ABSTRACT

YILMAZ, ZUHAL. Toward An Understanding of Students' Strategies on Reallocation and Covariation Items: In Relation to an Equipartitioning Learning Trajectory. (Under the direction of Dr. Jere Confrey).

The aim of this study is to inform the Equipartitioning Learning Trajectory (ELT) in terms of level descriptions for covariation and reallocation and identify the factors (item-stem formats, item types, item classes, manipulative usage) that affect the participants' performance, and mathematical strategies at those levels. In addition, this study aims to identify the reasons why younger participants perform poorly on those levels in ELT.

In this study, Piagetian clinical interview methodology (Opper, 1977) was used. Clinical interviews were conducted with 23 students in grades K–7 with newly developed reallocation and covariation items. All interviews were videotaped, observation notes were taken and the participants' written work was collected. Then, an analytical model (Powell, Francisco, & Maher, 2003) was used analyze the video data as critical events were identified. Video data were coded to identify mathematical strategies, misconceptions, mathematical representations, and critical events from each interview were transcribed.

The findings from this study indicate that there is a significant difference in younger and older participants' mathematical strategies on given reallocation items. Younger participants demonstrated a significant change in their problem-solving strategies and performance when presented with manipulatives. Students of all ages demonstrated changes in their problem-solving strategies when item-stems were presented in different formats and from varied item classes. Based on this study, decisions are being made to limit reallocation task classes to the reallocation departure and reallocation uneven share. Furthermore, the

results show the critical role that reallocation serves as a bridge between fair sharing and reassembly and division/ multiplication.

Toward An Understanding of Students' Strategies on Reallocation and Covariation Items: In Relation to an Equipartitioning Learning Trajectory

by
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A thesis submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the degree of
Master of Science

Mathematics Education

Raleigh, North Carolina

2011

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DEDICATION

For my family, teachers, students and colleagues, and those all others make my life meaningful.

BIOGRAPHY

Zuhal Yilmaz was born on June 17, 1986 in Rize,Turkey. Zuhal grew up in Mediterranean region of Turkey and graduated from Isparta Murside Ermumcu Teacher Training High School in Turkey in 2004. She attended one of the best universities Bosphorus University in Istanbul, Turkey and graduated with a honor degree in July 2009 with Bachelor of Education degree in mathematics and Bachelor of Education degree in Science. Upon graduation, Zuhal applied for Master of Science program in Mathematics Education in North Carolina State University in 2009. He worked as a graduate research assistant on “Diagnostic e-Learning Trajectories Approach” project under the Direction Dr. Jere Confrey. From this work and other studies in her master degree, Zuhal presented at national and international conferences and submitted publications to journals.

After receiving her MS degree, Zuhal plans to pursue a doctorate degree in mathematics education.

ACKNOWLEDGMENTS

My deepest thanks to:

My advisor, Dr. Jere Confrey for believing in me in my health recovery process and giving me a second chance to pursue my degree, her time and feedback on the ideas in this thesis, and opportunities for invaluable research experiences

My committee, Dr. Allison McCulloch for her time, encouragement and valuable feedbacks, Dr. Paola Sztajn and Dr. Roger Woodard, for their insights, support, and belief;

My mentors and friends at NC State and UNC, for advising, supporting and respecting: Dr. Alan Maloney, Dr. Hollylynne Lee, Dr. Karen Keene, Dr. Kenny Nguyen, Dr. Holt Wilson and Dr. Karen Hollebrands;

My fellow graduate students: Ayanna Franklin, special thanks for her support and belief in me, Nadia Monrose, special thanks for her encouragement, Cyndi Edgington, Drew Corley, Shirley Varela, Erin Krupa, Marrielle Myers, Ryan Pescosolido, and Shayla Thomas;

The teachers, the students and the schools participating in my study: Principal Mussarut Jabeen, Sister Hamida, and Sister Aishia…

My dear family: My mother and father thanks for the support, belief in me and the love, my sister and her husband and my brother thanks for encouragement and listening me, my nephew thanks for making me laugh and feel better

My dear friends: Special thanks for Sister Neriman for her support, encouragement and belief in me, and for never leaving me alone, Ilkay Ozsev Yuksek, Emine Yaylali, Kubra Bal ve ones that I cannot list

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CHAPTER 1

INTRODUCTION

Rational number reasoning (RNR) is one of the most important conceptual fields in mathematics education. The number of studies conducted on RNR reflects the importance to students of learning its concepts. According to Maloney and Confrey (2010) RNR "underpins competence in algebra, higher mathematical reasoning and the qualitative competence required in science" (p.1). Moreover, it is strongly associated with essential areas in mathematics such as division and multiplication, ratios, proportions and rates, fractions, areas and volumes, similarity and scaling, and decimals and percentages (Confrey, 2008).

RNR serves as a basis for many related mathematical content areas however it is remarkable that many students have difficulty obtaining conceptual knowledge about rational numbers (Confrey, 2008). State assessments and researchers have documented that students often fail to internalize a workable concept of rational numbers (Behr, Washsmuth, Post, & Lesh, 1984). Smith (1995) remarked that "[rational numbers] are a central topic in precollege mathematics, yet many students have difficulties learning even their most basic properties"(p.11). In order to establish better learning environments and assessment systems, it is important to investigate and understand students' thinking about rational numbers and how they perceive them in different contexts.

The DELTA Project

Diagnostic E-Learning Trajectories Approach (DELTA) to Rational Number Reasoning project at North Carolina State University[1], under the direction of Drs. Jere Confrey and Alan Maloney, has created a proposed learning trajectory [2](Confrey, Maloney, Nguyen, Mojica, & Myers, 2009) for equipartitioning (Confrey et al., 2009). Equipartitioning means " Cognitive behaviors that have the goal of producing equal-sized groups (from collections) or equal-sized parts (from continuous wholes) or equal-sized combinations of wholes and parts, such as is typically encountered by children initially in constructing "fair shares" for each of a set of individuals"(Confrey et al., 2009, p. 2).

The equipartitioning learning trajectory (ELT) is comprised of sixteen proficiency levels. Also there is a second set of variables referred to as task classes. These task classes delineated from existing literature, analysis of clinical interviews. ELT has 13 task classes which are determined by the tasks that are given to students at particular levels. For instance, in level 9-reallocation students only got task A items in which students are dealing with equipartitioning collections. On the other hand in level 2 students are not presented with task class A items rather than they presented with task classes B, C,D, E, F,G,H, and I which are all sharing wholes tasks classes. Those task classes ordered in terms of sophistication. For instance odd splits on circles (task class H) is harder than odd splits on rectangle (task class F). Two of the levels reallocation and covariation are the focus of my study. Reallocation

[1] Former and current members of the DELTA research group are Dr. Kenny Nguyen, Dr. P. Holt Wilson, Dr. Gemma Mojica, Nadia Monrose, Cyndi Edgington, Ayanna Franklin, Marrielle Myers and Ryan Pescosolido. This work is conducted in collaboration with other members of the team and build on the Dr. Jere Confrey' s work on equipartitioning

[2] See Table 1: Two-Dimensional Matrix Display for the Equipartitioning Learning Trajectory

means that students demonstrate a strategy that requires distribution of extra shares among the existing number of people in the case of sharing collections. It could be investigated by giving students tasks to solve such as 1) A collection that has been already equipartitioned among a certain number of people (p) were given to a student. Then they are told that a certain number of people arrive or leave, and are asked to create fair shares for the new number of people. 2) Students were presented with a set of collection that has not been fairly shared among existing number of people (p). Then they are asked to remediate uneven shares and make them fairly shared among the same amount of people (p).

Another level was called covariation. It meant that as the number of the objects and people changed one preserve the relation (ratio) between these two different quantities. In order to study the learning trajectory, a set of 128 assessment items were created and given to approximately 5000 students from grades 1-7 by DELTA team. Based on an analysis of the item difficulty using Based on the Item Response Theory (IRT) analysis and descriptive analysis, we found that the items from two proficiency levels were not functioning as conjectured. In addition limited amount of item [3]were used to represent the different reallocation task classes items and they did not presented to grades K-2 (Pescosolido, 2010). For instance for reallocation level, the crayon item (Appendix A[4]) was given grades 3-7 and based on the IRT analysis it identified as "hard" for younger students (logit 1.1 for grade band 234 and 345 in Wright maps). Also, only 15.9% of the students from grades 3-7

[3] In total three items were presented for each reallocation task classes items (See Appendix A)
[4] Appendix A includes all covariation and reallocation field test items and percentage for correct responses for each item

responded correctly to this item. For the all covariation items only 17.8% overall responses were correct (Pescosolido, 2010).

Based on these findings some of the questions discussed were

(a) What should be the characteristics of newly developed reallocation and covariation items so that they make the items accessible to students?

(b) Why did the covariation and reallocation items behave so differently than expected in the field test?

(c) Can younger students (grades K-2) solve the reallocation items with different representations if those items were presented in field test?

(d) How does reallocation inform the ELT?

(e) Why do both younger and older students perform poorly on covariation items?

A detailed study on reallocation and covariation levels for students in grades K–7 was designed to undercover the reasons why the items did not function as conjectured, why we did not give the reallocation items to younger participants (K–2), and why the younger participants performed poorly on the field test items on reallocation and covariation. This study was designed to inform level descriptions and outcome measures for covariation and reallocation. A detailed analysis of the participants' actions in clinical interview settings would inform us on how reallocation and covariation related to RNR and how equipartitioning concepts supported participants' abilities to use reallocation and covariation strategies as they worked on the items.

