K]\ [X^k] \qquad [_LM^k]\ \ [X^k]^T\ [_0M]\ [X^k]$$

If we choose $[X^k]$ as the solution of

$$(11) \qquad [_0K]\ [X^k] = [_0M]\ [_0\Phi^{k-1}]$$

eq.(8), (9), (10) and (11) define a subspace iteration (Clough & Penzien 1975), (Bathe 1982).

If $[_0\Phi^1]$ stores q starting iteration vectors linearly independent (spanning the starting subspace E_q^1), not orthogonal to any one of the required eigenvectors, the sequence of subspaces E_q^k converges to E_q^∞. Therefore $[\Omega^k] \rightarrow [\Omega] \qquad [_0\Phi^k] \rightarrow [_0\Phi] \qquad$ as $k \rightarrow \infty$

If matrix $[_0M]$ is positive semidefinite, the starting iteration vectors must be selected to lie in E_e and they must be linearly independent when considering only the mass degrees of freedom.

In order to obtain a matrix of starting vectors satisfying the above conditions and being a good approximation of the first q eigenvectors, we can use eq.(8) in which we adopt, as Ritz basis vectors $[X^1]$, those used by one of the known techniques of dynamic condensation. A good choice of $[X^1]$, generating a good set of starting vectors, can substantially reduce the number of iterations needed to achieve the convergence.

In a multilevel substructure system, the transformation (8) can be carried out by L successive transformations (one transformation per level) of the form

$$(12) \qquad [_\ell\Phi^k] = [_\ell X^k]\ [_{\ell+1}\Phi^k]$$

The complete transformation matrix $[X^k]$ of eq.(8) is given by

$$(13) \qquad [X^k] = \Pi_{\ell=0}^{L}\ \ [_\ell X^k]$$

The $[_{\ell+1}K^k]$ and $[_{\ell+1}M^k]$ matrices of the structure (assembled in the level $\ell+1$) are the projections, onto the subspace spanned by the Ritz basis vectors $[_\ell X^k]$, of the operators $[_\ell K^k]$ and $[_\ell M^k]$ of the level ℓ (that is the projections of the matrices $[_0K]$ and $[_0M]$ onto the subspace spanned by the basis vectors $[_\ell Y^k] = \Pi_{i=0}^{\ell}\ [_iX^k]$). Therefore $[_{\ell+1}K^k] = [_\ell K^{*k}]$, $[_{\ell+1}M^k] = [_\ell M^{*k}]$.

Assume to apply to each of the substructures s of the same level ℓ a transformation

$$(14) \qquad [^s_\ell\ \Phi^k] = [^s_\ell\ X^k]\ [^s_{\ell+1}\ \Phi^k]$$

that produces the same change of coordinates (an exact change) on the attachment degrees of freedom and an approximate change ("condensation") on the internal d.o.f.s. The substructure matrices obtained by such transformation can be assembled to form matrices $[{}_{\ell}K^{*k}]$ and $[{}_{\ell}M^{*k}]$ which are the projections (of matrices $[{}_{\ell}K^{k}]$ and $[{}_{\ell}M^{k}]$ that can be obtained using eq.(12), in which $[{}_{\ell}X^{k}]$ is the solution of

$$(15) \qquad [{}_{\ell}K]\,[{}_{\ell}X^{k}] = \sum\,[{}^{s}_{\ell}\,K^{k}]\,[{}^{s}_{\ell}\,X^{k}] = \sum\,{}^{s}_{\ell}\,F$$

It follows from eq.(13) that the complete transformation (8) can be obtained by transformations (14) applied to the single substructures and by assemblages executed on groups of substructures (according to the selected tree organization). The simplest and most convenient choice for the attachment d.o.f. transformation is to maintain the original system of coordinates (usually nodal coordinates) (in addition to its simplicity, it allows to keep inaltered, as attachment d.o.f.s of the structure in the level L, those degrees of freedom on which constraint conditions have to be applied, if rigid body modes must be eliminated).