The Problems

An examination of the results from the first field test of the DELTA ELT showed the need for further examination of the participants' strategies in two particular levels:

> *Level 9:* Demonstrate and justify how extra shares can be redistributed for fewer people (additive changes) sharing collections (equipartitioning over breaking to quantify compensation).
>
> *Level 14:* Make factor or split-based changes in a number of objects, number of people sharing, the size of fair shares, or any combination thereof and predict the effects on the other variables (direct, inverse, and covariation to quantify compensation).

The first proficiency level requiring further examination was reallocation. *Reallocation* is a strategy that requires demonstration and justification of how extra shares can be distributed among the remaining number of receivers when an additive change occurs in the number of the receivers in the case of sharing collections. In addition it requires demonstration and justification of remediating unfair shares into fair shares among the existing amount of receivers. For Level 9, the DELTA group had only included one item for each reallocation task class that could be solved using a reallocation strategy (Appendix A). Whether items with different attributes (with different item stems) could be included to encourage students to use reallocation strategies needed to be explored. For this study, I examined the conjectured instances in which students might use reallocation strategies while dealing with mathematical tasks. This research examined three possible instances of reallocation strategy use:

(a) *Reallocation departure:* Given a set of objects already fairly shared among a number of people, participants adjust the shares based on the *departure* of one or more people.

(b) *Reallocation arrival:* Given a set of objects already fairly shared among a number of people, participants adjust the shares based on the *arrival* of one or more people.

(c) *Reallocation uneven shares:* Given a set of objects *unfairly* shared among a number of people but that could have been fairly shared, participants adjust the shares to obtain fair shares.

In the previous DELTA field test, reallocation items were not given to participants in grades K, 1, and 2. Investigation of whether younger students could solve reallocation problems was proposed.

The second proficiency level requiring further examination was covariation. *Covariation*, in regards to preserving the size of a share, is a direct relationship between the number of receivers and the total amount of a substance or set of units associated with that number of shares. To preserve the size of each share, a factor-based change in the number of wholes (the collection) is accompanied by the same factor-based change in the number of people. For example, two people fairly share three boxes of cereal. When the number of people is doubled, the number of cereal boxes should also be doubled to preserve the size of the share a person gets in both situations. Eventually, covariation becomes an important concept for developing a complete understanding of ratio.

Ratio is one of the most difficult concept for children to understand and learn (Behr, Harel, Post & Lesh, 1992; Resnick & Singer, 1993; Streefland, 1985). In the literature (Confrey & Smith, 1995; Resnick & Greeno, 1990; Resnick, &Singer, 1993; Streefland, 1984) main questions were why children have a hard time learning the ratio? Which characteristics make the ratio concept hard to learn? Rensnick et al. (1993) argued that children have some protoquantitave schemas that form a basis for them to understand ratio .The first schema that serve a basis for protoquantitative origin for ratio is a fitting-ness schema which is "… the idea that two things go together based on an external dimensions" (Resnick et al., 1993, p. 101). In addition covariation is the second schema that forms a basis for protoquantitative knowledge. They defined covariation as " … the idea that tow size-ordered series covary, either directly or inversely" (p.107). They stated early recognition of direct covariation may lay in early social experiences like larger place to sit for larger people.

Children's daily life experiences with numbers, relations (increase, decrease etc.) may constitute a potential base for future mathematics learning (Resnick et al., 1993). For instance, children share a pie among a certain number of people and although at younger ages they may not consider the quantities, they consider being fair. They asserted that since children's early ability to reason non-numerically about the relation amount of object (a part of a pie) given to one person form a set of relational schemas in their thinking that finally attach to formal mathematical understanding. In our DELTA work, compensation (Confrey et al, 2009; Maloney & Confrey, 2010) is one of the best examples for this case. In this level students describe both qualitatively (increase, decrease) and quantitatively (factor base change, give numeric explanations) changes in the size of a share when the number of people (share the same amount) change. Piaget, Grize, Szeminska and Bang (1977) used a fish problem to understand how students reason about the relation between quantities. In the problem different size of three fishes (Fish A=5cm, Fish B=10cm and Fish C=15cm)were fed with food. Children can see the comparative relation between the sizes of fishes and amount of food needed to feed a fish. Children stated the relation between quantities qualitatively. For instance, if the size of the fish is smaller than another fish, the first fish should get less amount of food. If children can carry these relations and apply into numerical context, this may lead the usage of direct covariation approach. Once children understand numbers as mental entities, they start to reason about the relations between numbers and carry this reasoning into operations (Confrey, personal communication 2011). Forming these mental structures to express relations and think about mathematics as functions and relational reasoning among numbers leads to a ratio understanding (Confrey & Smith, 1994; Confrey & Smith, 1995; Resnick et al., 1993).

According to Lesh, Behr, and Post (1987), a ratio target to show the same relation by using a different representation for a given relation in one representational system. Confrey and Smith (1995) found that splitting was a primitive basis for multiplication and division (Fischbein, Deri, Nello, & Marino, 1985). They defined the splitting action as " . . . creating equal parts or copies of an original . . . an operation that requires only recognition of the type of split and the requirement that the parts are equal"(Confrey & Smith, 1995, p. 294). Then they emphasized that "the action of counting is associated with one view of the development of adding and subtracting, the action of splitting is associated with one view of the development of multiplying and dividing"(Confrey & Smith, 1995, p. 295). Their analysis of splitting was carried further into the development of ratio understanding in that splitting provided a basis for developing a complete ratio understanding. At its initial level, Confrey and Smith (1995) defined *ratio* as "a description of the invariance across a set of proportions" (p. 73). They also believed that ratio reasoning, multiplication, and division needed to be developed simultaneously.

In the first DELTA field test, six covariation items (Appendix A) assessed participants' performances on Level 14. The analyses showed that participants' performed poorly on the covariation items:

> Examination of the parameters provided in [the linear regression] model can identify trends in the students' performance. A positive parameter indicates a factor related to an increase in item difficulty (resulting in a lower success rate) and vice versa for a negative parameter. Items from learning trajectory levels 5, 8, and 10–16 proved to be significantly more difficult than items from level 1 (Pescosolido, 2010, p. 96).

Because there was not enough variation in the first field test's results from the covariation level, there was not enough evidence to show the relation between participants' proficiencies in equipartitioning and their abilities to use covariation in the presented field test items. In this study, I addressed the possible reasons for this including whether or not grade level affects participants' performance levels as they work on covariation items. The DELTA field test indicated a need for the further investigation of these two critical topics, and by studying the results from the first field test; I proposed a more specific set of research questions to investigate.

Research Questions

Shavelson and Towne (2002) discussed three key questions that guide the methodology of social science research. They are

1. Description: What is happening?
2. Causation: Is there a systematic reason why it is happening?
3. Mechanism: How is it happening?

The third question aided the formulation of my research design and questions that focused on the need areas of covariation and reallocation concepts. This question led me to identify the participants' strategies and errors systematically. Then examine the reasons for the systematic errors and identify the possible misconceptions and learning barriers of the participants. Finally, this helped me to identify in what conditions and factors (e.g., item type, item-stem format, grade effect) participants were unable to solve the given items, could demonstrate correct strategies, and exhibited systematic errors and misconceptions. Each research question and conjecture will now be discussed.

Research Question 1

What are the observed differences and similarities among participants' strategies in earlier grades (K–4) and higher grades (5–7) while solving reallocation items and sharing collections items based on evidence from clinical interviews? Is there any systematic difference observed among participants'' strategies at different grade levels when solving reallocation items? If yes, what might be the reasons for these differences? (item class types, item-stem format, item type)

Conjectures: Younger participants (K–4) will tend to use reallocation and older participants (5–7) will usually use multiplication and division facts. The possible reason for this difference in strategies is that older participants will have received more formal instruction on multiplication and division than younger participants. Item attributes (i.e., item-stem formats, item types, item classes) may affect participants' strategies while dealing with the reallocation items in clinical interviews. Participants tend to use reallocation on reallocation strategy departure task class items while they tends to use collection strategy on reallocation arrival and reallocation uneven share task classes items. If the item stem includes an organized picture of the initial shares for all task classes, participants more likely to use a reallocation strategy since a picture of the initial shares help them to act on the shares as they reallocate. Multiple method and structured item types will elicit both reallocation and collection strategy as participants worked with presented item. Since in structured items both collection and reallocation strategies were described in item stem and asked for choosing the correct strategy (ies) and multiple method items were asked for two different ways to solve presented reallocation item.

Research Question 2

What is the effect of manipulative use on younger participants' (grades K–2) strategies while solving reallocation items? Is there any difference between participants' abilities to solve reallocation items when they do and do not have manipulatives?

Conjectures: Young participants who cannot solve reallocation problems with a paper-pencil assessment will be able to solve the same reallocation problems with the use of manipulatives.

Research Question 3

What are participants' performance levels and mathematical strategies in solving covariation items? Is there any systematic difference observed among their performances at different grade levels while dealing with covariation items?

Conjectures: The older participants are more likely to perform better on covariation items. For covariation items, when the item stem includes a table representation, participants will search for a pattern in the given table to solve the problem without having a correct justification. When the item stem does not include a table, participants will use a splitting strategy. First, they try to find the each person share and then use this amount to to find the amount of share for existing amount of people.

Thesis Overview

A review of relevant literature including the concepts of equipartitioning, learning trajectories (LTs), and concrete objects in mathematics education is given in Chapter 2. Chapter 3 introduces a detailed discussion of the methods used to design and implement the study, along with the analysis strategies for the resulting data set. Chapter 4 contains a qualitative analysis of the participants' thought processes and mathematical strategies and an in-depth exploration of the plausible connections among relevant factors. Finally, Chapter 5 addresses the implications of the qualitative and quantitative results. A discussion of the participants' thought processes and strategies used in equipartitioning relates these findings to the central research questions and the DELTA team's understanding of levels 9 and 14 of the ELT. It also addresses future research implications.

CHAPTER 2

EQUIPARTITIONING LITERATURE

Understanding RNR is a challenge for many students. Many researchers (Carpenter, Fennema, & Romberg, 1993; Confrey 2008; Gelman, 2000) in the RNR field have provided records of the learning difficulties and misconceptions students must overcome to have a complete understanding of rational number. Several have indicated that students use their knowledge about whole numbers to reason about rational numbers, which gives rise to many misconception(Moskal & Magone, 2000; Resnick et al., 1989). Some of the major misconceptions are the result of students applying natural number reasoning to decimal numbers. For instance, some students believe that multiplication always makes the resulting number bigger while division makes it smaller (Fischbein et al., 1985) or that the more digits a number has, the larger it is (Stacey & Steinle, 1998) or that the product of factors is their sum (Empson & Turner, 2006; Wilson, in progress). Each of these misconceptions is connected to principles only applicable to natural numbers except the last one which emphasizes additive reasoning.

Researchers have interpreted rational numbers in various ways. For nearly thirty years, different constructs have been related to rational numbers. One recurring theme is *partitioning*. Partitioning can be defined as dividing a whole into parts (Kieren & Nelson, 1981). Several studies have been conducted to understand children's partitioning strategies for dividing a continuous whole into equal parts (Charles & Nason, 2000; Lamon, 1996; Pothier & Sawada, 1983; Squire & Bryant, 2002). According to Kieren (1993), the development of a deeper understanding of rational number knowledge is strongly related to the partitive quotient construct. Differing from early perspectives on partitioning, McGee, Kervin and Chinnapan (2006) defined partitioning as dividing an object or objects into exhaustive and disjoint number of pieces. McGee et al. (2006) pointed the one criterion for partitioning that these disjoint pieces should exhaust the whole object. English and Halford (1995) stated that the parts should not overlap and the whole object should be exhausted – a representation of a division. Another important construct revealed from the research related to the acquisition of rational number understanding is that "the part-whole construct of a fraction represents one or more parts of a unit that have been divided into some number of equal-sized pieces" (McGee, Kervin, & Chinnappan, 2006, p. 362).

The DELTA research group completed a comprehensive literature synthesis on partitioning. In this synthesis, Confrey (in progress) integrated the literature on partitioning into four—later collapsed to three—cases, which are described in more detail in the following sections: a collection and a single whole and multiple wholes that produced proper fractions and multiple wholes that produced improper fractions. The last two were collapsed so that describe their interconnections could be described more clearly. Confrey disagreed with some researchers' practices of defining partitioning as breaking or sub-grouping into uneven groups and associating partitioning with both part–part and part–whole relations (Steffe, 2004). She related partitioning only to fair shares (2008, 2009, 2009b). To clarify, Confrey introduced a new concept of *equipartitioning* and only used it for the case of creating equal-sized groups. Her research group defined equipartitioning as:

> Cognitive behaviors that have the goal of producing equal-sized groups (form collections) or equal-sized parts (from continuous wholes), or equal-sized combinations of wholes and parts, such as is typically encountered by children initially in constructing “fair shares” for each of a set of individuals. (Confrey, Maloney, Nguyen, Mojica, & Myers, 2009b)

This definition of equipartitioning that covers the decomposition of any size, shape, partition of wholes and collections, and the nature of synthesis led the DELTA research team to delineate equipartitioning into four cases: Cases A, B, C, and D.

Case A is formed by creating equal-sized groups from a *collection,* a group of indivisible wholes or discrete items. Case B involves creating fair shares of a single continuous whole, most commonly known as *unit fractions*. The remaining problem types, Cases C and D, indicate sharing multiple wholes among several people. Case C yields proper fraction outcomes, and Case D gives improper fraction outcomes. The next section will examine each case in detail based on the relevant literature and how the DELTA team has used them.

Case A: Equipartitioning Collections

Equipartitioned collections potentially convey important mathematical understanding such as ratio reasoning (how many items per person), fractions as numbers, and *a*/*b* as operators as shown in the RNR map created by Confrey (Confrey, et.al, 2009).

Many studies have shown that children are most successful at sharing collections among two or more people (Confrey et al., 2009; Confrey et al., in review; Pepper & Hunting, 1998). They can distribute items such as biscuits, toys, and jelly beans fairly by using different strategies. Studies found that *dealing* is a commonly used strategy among children that allows them to share items fairly among multiple people (Davis & Pitkethly, 1990; Pepper & Hunting, 1998). Some studies have ranked children's fair share strategies as (1) systematic and successful, (2) unsystematic and successful, and (3) unsystematic and unsuccessful (Confrey et al., in review; Davis & Pitkethly, 1990; Hunting & Sharpley, 1991).

Although children's dealing strategies have been systematically researched in the literature for degrees of success, the DELTA research team created several items and conducted additional clinical interviews to explore children's strategies and responses when asked to fairly share a collection among a specific amount of people. In systematic dealing strategies, students tended to use one-to-one correspondence, many-to-one correspondence, or simultaneous distribution for two-splits (halving) between items and recipients until all items in the collection were reallocated. For instance, students were asked to share 15 candies among three people. In Figure 1, a student deals one candy into each person's hand, demonstrating one-to-one correspondence.

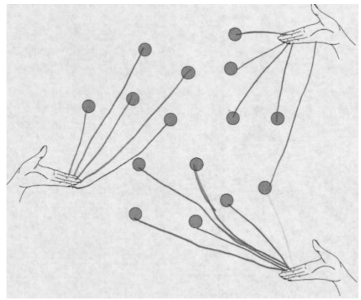


Figure 1. One-to-one correspondence in systematic dealing.

As students gained proficiency, they resorted to using composite units (many-to-one) to allocate the collection among two or more people, sometimes dropping back to individual units as they approached completion. In Figure 2, a student gave five candies to each recipient at one time; five candies are the composite unit in this instance.

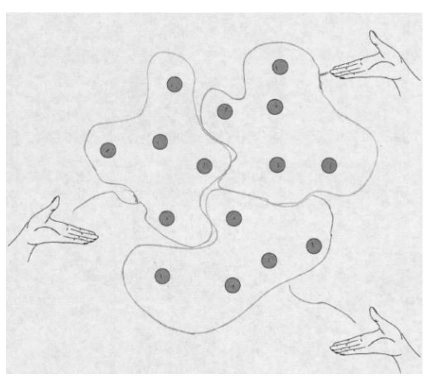


Figure 2. Composite unit distribution.

In Case A, students justified their fair shares in multiple ways. They used counting of items per pile or group, height comparisons of piles, and visual comparisons of pile heights to justify their fair shares (Hunting & Sharpley, 1991). Pepper (1991) stated that some children understand that systematic dealing results in fair shares while others are not aware of this fact. The children, who used systematic dealing, may believe that their shares are fair since they completed one round and end up in the right place. Even though, they ended in the right place, they may want to justify that their shares are fair with counting. On the other hand, those children unaware of the properties of systematic dealing need to verify the equality of shares by using counting, height comparisons, and visual comparisons. The use of counting to justify fair shares generated research questions about whether there is a discernible cognitive relation between counting and fair-sharing competency. Pepper and Hunting (1998) reported that sharing discrete items does not significantly relate to the counting ability level of children. This indicates that poor counters can share discrete items fairly. The previously stated findings reinforce the DELTA team's conjecture that "counting and sharing have distinct cognitive roots" (Confrey, et al., 2009).

In addition to sharing collections strategies, students developed other strategies for solving tasks related to Case A. In this study, I examined a specific strategy from DELTA called *reallocation*. Reallocation refers to the strategies used to generate fair shares from existing objects when given an amount of objects to fairly share among a number of people with additional people arriving or leaving. For instance, if 20 coins are fairly shared among a group of five people then each person gets four coins. If one person leaves the group, reallocation happens when children fairly share the extra four coins among the remaining four people rather than regrouping all the shares to form a collection again and then dealing the 20 coins to the four remaining people. Reallocation can also be observed when more people arrive or when a given amount of objects are unfairly shared among a number of people and the participant must demonstrate strategies for reallocating the coins to generate fair shares. In the DELTA research for Case A, both the original distribution of coins and their reallocation had to result in integral numbers of items being given to each person sharing them.

To illustrate this last situation, imagine a group of three people with a total of 21 coins. The shares are unequal with the first person having four coins, the second person having ten coins, and the last person having seven coins. The students could make these unfair shares into fair shares by using the following strategy. First, they could determine that every person had at least four coins. Then they could redistribute the extra coins (6 coins + 3 coins = 9 coins) from the second and third persons' shares among the three people (9 coins / 3 people = 3 coins / 1 person). Then they could add three coins to each person's existing fair share of four coins. Eventually, they would end up with seven coins per person. All the cases (i.e., a person leaves, a person comes, reallocating unfair shares) anticipate a nascent form of the distributive property that describes how equipartitioning distributes over breaking rather than describing how multiplication distributes over addition (Confrey, Maloney, Nguyen, Wilson et al., 2009). In case D tasks, children used benchmarking strategies, after all cycles of dealing one size of benchmarking are finished and leaving a remainder (n parts less than number of people p). For instance, a student asked for fairly shared objects(m=p+1) among number of people (p). Students stated their answers in terms of mixed numbers. They first gave one whole to each person then they add $1/p^{th}$. We can mathematically express this in an example. Eight pizzas fairly shared among six people so a student can show this nascent form of distributive property as; $4/3= (3+1)/3=3/3+1/3=1+1/3$.