The transformation (14), satisfying the above conditions, has the form

$$(16) \qquad \begin{bmatrix} \Phi_{mq} \\ \Phi_{nq} \end{bmatrix} = \begin{bmatrix} I & 0 \\ r_{nm} & X_{ng} \end{bmatrix} \begin{bmatrix} \Phi_{mq} \\ \Phi_{gq} \end{bmatrix}$$

where $g \leq n$.

If the degrees of freedom g are different for each substructure, the Ritz forces defined by eq.(15), corresponding to these d.o.f.s, must be assembled on different columns. If, on the contrary, the degrees of freedom g (internal Ritz coordinates) are the same for all the substructures, the Ritz forces can be assembled on the same columns. Consequently the vectors $[{}_{\ell}X^{k}]$ obtained by solving eq.(15) produce a transformation identical to eq.(16), in which

$$(17) \quad [r_{nm}] = \begin{bmatrix} {}^{1}r_{nm} \\ {}^{2}r_{nm} \\ \cdots \\ {}^{N}r_{nm} \end{bmatrix} \qquad (18a) \quad [X_{ng}] = \begin{bmatrix} {}^{1}X_{ng} & & & \\ & {}^{2}X_{ng} & & \\ & & \cdots & \\ & & & {}^{N}X_{ng} \end{bmatrix} \qquad (18b) \quad [X_{ng}] = \begin{bmatrix} {}^{1}X_{ng} \\ {}^{2}X_{ng} \\ \cdots \\ {}^{N}X_{ng} \end{bmatrix}$$

In the case of validity of eq.(18a) (internal Ritz coordinates different for each component substructure), the condition of linear independency for vectors $[{}_{\ell}X^{k}]$ imposes that all the matrices $[{}^{s}X_{ng}]$ be linearly independent. Eq.(18b), instead, only requires linear independency for the whole matrix $[X_{ng}]$.

Matrix $[r_{nm}]$ is chosen to be the solution of

$$(19) \qquad [K_{nm}]\,[I] + [K_{nn}]\,[r_{nm}] = [0]$$

Therefore $[r_{nm}] = -[K_{nn}]^{-1}\,[K_{nm}]$ is the influence matrix of internal displacements due to unit attachment displacements.

Eq.(16) and (19) allow to adopt, as basis vectors for the construction of starting vectors $[\Phi^{1}]$, the following

$$(20) \qquad [X_{ng}] = [0] \qquad\qquad (21) \qquad [K_{nn}]\,[X_{ng}] = [M_{nn}]\,[X_{ng}]\,[\Omega_{g}]$$

$$(22) \qquad [X_{ng}] = [I]$$

Eq.(20), that is Guyan reduction (Guyan 1965), is acceptable if a sufficient number of Guyan "master" d.o.f.s are included among the attachment d.o.f.s, Eq.(21), that is Hurty component mode synthesis (Hurty et al. 1971), is good if enough fixed attachment eigenvectors are used (they need not to be very accurate, as eq.(16) is only used to determine starting vectors). Eq.(22) can be used (as the less expensive) for very small substructures (that are useful, in structural models, for special purposes). In all of the above cases $[X_{ng}]$ has the form of eq.(18a) (and in all of them linear independency is assured).

For the successive iterations the vectors $[X_{ng}]$ will be selected to satisfy eq.(11), as explained in the next section, and they will have the form of eq.(18b).

4 SUBSTRUCTURE CONDENSATION

The projections of the matrices [K] and [M], partitioned in the form (3), obtained applying eq.(16), are

$$(23)\quad [K^*]=\begin{bmatrix} K^*_{mm} & K^*_{mn} \\ K^*_{nm} & K^*_{nn} \end{bmatrix} = \begin{bmatrix} S_{mm} & 0 \\ 0 & S_{gg} \end{bmatrix} \qquad [M^*]=\begin{bmatrix} M^*_{mm} & M^*_{mn} \\ M^*_{nm} & M^*_{nn} \end{bmatrix} = \begin{bmatrix} A_{mm} & A_{mg} \\ A_{gm} & A_{gg} \end{bmatrix}$$