Case B: The Equipartition of a Single Whole

Partitioning a single whole potentially contributes to the acquisition of RNR. Piaget, Inhelder, & Szeminska (1960) noticed that children used the following strategies when they evolved skills of partitioning a single whole. First, they observed children performing general fragmentation (chopping), and then the children progressed to making equal parts through dichotomous, trichotomous, or both methods of division. Finally, the children divided a whole into five and six equal parts (Piaget et al., 1960). This dividing process is important in acquiring an understanding of fractions. Piaget et al. (1960) stated that there are seven conditions for understanding fractions as; 1) A whole can be divided 2) A fraction can be used to determine number of parts 3) there is relation between number of the parts and number of the cuts 4) one should understand the conservation of the whole 5) one should create the correct number of pieces 6) one should create equal-sized pieces 7) one should exhaust the whole object. In our DELTA work the last three of which are essential for the correct equipartitioning of a continuous whole since as students create the correct number of parts they realized the relations between the number of the cuts and parts (on rectangles). In addition, as they exhaust the whole they realize the whole can be divided and it remains invariant.

In subsequent research, Pothier and Sawada (1983; 1989) created a new framework for studying the partitioning skills of children. The framework is based on the number of people who share a continuous single whole, which constitutes Case B tasks in the DELTA equipartitioning research. According to this framework, children first learn a halving mechanism (Pothier & Sawada, 1983). Though they can construct half of rectangular and circular regions, the regions created at this level are not necessarily fair shares and children refer to any fragmented parts of the whole as a half regardless of whether they are equally divided or not. In the next step, called *algorithmic halving*, children use repeated halving to produce 2 to the *n*th power shares by doubling the number of partitions (Pothier & Sawada, 1983). At the evenness level, children use algorithmic halving strategies to split wholes for numbers other than powers of two. Figure 3 represents partitioning into sixths by using the halving strategy.

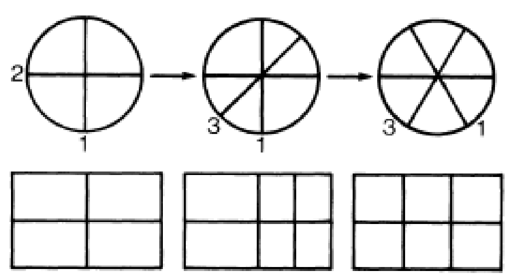


Figure 3. Partitioning into sixths (Pothier & Sawada, 1983).

According to Pothier and Sawada (1983), children at the evenness level cannot create some odd splits such as fifths and ninths. As a result, the level of oddness was formulated because the algorithmic halving strategy is not sufficient for obtaining fair odd-splits. At the oddness level, children use a counting algorithm to ensure that they have formed the correct number of shares (Pothier & Sawada, 1983). The last level of the partitioning framework is called *composition*. At that level of proficiency, children are aware that the counting algorithm for larger composite numbers of splits, such as 15 and 21, is not efficient. According to the DELTA conceptual mapping of RNR, children use the composition of factors to create larger composite number shares. For example, sharing a rectangular cake among 15 people can be accomplished in two ways. A child can make two vertical cuts yielding three pieces and five horizontal cuts yielding five pieces (Figure 4) or he or she can make 14 successive parallel cuts.

This final level of the framework requires the child to see the efficiency in the first strategy as compared to the second one and choose it. In the DELTA project, criticisms were made of Pothier and Swada's composition of factor choices, because it seemed to assume that children applied a concept of a factor to the situation. Rather, DELTA has argued that the concept of a factor derives from the composition of splits and hence renamed the postulate to permit the meaning of a factor to evolve from the act of composition of splits (Maloney & Confrey, 2010). Eventually students develop an understanding about multiplication facts and factors through composition of splits

Figure 4. Composition of factors of 15 splits, 3 x 5.

Cases C and D: Equipartitioning Multiple Wholes Among Multiple People

Many studies have discussed young children's partition strategies for continuous wholes in relation to their fractional understanding (Charles & Nason, 2000; Lamon, 1996; Toluk & Middleton, 2001). Lamon (1996) identified several strategies that children might use while partitioning multiple continuous wholes such as allocating wholes then fairly sharing the remaining parts (preserved-pieces), partitioning the wholes then allocating each share from each whole (mark all), and partitioning each share then allocating the shares (distribution). Charles and Nason (2000) developed a taxonomy that highlighted the young children's partitioning strategies involved in abstracting "partitive fraction quotient construct" (p. 192). McGee, Kervin and Chinnappan (2006) operationally defined *partitive quotient construct* as "the process in which one starts with two quantities, *x* and *y*, treats *x* as the dividend and *y* as the divisor, and by the operation of partitive division obtains a single quantity *x*/*y*" (p. 364). For instance, three pizzas shared among four people yields the fraction 3/4. The partitive quotient construct of fractions is important to "construct conceptual mapping between (1) the number of people *(y)* to the fraction name in each share *(yths),* and

(2) the number of objects being shared *(x)*and the number of *yths* in each share" (Charles & Nason, 2000, p. 194).

Although the importance of partitioning concepts for acquiring RNR is clearly emphasized in the literature and the DELTA research work carried this work further by defining and addressing the importance of equipartitioning concept. Confrey (1988) proposed splitting as an operation which results in creating equal-sized groups or parts. She identified and described different cognitive roots between counting and splitting. She deeply explained the difference between those roots as, in splitting worlds one can make comparisons and examine changes by relying on ratio relation between quantities (geometric sequences) while in counting world one can rely on, repeated iterations and arithmetic sequences (Confrey, 1988; Confrey & Smith 1995). She also claimed that splitting as an operation was a precursor to partitive division, inverse of the operation can be coded as multiplication which is not based on repeated addition. DELTA research team took this splitting conjecture and examined how this work of Confrey related with equipartitioning. If one created equal-sized parts on a whole or a collection, one has performed one split. So this means that number of parts defined the number of splitting action. For instance, one can fairly share a rectangle into five parts, which means one has made five-splits on the rectangle. This single split at a time can be an early explanation for fractions. As a result, Confrey and Smith (1994) stated, "by developing the construct of splitting in the curriculum, one can establish a more adequate and robust approach to such traditionally thorny topics as ratio and proportion, multiplicative rate of change, exponential functions, and so on . . ." (p. 1, as cited in Pescosolido, 2010).

Learning Trajectories Literature

This section of the literature review examines LTs in detail including various definitions of LTs, the necessary constructs for these definitions, the current uses and benefits of LTs, and the DELTA ELT.

Learning progressions (LPs), also called LTs, can be conceptualized in many ways (Clements, Sarama, & Julie, 2009). Smith et al. (2006) defined LPs as “successively more sophisticated ways of reasoning within a content domain that follow one another as students learn” (p. 1). More recently, Corcoran, Mosher, and Rogat (2009) defined an LP as “a hypothesized description of successively more sophisticated ways student thinking about an important domain of knowledge or practice develops as children learn about and investigate that domain over an appropriate span of time” (p. 37). As this definition of LP stated, and the literature has documented, the nature of LPs is mainly hypothetical (Duncan, 2009; Simon, 1995). It is rooted in Simon’s (1995) Hypothetical Learning trajectory (HLT) definition that refers to “. . . the teacher’s prediction as to the path by which learning might proceed. "It is hypothetical because the actual learning trajectory is not knowable in advance [and] it characterizes an expected tendency” (p. 135). According to Simon (1995), an LT comprises the connection between teachers’ knowledge and students’ actions around three elements: “a learning goal, activities intended to address that learning goal and a hypothetical process by which a student will use the activities in an attempt to reach the learning goal” (p. 136). Clements and Sarama (2004) slightly changed these three elements, stating that “a complete hypothetical learning trajectory includes the learning goal, developmental progression of thinking and learning, and sequence of instructional tasks” (p. 84).

In contrast to Simon's (1995) and Clements and Sarama's (2004) perspectives, Battista (2004)perceived LTs differently. He proposed Cognitive Based Assessment (CBA) is as a conceptual framework for understanding students' learning progress. In it are levels of sophistication for students' conceptualization and reasoning about a certain mathematics topic. These levels can be stated as "(a) starts with the informal, pre-instructional reasoning typically possessed by the student, (b) ends with the formal mathematical concepts targeted by instruction, and (c) indicates cognitive plateaus reached by students in moving from (a) to (b)" (Battista, 2004, p. 186). Battista (2004) proposed the term of *cognitive terrain* to describe how students enter several plateaus of knowledge about a certain mathematics topic while going through different LTs. Since each student has different backgrounds, initial knowledge and thought processes, the way student progress on this terrain may differ. The students' levels of sophistication in thinking describe not only cognitive terrain but also the students' conceptualizations and reasoning, the learning obstacles they face during the learning process, and their degrees of achievement.

Parallel to Battista's interpretation of the levels of sophistications found in LTs, Confrey (2006) embedded LTs within conceptual corridors. In these conceptual corridors, students typically encounter various learning obstacles, landmarks that imply multiple possible ways to reach learned ideas. Figure 5 shows the embedded LTs within conceptual corridors (Confrey,2006).

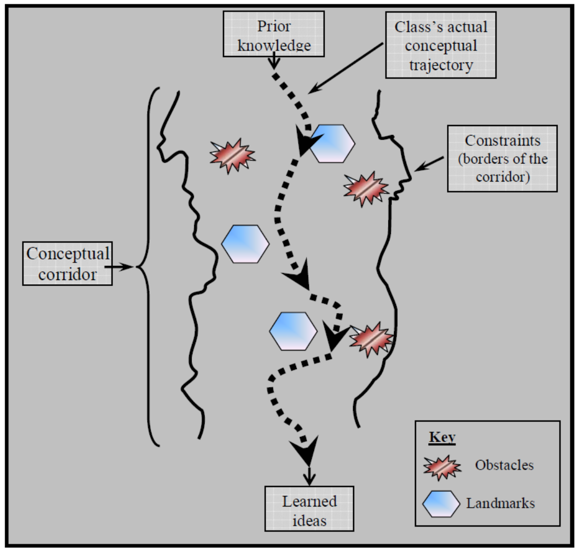


Figure 5. Confrey's conceptual corridors through learning trajectories *Source*: Confrey, 2006.