where

$$(24)\quad \begin{aligned} [S_{mm}] &= [K_{mm}] + [K_{mn}]\,[r_{nm}] \\ [S_{gg}] &= [X_{ng}]^T\,[K_{nn}]\,[X_{ng}] \\ [A_{mm}] &= [M_{mm}] + [M_{mn}]\,[r_{nm}] + [r_{nm}]^T\,[M_{nm}] + [r_{nm}]^T\,[M_{nn}]\,[r_{nm}] \\ [A_{mg}] &= \left([M_{mn}] + [r_{nm}]^T\,[M_{nn}]\right)[X_{ng}] = [A_{mn}]\,[X_{ng}] = [A_{gm}]^T \\ [A_{gg}] &= [X_{ng}]^T\,[M_{nn}]\,[X_{ng}] \end{aligned}$$

Note that matrices $[K^*_{mn}]=[K^*_{mg}]=[S_{mg}]=[0]$ must be put into eq.(2). Remembering equations (1), (2) and (4) we can write

$$[K_{mn}] = [\; K_{ma} \quad 0 \;] \qquad [M_{mn}] = [\; M_{ma} \quad M_{mb} \;]$$

$$[K_{nn}] = \begin{bmatrix} K_{aa} & 0 \\ 0 & K_{bb} \end{bmatrix} \qquad [M_{nn}] = \begin{bmatrix} M_{aa} & M_{ab} \\ M_{ba} & M_{bb} \end{bmatrix}$$

$$(25)\quad \begin{aligned} [K_{mm}] &= \textstyle\sum [S_{mm}] & \qquad [M_{mm}] &= \textstyle\sum [A_{mm}] \\ [K_{ma}] &= \textstyle\sum [S_{ma}] & [M_{ma}] &= \textstyle\sum [A_{ma}] \\ & & [M_{mb}] &= [\; {}^1A_{mg} \quad {}^2A_{mg} \quad \dots \quad {}^NA_{mg} \;] \\ [K_{aa}] &= \textstyle\sum [S_{aa}] & [M_{aa}] &= [A_{aa}] \\ & & [M_{ab}] &= [\; {}^1A_{ag} \quad {}^2A_{ag} \quad \dots \quad {}^NA_{ag} \;] \end{aligned}$$

$$[K_{bb}] = \begin{bmatrix} {}^1S_{gg} & & & \\ & {}^2S_{gg} & & \\ & & \dots & \\ & & & {}^NS_{gg} \end{bmatrix} \qquad [M_{bb}] = \begin{bmatrix} {}^1A_{gg} & & & \\ & {}^2A_{gg} & & \\ & & \dots & \\ & & & {}^NA_{gg} \end{bmatrix}$$

Therefore

$$(26)\quad [r_{nm}] = \begin{bmatrix} -[K_{aa}]^{-1}\,[K_{am}] \\ [0] \end{bmatrix} = \begin{bmatrix} r_{am} \\ 0 \end{bmatrix}$$

The m attachment degrees of freedom of level ℓ and the portion a of the attachment degrees of freedom of level ℓ-1, which become internal in level ℓ, are not changed by the transformation (16). From eq.(25) and eq.(26) it follows that the matrices $[K_{mm}]$, $[K_{ma}]$, $[K_{aa}]$, $[r_{am}]$, $[K_{mn}]$, $[S_{mm}]$, $[r_{nm}]$, $[M_{mm}]$, $[M_{ma}]$, $[M_{aa}]$, $[A_{mm}]$ are constant during the iteration process. Moreover, all the matrices of level 0, excepted the matrices $[S_{gg}]$, $[A_{mg}]$, $[A_{gg}]$, are constant.

5 RITZ BASIS VECTORS FOR MDSI

Eq.(11), written in partitioned form, is

$$(27)\qquad \begin{bmatrix} {}_0K_{mm} & {}_0K_{mn} \\ {}_0K_{nm} & {}_0K_{nn} \end{bmatrix} \begin{bmatrix} X^k_{mq} \\ X^k_{nq} \end{bmatrix} = \begin{bmatrix} {}_0M_{mm} & {}_0M_{mn} \\ {}_0M_{nm} & {}_0M_{nn} \end{bmatrix} \begin{bmatrix} {}_0\Phi^{k-1}_{mq} \\ {}_0\Phi^{k-1}_{nq} \end{bmatrix} = \begin{bmatrix} {}_0P^{k-1}_{mq} \\ {}_0P^{k-1}_{nq} \end{bmatrix}$$

Applying the following exact transformation to eq.(27)

$$(28)\qquad \begin{bmatrix} X^k_{mq} \\ X^k_{nq} \end{bmatrix} = \begin{bmatrix} I & 0 \\ {}_0r_{nm} & I \end{bmatrix} \begin{bmatrix} X^k_{mq} \\ {}_0Y^k_{nq} \end{bmatrix}$$

we obtain (from the lower partition)

$$(29)\qquad [{}_0K_{nn}]\,[{}_0Y^k_{nq}] = [\,{}_0M_{nm} \quad {}_0M_{nn}\,] \begin{bmatrix} {}_0\Phi^{k-1}_{mq} \\ {}_0\Phi^{k-1}_{nq} \end{bmatrix} = [{}_0P^{k-1}_{nq}]$$

Matrix $[{}_0Y^k_{nq}]$ is the matrix $[X_{ng}]$ that must be put in eq.(8), written in the form of eq.(16).

Now we rewrite eq.(28) as

$$(30)\qquad \begin{bmatrix} X^k_{mq} \\ X^k_{nq} \end{bmatrix} = \begin{bmatrix} I & 0 \\ {}_0r_{nm} & {}_0Y^k_{nq} \end{bmatrix} \begin{bmatrix} X^k_{mq} \\ I \end{bmatrix}$$

Eq.(30), being $[Y^k_{nq}]$ the solution of eq.(29), is an exact transformation for eq.(27), whose application, together with the application of eq.(16) (written for iteration k-1), gives

$$(31)\qquad \begin{bmatrix} {}_0S_{mm} & 0 \\ 0 & {}_0S^k_{qq} \end{bmatrix} \begin{bmatrix} X^k_{mq} \\ I \end{bmatrix} = \begin{bmatrix} {}_0F^{k-1}_{mq} \\ {}_0F^{k-1}_{qq} \end{bmatrix} = \begin{bmatrix} {}_0A_{mm} & {}_0A^{k-1}_{mq} \\ {}_0A^{k-1}_{qm} & {}_0A^{k-1}_{qq} \end{bmatrix} \begin{bmatrix} {}_0\Phi^{k-1}_{mq} \\ {}_0\Phi^{k-1}_{qq} \end{bmatrix}$$

where

$$(32)\qquad \begin{bmatrix} {}_0F^{k-1}_{mq} \\ {}_0F^{k-1}_{qq} \end{bmatrix} = \begin{bmatrix} [{}_0P_{mq}]+[{}_0r_{nm}]^T[{}_0P^{k-1}_{nq}] \\ [{}_0Y^k_{nq}]^T[{}_0P^{k-1}_{nq}] \end{bmatrix} = \begin{bmatrix} {}_0F^{k-1}_{mq} \\ {}_0S^k_{qq} \end{bmatrix}$$

and $[S_{mm}]$, $[{}_0S^k_{qq}]$ are given by eq.(24), with $[X_{ng}]=[{}_0Y^k_{nq}]$.
With $[{}_0Y^k_{nq}]$ taken as the solution of eq.(29), eq.(30) can be written as

$$(33)\qquad \begin{bmatrix} {}_0S_{mm} & 0 \\ 0 & {}_0S^k_{qq} \end{bmatrix} \begin{bmatrix} X^k_{mq} \\ {}_0X^k_{qq} \end{bmatrix} = \begin{bmatrix} {}_0F^{k-1}_{mq} \\ {}_0S^k_{qq} \end{bmatrix}$$

Eq.(33) is the same as eq.(27), when eq.(29) is already solved.