In line with the previously discussed definitions, the DELTA research team constructed an LT definition of

> A researcher-conjectured, empirically supported description of the ordered network of constructs a student encounters through instruction (i.e., activities, tasks, tools, forms of interaction, and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation and reflection, towards increasingly complex concepts over time. (Confrey, Maloney, Nguyen, Mojica & Myers, 2009, p. 2)

The DELTA definition of LT shares some commonalities with those found in the literature. Using the existing research synthesis as a basis, DELTA's LTs identify a particular domain and a goal level of understanding; serve as effective starting points in relation to the different background experiences of the students, and progress from simple to complex cognitive states (Maloney & Confrey, 2010). However, there are pointed differences that distinguish DELTA LTs from those found in the literature. They are seen in the following terms:

- *Researcher-conjectured* refers to the fact that LTs are models created by researchers of students' likely paths.
- *Empirically supported* refers to a three-step process: reviewing the literature, asking outside experts to review the syntheses, and conducting further studies.
- *Through instruction* is the recognition that students will only progress if provided appropriate opportunities, technology, and tools to learn the material and that the sequence of those activities must be designed intentionally to support the trajectory.
- *Through successive refinements* indicates the needs for students' active involvement in the learning process and engagement in cycles of problem-solving behavior.

(Confrey, Maloney, Nguyen, Mojica, 2009, p. 2–3).

The DELTA team's construction of the ELT incorporates these pointed differences. The DELTA method (Maloney and Confrey, 2009) for constructing ELT, task classes, and assessment items is adapted from evidence-centered assessments (Mislevy, 2003; Wilson, 2005). Figure 6 conveys this methodology that has been designed by Confrey and Maloney (in press). The blue arrows show the direction of the interaction effect between elements in the model. The red arrows show the possible theoretical interactions between the elements (Pescosolido, 2010).

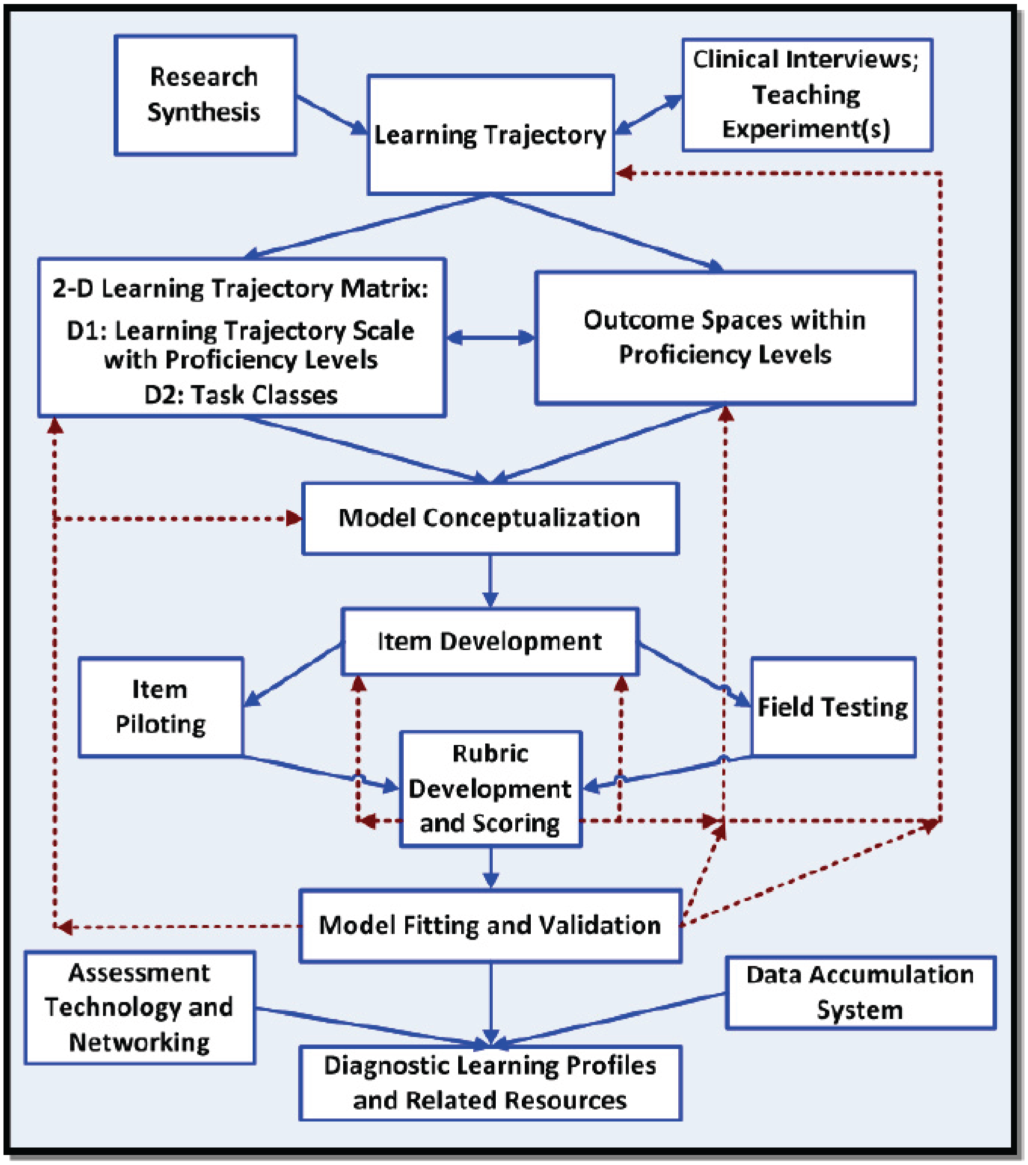


Figure 6. DELTA methodology for creating LTs. *Source*: Confrey & Maloney, in press.

The DELTA method for the construction of ELTs started with a synthesis of the research literature on RNR. With it, our theoretical perspectives, and our experiences with children learning mathematics, the DELTA team identified an initial ELT. After completing the synthesis, cross-sectional clinical interviews were conducted to elaborate on the initial LT by filling in the gaps. Paper-pencil assessment items were then developed and piloted. Based on the students' responses, outcome spaces were formulated for each proficiency level of the resulting LTs. At last, field tests items were distributed to schools in four North Carolina school districts (Maloney & Confrey, 2010).

It was essential for the team to collect empirical evidence from the students' answers to field test items about different solution strategies and students' misconceptions to properly build the scoring rubrics. With the help of that evidence, the team iteratively tested paper-pencil assessments items and made revisions to the LT. As Duncan (2009) stated, "the process of development and validation for LPs are intertwined and occur through iterative cycles of empirical testing and theoretical revision and refinement" (p. 608). After the field test items were examined, they were categorized based on the quality of responses, clarity of questions, and the matching degree of the item with the appropriate (Maloney & Confrey, 2010). The research team examined all the students' responses to a particular item, developed a rubric, and found a range of student response exemplars (Maloney & Confrey, 2010). The responses to each item were scored. During the scoring process, the rubrics were used to score justifications, solutions, or both for the assessment items. After the rubrics were examined and discussed, the research team constructed a revised, two-dimensional matrix display for the ELT. The vertical dimension represents the progression of the LT's proficiency levels with the sophistication increasing from bottom to top, and the horizontal dimension represents the task classes. The equipartitioning tasks were broken into different task classes. Those task classes determined in order of the previously mentioned equipartitioning cases A, B and C/D as left to right in the horizontal dimension of ELT (Confrey and Maloney, 2010). Table 1 represents the current ELT after all the revisions and refinements have been made.

Table 1

Two-Dimensional Matrix Display for the Equipartitioning Learning Trajectory

		Task Classes												
	Proficiency Levels	Collections **A**	2-split (Rect/Circle) **B**	2^n split (Rect) **C**	2^n split (Circle) **D**	Even split (Rect)**E**	Odd split (Rect) **F**	Even split (Circle) **G**	Odd split (Circle) **H**	Arbitrary integer split **I**	$p = n + 1$; $p = n$-1 **J**	p is odd, and $n = 2^i$ **K**	$p >> n$, p close to n **L**	all p, all n (integers) **M**
16	*Generalize*: a among $b = a/b$													
15	*Distributive property*, multiple wholes													
14	*Direct, Inverse, and Covariation*													
13	*Compositions* of splits, multiple wholes													
12	Equipartition *multiple wholes*													
11	Assert *continuity principle*													
10	*Property of Equality for Equipartitioning (PEEQ)*													
9	Redistribution of shares (*quantitative*)													
8	Factor-based changes (*quantitative*)													
7	*Compositions* of splits; *factor-pairs*													
6	*Qualitative compensation*													
5	*Reassemble*: n times as much													
4	*Name* a share with respect to referent unit													
3	*Justify* the results of equipartitioning													
2	*Equipartition single wholes*													
1	*Equipartition Collections*													

Note. Adapted from Maloney & Confrey, 2010. Rect = rectangle.

Based on this iterative process, the DELTA team recognized a need to examine Level 14 and Level 9 of the field test items more extensively by using the DELTA methodology: research synthesis, item development, and clinical interviews. My study addressed the literature review and clinical interviews on these two topics and will contribute to the DELTA teams' understanding of those levels.

Manipulative Usage in Mathematics Education

The effects of manipulative usage on students' understanding of mathematics have been discussed intensely for several years. This section gives a brief summary of the existing research on manipulative usage. Learning theorists believe that children's internalized concepts evolve through interaction with the environment and concrete materials are the tools for this interaction. This belief has been conveyed in several ways.

Piaget (1971) suggested that through the reconstruction of reality, children develop their internalized concepts. Dewey (1938) believed that in a learning environment, a student needs firsthand practice so he can turn back and evaluate, make decisions about what is beneficial and useful to remember, and transfer this knowledge and experience to perform another activity in new settings called *experiential learning*. These theories on learning argue that children should actively construct their knowledge instead of having information imposed on them and hoping it will be internalized.

According to Piaget and Szeminska (1952), students move from concrete to pictorial thinking and then to abstract thinking in mathematics. In parallel with Piaget's beliefs is an accumulated body of evidence from research on manipulatives that indicates several benefits of manipulative usage in mathematics learning and achievement (Burns, 1996; Fennema, 1972; Sowell, 1989).

Piaget and Szeminska (1952) claimed that children need concrete physical objects to develop abstract mathematics concepts. According to Heddens (1986), manipulatives are effective tools for children to move from concrete to abstract level of mathematics understanding. Manipulation of concrete objects, sorting, touching, breaking actual physical objects helps students to develop clear mental images and communicate abstract level of understanding. Students have limited experiences with manipulation of actual physical objects, have difficulties while communicating their mathematical thinking and moving into abstract level of thinking. Thus, mathematics manipulatives are effective tools that demonstrate or model abstract mathematical concepts and several aspects of the real world. Post (1981) suggested that if the model functions parallel to its original, “it becomes possible to manipulate and use the model to make conclusions and/or predictions about its counterpart in the real world” (p. 110). According to Lesh (1979, as cited in Post 1981), manipulative materials play a mediator role between the real world and the mathematical world. Figure 7 represents the relation between the mathematical and real world.

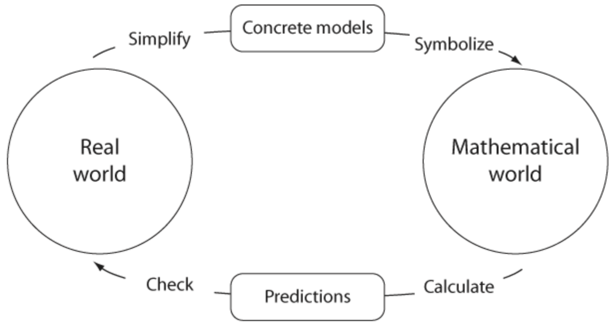


Figure 7. An extended relationship between the real and mathematical worlds. *Source*: Post, 1981, p. 111.

According to Piaget's (1945, 1962) theory on children's acquisition of logicomathematical knowledge, I examined the effects of manipulative usage on younger students' strategies while solving reallocation items. Piaget stated that logicomathematical knowledge "consists of mental relationships, and the ultimate source of these relationships is in each individual" (as cited in, Kato, Kamii, Ozaki, & Nagahiro, 2002, p. 33). This theory argues that children construct logicomathematical knowledge through their own thinking and that manipulatives are desirable when they encourage children to think (i.e., to make relationships through constructive abstraction) during problem solving. Kamii, Lewis and Kirkland (2001) addressed how mathematical relations and meanings are not attached to actual physical items; children do not obtain these relations and meanings through abstraction from items itself. Thus, the most challenging issue in using concrete items in mathematics education is facilitating students' abstraction from the modeled mathematical situation. In other words, the challenge is in facilitating students' ability to transfer what they model with manipulatives into meaningful mathematical learning experiences.

According to Sarama and Clements (2009), "manipulatives are meaningful for learning only with respect to learners' activities and thinking" (p.148), which implies the proper usage of manipulatives. Hiebert and his colleagues (1997) explained the proper usage of mathematical tools thusly,

> Mathematical tools should be seen as supports for learning. But using tools as supports does not happen automatically. Students must construct meaning for them. This requires more than watching demonstrations; it requires working with tools over extended periods of time, trying them out, and watching what happens. Meaning does not reside in tools; it is constructed by students as they use tools. (p. 10)

From this discussion, we can conclude that although manipulative usage is important for mathematics learning, the physicality of manipulatives do not convey mathematical meaning. Thus, educators must make sure their students reflect on their actions while modeling a mathematical situation by using manipulatives. They should attach mathematical understanding to their actions. From this point of view, some researchers have questioned manipulatives use in mathematics education because the improper use of manipulatives may harm students' learning. For instance, Gravemeijer (1991) argued that " . . . even if children begin to make connections between manipulatives and nascent ideas, physical actions with certain manipulatives may suggest different mental actions than those students are to learn" (as cited in, Sarama & Clements, 2009, p. 146). The author gave an addition operation example to represent this mismatch when students perform addition operations by using a number line:

> When adding 6 + 3, students locate 6, count ‘‘one, two, three,’’ and read the answer, ‘‘9.’’ This usually does not help them solve the problem mentally, for to do so they have to count ‘‘seven, eight, nine’’ and at the same time count the counts—7 is one, 8 is two, and so on. These actions are quite different. (Gravemeijer, 1991; see also Sarama & Clements, 2009, p. 146).

Summary of Introduction and Literature Review Chapters

DELTA definition of a learning trajectory strongly emphasize further revisions and refinements on ELT which are heavily relied on both existing literature and test of conjectures and collecting empirical evidence in studies. The IRT analysis and descriptive analysis of the field test item data revealed a need for further examination on two specific levels of ELT: Level 9 reallocation and level 14 covariation. Students perform poorly on those levels and in the field test younger student (Grades K-2) were not presented with those items. In order to identify the reasons for those problems first I discussed the related literature on equipartitioning and learning trajectories to frame my study to address further revisions and refinements on particular levels of ELT. Second I made a search on theoretical perspectives about usage of manipulatives in mathematics education to make the mathematics in presented items accessible to younger students in my study. Third, I addressed the early roots of ratio reasoning and identified why ratio concept is a hard for students to learn from literature that can inform my study in terms of item development and clinical interview settings for covariation level of ELT.

As a result, this study explored the possible factors (item-stem formats, item types, item classes, manipulative usage) that affect the participants' performance and mathematical strategies at those levels to inform the ELT in terms of level descriptions for covariation and reallocation. Moreover, this study explored the reasons why younger participants perform poorly on those levels of the ELT.

CHAPTER 3

METHODOLOGY

This study was designed to analyze students' strategies and cognitive processes while dealing with reallocation and covariation items in clinical interviews. Although students' sometimes use the same and sometimes use different strategies for solving the same or parallel reallocation or covariation items, their cognitive processes, mental structures, or reasons may differ, which this study examined.

In this study, I conducted clinical interviews to gain insight into what the students were thinking as they solved reallocation and covariation items. I used them as not only a way to understand and examine how the students worked through a given task, but also as a device to assess the factors that might affect their strategies while doing so. Piaget (1975) pioneered clinical interview techniques "to study the form of knowledge structures and reasoning processes" (as cited in Clements, 2000, p. 547). Clements (2000) stated that this research evolved into several types of interviewing techniques including open-ended interviews and think-aloud procedures. According to Opper (1977) the main character of this methodology is that it constitutes hypotheses- testing situations, observing children performance on specific tasks. Clinical interview methods are more powerful than nonclinical methods because they "include the ability to collect and analyze data on mental processes at the level of a subject's authentic ideas and meanings, and to expose hidden structures and processes in the subject's thinking that could not be detected by less open-ended techniques" (Clements, 2000, p. 547). Clinical interviewing was a powerful technique to use in this study because I diagnosed students' reasoning, mental structures, and strategies used on the reallocation and covariation items as solved in a flexible environment. In a clinical setting, further probing and questioning provided a better understanding of how students' thought about the tasks and items. Additionally, the flexibility of introducing more than one task or item to a student helped the interviewer grasp the scope of their thought processes and clarify the "obscure points or inconsistencies [authenticity or stability] in . . . [students'] previous responses" (Clements, 2000, p. 547). The study had three main research questions and related conjectures for each. The clinical interviews were conducted to test these conjectures through analysis of video data of each interview.

Clinical interviews were helpful for the following reasons:

(a) To examine the cognitive processes students used to arrive at their answers. Clinical Interviews permit one to closely monitor and probe the students' cognitive processes while dealing with the assessment items, even if the students produced the identical correct answer on a given assessment item.

(b) To gain insight into which factors may have hindered younger students' abilities to solve paper-pencil assessment items. They helped identify students' cognitive abilities, misconceptions, or reading difficulties as they worked on the items.

(c) To address what reallocation concepts may mean for the ELT by closely examining students' strategies and cognitive processes on given assessment items through their videos, verbal and nonverbal explanations, and written work.

(d) To provide rich information about students' thought processes in solving the reallocation and covariation items on the paper-pencil field test.

This study examined the factors, item-stem formats, and manipulatives usage in tasks using the clinical interview method particularly for cases of reallocation items that may have affected students' performances and strategy selections. Eventually, this study will inform the DELTA teams' understanding of Levels 9 and 14 of the ELT. The main aim of this chapter is to describe this study's methods for assessment item creation and clinical interviews protocol, participant selection, and the scope of the research, data collection and the process used for the analysis of data collected from the students' responses.

Assessment Item Creation

To draw valid and reliable inferences from assessment scores, it is important to construct good assessment items. Constructing good assessment items is difficult because writing a precise and targeted item is challenging (Osterlind, 1998).

First, the DELTA research team reviewed each of the items at Levels 14 and 9 from previous field tests. Revisions were made on those items based on the analysis of previous clinical interviews. The wording and format of four of those items were changed. Researchers have found that the wording and format of an item can greatly influence examinee's psychological perspectives while considering a response (Wolf, 1995). After that revision, the team examined research frameworks for constructing assessment tasks that would elicit insights into children's strategies and knowledge.