But eq.(33) can be rewritten (remembering eq.(1) and eq.(23)), in the partitioned form of eq.(3), as

$$(34)\qquad \begin{bmatrix} {}_1K_{mm} & {}_1K_{mn} \\ {}_1K_{nm} & {}_1K^k_{nn} \end{bmatrix} \begin{bmatrix} X^k_{mq} \\ X^k_{nq} \end{bmatrix} = \begin{bmatrix} {}_1M_{mm} & {}_1M^{k-1}_{mn} \\ {}_1M^{k-1}_{nm} & {}_1M^{k-1}_{nn} \end{bmatrix} \begin{bmatrix} {}_1\Phi^{k-1}_{mq} \\ {}_1\Phi^{k-1}_{nq} \end{bmatrix} = \begin{bmatrix} {}_1P^{k-1}_{mq} \\ {}_1P^{k-1}_{nq} \end{bmatrix}$$

whose lower partition, reapplying the exact transformation (28), becomes

$$(35)\qquad [{}_1K^k_{nn}]\,[{}_1Y^k_{nq}] = [\,{}_1M^{k-1}_{nm} \quad {}_1M^{k-1}_{nn}\,] \begin{bmatrix} {}_1\Phi^{k-1}_{mq} \\ {}_1\Phi^{k-1}_{nq} \end{bmatrix} = [{}_1P^{k-1}_{nq}]$$

The eq.(35), making use of equations (25), becomes

$$(36)\qquad \begin{bmatrix} {}_0S_{aa} & 0 \\ 0 & {}_0S^k_{qq} \end{bmatrix} \begin{bmatrix} {}_0Y^k_{aq} \\ {}_0Y^k_{qq} \end{bmatrix} = \begin{bmatrix} {}_0A_{aa} & {}_0A^{k-1}_{aq} \\ {}_0A^{k-1}_{qa} & {}_0A^{k-1}_{qq} \end{bmatrix} \begin{bmatrix} {}_0\Phi^{k-1}_{aq} \\ {}_0\Phi^{k-1}_{qq} \end{bmatrix} = \begin{bmatrix} {}_0F^{k-1}_{aq} \\ {}_0S^k_{qq} \end{bmatrix}$$

The solution of eq.(36), that is of eq.(35), giving $[{}_0Y^k_{qq}]=[I]$, does not modify the solution of eq.(29), but provide us with the solution of the rows of the upper partition of eq.(27) that correspond to those d.o.f.s that become internal in the level 1. The transformation

$$(37)\qquad \begin{bmatrix} {}_1\Phi^k_{mq} \\ {}_1\Phi^k_{nq} \end{bmatrix} = \begin{bmatrix} I & 0 \\ {}_1r_{nm} & {}_1Y^k_{nq} \end{bmatrix} \begin{bmatrix} {}_1\Phi^k_{mq} \\ {}_1\Phi^k_{qq} \end{bmatrix}$$

with $[{}_1Y^k_{nq}]$ being the solution of eq.(35), operates, on the internal degrees of freedom of level 1, the same transformation that can be performed using, as Ritz basis vectors, the solution of eq.(27).

Repeating the same reasoning for superior levels, we see that, using transformations of the type (30) or (37) in each level, we can obtain the execution of the complete transformation required by eq.(9), (10) and (11). The practical process is obtained by transformations of the single substructures and partial assemblages throughout the substructure tree.

Observe that in eq.(36) we supposed that all the component matrices $[{}^sS_{qq}]$ could be assembled together. This derives from the fact that the partition of eq.(35) corresponding to the b d.o.f.s (see equations (25)) would be

$$(38)\qquad \begin{bmatrix} {}^1S^k_{qq} & & & \\ & {}^2S^k_{qq} & & \\ & & \cdots & \\ & & & {}^NS^k_{qq} \end{bmatrix} \begin{bmatrix} {}^1Y^k_{qq} \\ {}^2Y^k_{qq} \\ \cdots \\ {}^NY^k_{qq} \end{bmatrix} = \begin{bmatrix} {}^1S^k_{qq} \\ {}^2S^k_{qq} \\ \cdots \\ {}^NS^k_{qq} \end{bmatrix}$$

The use of the solution of eq.(38) corresponds to the summation of matrices $[{}^sS^k_{qq}]$, which is more efficiently performed during the assemblage phase. The superposition is based on the fact that, at convergence, all the Ritz coordinates $[{}^s_\ell\,\Phi_{qq}]$ are coincident with the matrix $[\Omega]$.

6 COMPUTER IMPLEMENTATION OF MDSI METHOD

The transformations (24), the assemblages (25), the solution of eq.(35) for each substructure of each level and the solution of the condensed eigenproblem (9) are all what must be implemented in a computer program.

The solution of eq.(35) (that only requires the calculation of $[Y_{aq}]$) cannot give any problem because matrices $[S_{aa}]$ are positive definite. Vectors $[Y_{aq}]$ can result to be linearly dependent, but this is not a problem because matrices $[S_{qq}]$ (that can be positive semidefinite) need only to be assembled together (see eq.(37)). The condensed eigenproblem is well conditioned if the starting vectors are calculated as suggested before (and convergence is achieved in a few iterations). Practical problems arise from the fact that transformations (24) can be very expensive, particularly if matrices must be stored on mass storage in blocks.

Moreover, a great amount of mass storage can be required by the substructure files storing all the matrices of eq.s(24)÷(36).

Finally, the iterative execution of all the matrix operations for all the substructures of the tree requires the repeated use of a considerable number of files.

Also thinking that the calculation of constant matrices is performed only once, the efficient execution of matrix calculations is mandatory. A good compromise between the need of optimization and the corresponding programming effort resulted to be the use of an extended concept of skyline (which is peculiar to symmetric matrices stored columwise), called "variable profile", to symmetric and non symmetric matrices stored columnwise or rowwise.

The efficient use of mass storage requires a special set of subroutines able to compact the data stored on files (rewriting of records of variable length without dead spaces or superpositions of data).

The treatment of the files of a complex tree of substructures need to be completely automatic. Therefore, the program must be provided of a set of subroutines for the automatic file management (with this capability of file manipulation, the use of substructures becomes as easy as the use of finite elements in a standard program).

If the program has this level of sophistication, the use of substructures, also for iterative solutions, can become competitive with standard methods. Of course, the advantages increase with the dimensions of the problem to be solved, till arrive to those dimensions for which only a substructure method is applicable. Beyond this limit, the choice is obliged.

7 SUMMARY AND CONCLUSIONS

An iterative procedure (called MDSI) has been presented for the calculation of the first eigenmodes of a structure subdivided into substructures organized in a multilevel tree. The method consist in a standard subspace iteration. The construction of the standard Rayleigh-Ritz problem of subspace iteration is performed by successive transformations and assemblages throughout the substructure tree. The efficiency of the method is based on two fact:

1) the transformations are performed using matrices of reduced size
2) excellent starting vectors can be calculated executing a first iteration in which traditional dynamic condensation methods are used.

The only drawback is that the method, to be actually competitive for medium size problems, must be implemented using an elevated degree of sophistication in the programming technique.

REFERENCES

Clough, R.W. & J. Penzien. 1975. Dynamics of Structures. Mc Graw-Hill.
Bathe, K.J. 1982. Finite element procedures in engineering analysis. Englewood Cliffs, New Jersey: Prentice-Hall.
Dickens, J.M. & E.L. Wilson. 1980. Numerical method for dynamic substructure analysis, p.120-135. UCB/EERC-80/20. University of California, Berkeley, California.
Guyan, R.J. 1965. Reduction of Stiffness and Mass Matrices. AIAA Journal 3: 380.
Hurty, W.C., J.D. Collins & G.C. Hart. 1971. Dynamic analysis of large structures by modal synthesis techniques. Computers and Structures 1: 535-563.