Initially, the team developed 28 new reallocation items and 11 new covariation items. The emergent nature of the research design led to revisions and refinements of the existing items, and12 new items were designed to elicit specific strategies and responses at the reallocation and covariation levels of the ELT. Of the 40 reallocation items, 16 were selected for use in the clinical interviews as were six of the 12 covariation items.

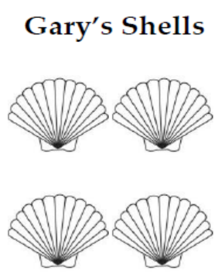
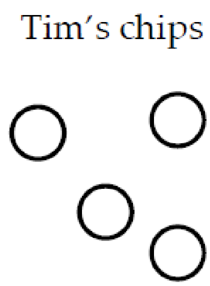
Item-Stem Format Types

In this study, there were two main item-stem format types for the newly developed reallocation items. The first format included a picture of the initial share of coins. For example, students were told that 12 coins were fairly shared among three pirates, and each pirate's fair share was shown in the picture below or above the item. The item stem showed each pirate's initial share: four coins. Then the task stipulated that one of the pirates left the group, and the students needed to find the new fair share for each remaining pirate. The fair shares were oriented in two formats. First, the fair share was represented in an organized way such as an array or stacked format. Then, the fair share was represented in an unorganized way such as randomly dispersed coins.

For the second format, the item stem did not include a picture of the initial share. For example, students were told that 12 shells were fairly shared among three friends, and the question stem did not include a picture of each friend's initial fair share: 4 shells. Instead, the question gave an explicit verbal description of each friend's share. Table 2 shows an example of the newly developed reallocation items for each item-stem format type. The examples in the table represent a single person's share as given in the verbal descriptions of either 12 shells fairly shared among three friends, with each friend receiving four shells in their fair shares, or 12 chips shared among four friends and "Tim" getting four chips.

Table 2

Examples of Item-Stem Format Types for Reallocation Items

ITEM-STEM FORMAT		
With Picture of Initial Share		**Without Picture of Initial Share**
Organized Layout	***Unorganized Layout***	
		. . . The three friends shared the shells/chips so each one of them got four shells . . .

There were two main item stem format types for the newly developed covariation items. The first format included a table representation of the collection amount shared and the amount each receiver had. The items that included table representation included one pair of values number of items/objects and number of receivers. Moreover, the ratio unit was not explicitly stated in the item stem. The second format did not include a table representation. Table 3 shows an example of the newly developed covariation items for each item-stem format type.

Table 3

Example of Item-Stem Format Types for Covariation Items

<table>
<tr><th colspan="2">ITEM-STEM FORMAT</th></tr>
<tr><th>With Table</th><th>Without Table</th></tr>
<tr><td>Matt knows that 6 carrots will feed 4 rabbits if they are shared fairly. Predict the number of carrots needed for each number of rabbits listed in the table, so each rabbit will get the same share of carrots. Explain how you figure out your answer.
<table><tr><th>Number of rabbits</th><th>Number of carrots</th></tr><tr><td>2</td><td></td></tr><tr><td>4</td><td>8</td></tr><tr><td>8</td><td></td></tr></table></td><td>It was time for breakfast at Gracie’s slumber party. Gracie knew that one box of cereal will share fairly among the three of them. As they were about to eat, six more friends came over. How many boxes of cereal will she need so that everyone gets the same fair share?</td></tr>
</table>

Item Types

We used three item types for the newly developed reallocation items: open-ended, structured, and multiple methods items. One item type was used for the covariation items: Open ended. Open ended items were presented to see directly students strategies. Multiple method items were presented to see whether students can produce both reallocation and collection strategy and structured items were presented to see whether students made sense of both described reallocation and collection strategy.

Open-ended items give enough space in an item stem for students to show their strategies for creating fair shares. Structured items ask students to pick one strategy, described in the question stem, that they think will yield fair shares and be more effective. A structured example might be of five friends who fairly share 40 marbles, and one named Alec wants to leave the group. After Alec leaves the group, the question stem describes two methods for fairly sharing the existing marbles among the remaining friends. The first method might be that Danny says they should put all of the marbles back in a pile in the middle and share them evenly again. The second method might be that Tyson says they should all keep the marbles they already have and just share Alec's marbles amongst them. The students selected one of the methods and justified their selections as to why they believed it yielded fair shares and why it was an effective strategy. The multiple method items asked students to show two different strategies for creating new fair shares.

Additionally, all these item types included justify and construct items. *Justify* items included space on the item page so participants could answer the item and explain their reasoning and answers adequately. For instance, they could explain why their reallocation strategy yielded a fair share when a person came into an existing group. *Construct* items asked the students to simply construct fair shares after one person arrived or left or to construct fair shares from existing, unequal shares among the existing number of people within a group.

Item Classes

As previously discussed, this study examined different situations in which students might use reallocation strategies to generate fair shares. The reallocation items developed after the first field test reflected such situations for three allocation item class types as shown in Table 4.

Table 4

Reallocation Item Classes

Reallocation departure task	Start with initial fair shares, additional people depart, then ask for the generation of new fair shares.
Reallocation arrival task	Start with initial fair shares, additional people arrive, then ask for the generation of new fair shares.
Reallocation uneven shares	Start with initial unfair shares, then ask for the generation of fair shares among the existing people.

Participants and Scope of the Research

Repeated cycles of interviews with 23 participants were conducted: one kindergartener, two first graders, three second graders, two third graders, two fourth graders, two fifth graders[5], seven sixth graders, and four seventh graders. Fifteen of the participants were from public schools in North Carolina, and eight of them were students at a private school.

Initially, the sample was purposely selected from both middle school and elementary school grade levels. Table 5 shows the distribution of the participants in the first and second set of interviews. In my study, initially I conducted first set of clinical interviews with both reallocation and covariation items. Based on the analysis of the items, I decided to conduct a second sets of interviews with just reallocation items. In the second set of interviews, I worked with participants from K-2. Participants from this grade band worked with manipulative materials as they worked on the presented reallocation items since in the first set of interviews with paper pencil reallocation items typically they could not solve the problems. Manipulative usage might make the mathematics accessible to these younger participants. I also worked with participants from grades four and five. Participants from this grade band presented with just reallocation multiple method and structured items. The reason for this selection was I would like to see whether students in transition to early grades to middle grades can produce both reallocation and collection strategy on the presented items.

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[5] One of the 5th graders did not participate in the interview. I asked the reason to his teacher and his teacher said "before this interview, he had two tutoring session for mathematics and he might be bored". As a result, I reported findings related to one 5th grade student clinical interview data.

Table 5

Distribution of the Participants in the 1st and 2nd Set of Interviews [6]

	6th Grade		**7th Grade**	
	High Achiever (Pre-Algebra)	Regular	High Achiever (Algebra)	Regular
Number of Students (1st Set of Interviews)	3	4	-	4
	Kindergarten	**1st Grade**	**3rd Grade**	**4th Grade**
	High Achiever	Regular	High Achiever	Regular
Number of Students (1st Set of Interviews)	1	1	1	1
	Kindergarten	**1st Grade**	**2nd Grade**	**3rd Grade**
Number of Students (2nd Set of Interviews)	1	1	3	1
	4th Grade	**5th Grade**	**6th Grade**	**7th Grade**
Number of Students (2nd Set of Interviews)	1	1	-	-

[6] In the first sets of interviews participants presented with both covariation and reallocation items. In the second sets of participants presented with just reallocation items. The same Kindergarten students participated in both first and second set of interviews.

Three high achievers were selected in sixth grade pre-algebra, four regular achievers from the sixth grade, and four regular achievers from the seventh grade to see whether their levels of knowledge and achievement affected their strategies while dealing with reallocation and covariation items. At the same time, one kindergartner, one first grader, one third grader, and one fourth grader were interviews to determine if younger students' mathematical solution strategies for the reallocation and covariation differed from older students' strategies. In addition, the interviews aimed to identify the differences between younger (K–4) and older students' (5–7) abilities to solve both types of item. The reason for choosing these numbers and grade levels of participants was to answer the first and third research questions:

> *Research Question 1:* What are the observed differences and similarities among participants' strategies in earlier grades (K–5) and higher grades (6–7) while solving reallocation items and sharing collections items based on evidence from clinical interviews? Is there any systematic difference observed among participants' strategies at different grade levels when solving reallocation items? If yes, what might be the reasons for these differences?
>
> *Research Question 3*: What are participants' performance levels while solving covariation items? Is there any systematic difference observed among participants' performance at different grade levels while dealing with covariation items?

Based on the preliminary analysis of these interviews and my increased understanding of the possibilities that arose from the emerging nature of a research design, the new research questions were added and related sampling methods used more *theoretical sampling* (Glaser & Strauss, 1967). In this sampling technique, the “selection of participants is directed by the emerging analysis, and the theory being developed from data is subsequently modified by data obtained from the next participants” (Richard, 2006, p. 76). The emerging analysis from the first data sets from the clinical interviews also led to search for additional younger participants. As a result, the scope of the research was expanded and younger participants were recruited after the first sets of interviews were completed. Based on the results of previous interviews, the item stem was affecting students’ strategies, and some of younger children were unable to solve the paper-pencil reallocation item. So a second set of interviews were conducted with one kindergartener (the same participant as in the first set), one first grader, three second graders, one third grader, one fourth grader, and two fifth graders with just the reallocation items.

For grade levels K–2 students used manipulative materials while dealing with the reallocation items, but in grade levels 4–5, students did not use manipulative materials in the second sets of interviews. As a result, two phases of research were undertaken that affected the sample selection and research questions. First, a new research question was added to address *shadowed data* that provided “negative cases that are contrary to the emerging theory but not yet encountered” (Morse, 2001, p. 291) and the first research question was modified . New modified research question was:

> *Research Question 1:* What are the observed differences and similarities among participants’ strategies in earlier grades (K–5) and higher grades (6–7) while solving reallocation items and sharing collections items based on evidence from clinical

interviews? Is there any systematic difference observed among participants' strategies at different grade levels when solving reallocation items? If yes, what might be the reasons for these differences? (Item class type, item-stem format, item types)

Research Question 2: What is the effect of manipulative use on younger participants' (grades K–2) strategies while solving reallocation items? Is there any difference between participants' abilities to solve reallocation items when they do and do not have manipulatives?

These shadowed data led the research to another direction for theoretical sampling that is addressed in-depth in Chapter 4. More elementary school participants were selected and worked with with and without manipulatives for use in solving the items in the interviews.

Data Collection

To gain rich aural and visual data about the participants' strategies for solving reallocation and covariation items, video recordings of each clinical interview and the participants' written work were taken and transcribed. Each of the participants and their families were informed of interviews, what it meant to be videotaped, and the intended usage of these recordings. All 24 participants and their parents gave their consent to be videotaped. In the consent forms, "progressive levels of consent" (Roschelle, 2000, p. 726) were specified. The participants and their families consented to their data being used for "small research group use" and "scientific conferences and meetings" (Roschelle, 2000, p. 726).

Video recording was used for this study for observation purposes only, because only a limited amount of information during the each interview session could be captured. Without a recording, the observational data from the each sessions with each participant could not be reexamined. Through employing video as a data source, a detailed and fascinating description (e.g., their thought processes, strategies, difficulties, etc.) of the participants in the clinical interview setting while completing the mathematical tasks (Powell, Francisco, & Maher, 2003) could be given. These advantageous benefits of video data helped detail answers to the research questions to emerge.

Although video recorded data have unique riches and advantages, Bottorff (1994, as cited in Powell et al., 2003) listed three limitations of video data: it’s “capable of selectivity because of mechanical limitations; incapable of discerning subjective content of behavior being recorded; and usually unable to convey historical context of captured behavior” (p. 408). To overcome these limitations in this study, appropriate technological tools for videotaping were selected and systematically recorded. In addition to this systematic coding and recording of each video, attention was paid to the recording quality and learned competent videography techniques before starting the recording of the clinical interviews (Roschelle, 2000).

The clinical interviews were conducted one student at a time and were not timed. Video cameras were set up to capture each interview session for each student. For most of the videotaping, the camera was set up to capture both the students’ actions and their written work. The students were encouraged to speak openly, and the interviewer asked questions to illuminate their thought processes during the interviews. At the beginning of each interview, students were warmed up with questions that helped them to render their thoughts and feel comfortable verbalizing them during the rest of the interview session. Some of the questions were (a) How was your day in school today? and (b) Which course do you like most in the school? After establishing a rapport with the students, the interviewer introduced the task (Opper, 1977).

During the interviews, each student worked on each presented task in sequence. In the first set of clinical interviews, students were presented with six to ten items in a booklet. The booklets included six to seven reallocation items and three to four covariation items, though kindergarten and first-grade participants did not receive the covariation items. Each participant tried to solve the same three to five linking items, at least, and the rest of the items were parallel in terms of what was assessed.

At the beginning of each interview session, the conjectures guided the interviewer on how students' might process a given item, what kind of strategies they might use to solve it, and which factors might affect their solution strategies and mathematical reasoning. For each item, the interviewer asked questions that promoted insight into the students' thought processes, verbalizing their thoughts on the tasks. These questions also helped them to predict, observe, explain and justify their actions. Some of the eliciting questions used were (a) How do you know that your action creates fair shares? (b) What is the question asking for? (c) Can you explain to me what you are thinking? (d) Is there any other way to solve this problem? and (e) Why did you select this strategy instead of the other one?

The verbalization of thoughts was one of the most important data sources for understanding the students' thought processes. The eliciting questions helped one to infer students' "underlying mental processes" (Opper, 1977, p. 93) to understand "students' statements or judgments on each item" (p. 93). The other data sources for the students' mental processes were their actions, written work, and their manipulation of concrete materials.

In two interviews with a first-grade participant during the second set of interviews and one with a third-grade student during the first set of interviews, neither student wanted to go further on a given task and did not verbalize their thoughts. In those situations, the interviewer completed the interview and asked the participants if they wanted to work again another time without making them feel uncomfortable. In an interview with one sixth-grade participant and one seventh-grade participant during the first set of interviews, the video charge ran out so only the first 10 to 15 minutes of the interviews were recorded. For data analysis purposes, only the written work of these participants was used.

Data Analysis Method

Bottorff (1994) stated that video recording allows the researcher to reexamine the data as necessary. In addition, the nature of the video gives the researcher a flexible environment to play the recorded data such as in “real time, slow motion, frame by frame, forward and backward” (Bottotff, 1994, p. 246). This capability allows one to focus on the different and similar aspects of the participants’ behaviors through multiple viewing with other colleagues and at different times.

Multiple viewings of video data can serve as a means for triangulation in qualitative data analysis to strengthen the validity and reliability of the research. Mathison (1988) elaborated on the value of triangulation:

> Triangulation has risen [as] an important methodological issue in naturalistic and qualitative approaches to evaluation [to] control bias and establish valid propositions

because traditional scientific techniques are incompatible with this alternate epistemology. (p. 13)

In this study, repeated viewing of the video data and multiple viewing of the data with DELTA team "has potential to enhance the triangulation in data analysis" (Powell et al., 2003, p. 410). Each recorded interview was viewed twice. During the first viewing, I watched the procedures carefully. During the second viewing, I tried to identify the critical events and moments of each video. During this process, I revisited parts of the videos and watched them several times. In addition, shared viewing of the videos with the DELTA team enhanced the validity of my qualitative analysis of the video data.

According to Powell et al. (2003), the clear criteria for using video to capture data can be described through Erickson's (1992) perspective on the ethnographic analysis of video. Erickson posited that it is important to capture nonverbal interactions and identify accurate speech information. The video data allowed the examination of those factors moment by moment. To gain insight into both the explicit and implicit meanings of the participants' behaviors, responses, and strategies while solving the reallocation and covariation items, I used the last five levels of Powell et al.'s (2003) suggested analytical model: 1) Identifying critical events, 2) transcribing [necessary sections and constructing video clips], 3) coding, 4) constructing storyline, and 5) composing narrative (p. 413).

I started my data analysis from identifying critical events. Through the process of conducting each interview, a preliminary analysis of each interview made me familiar with their content. Multiple viewings of each video recording helped me revisit the first two levels of the model: “viewing attentively the video data” and “describing the video data” (Powell, et al., 2003, p. 154). Through following the model, I defined four critical events:

1. Changes in participants’ solution strategies in relation to item-task classes
2. Changes in participants’ mathematical strategies in relation to item attributes
3. Differences between younger and older students’ solution strategies and problem solving performances
4. Negative instances that are not confirmed my initial conjectures

Coding was one of the most important steps in my data analysis; it facilitated the interpretation of the video data. The coding schema emerged through the repeated viewing of each interview video and through shared viewing with DELTA team members to enhance my ability to formulate codes. In addition, my coding was directed by identified critical events and the research questions of this study. Table 6 represents my coding schema and explains them by using this item as an example:

> Gary, Nina, and Vera have all collected 12 shells of the same size and shape to use for a science project. The three friends shared the shells so each one got 4 shells. Mitch joins the group. After Mitch has joined the group, how can all of the children still end up with fair shares, using all the shells?

Table 6

Coding Schemas of the Study

Codes	Example
Mathematical Strategies/ Mathematical Behaviors	The participant used a reallocation strategy to generate new fair shares. She took away 1 shell from each person's share and gave them to Mitch. So everybody received three shells. The participant used a collection and then division strategy. She gathered all the collection back together and divided by four. So everybody received four shells. The participant used a collection and fair sharing strategy. She gathered all the collection back together and shared among four. So everybody received four shells
Mathematical Justifications/ Reasoning	The participant justified her new fair shares by counting each share.
Mathematical Representations	The participant used manipulatives to model the situation in two steps: 1) Modeled initial share for each person: Four shells per person. 2) Generated new fair shares by taking one shell from each person's share and giving them to Mitch. She mainly used reallocation strategies.

As my data analysis progressed to the level of forming a story line, I reexamined my research questions. The overall story line of the study describes how I tried to discover inductively what strategies the participants used, what their cognitive processes were while solving the reallocation and covariation items, and which factors affected their strategies as seen by grade bands: K–2 , 3–4, and 5–7. Then it describes how I moved into deductively verifying the observed differences and similarities in the participants' responses by grade band.

During the process of identifying critical events and codes, I sorted out some video clips from each interview that exemplified those critical events and codes. For reporting purposes in the findings of this thesis, I then transcribed some video clips and composed narratives about my interpretation of the data from them.

CHAPTER 4

FINDINGS

This chapter presents the findings from the analyses of video data regarding the primarily qualitative research questions.

The first section will report on the findings of the first and second research questions based on the analysis of the participants' responses to the reallocation items, viewing the data from the perspective of three grade bands: K–2, 3-4, and 5-7. The participants were divided into these grade bands because the participants grouped in each exhibited similar behaviors and strategies. The second section of this chapter will report on the results from the analysis of the participants' responses to the covariation items in the last research question.

Analysis of Participants' Responses to Reallocation Items

This section will report on findings by breaking them down into three grade bands: K–2, 3–4, and 5–7. These grade bands were selected for the following reasons:

(a) In each grade band, the students exhibited similar behaviors as they worked on the reallocation items.

(b) Only participants in Grade Band K–2 were presented with manipulative materials as they worked on the reallocation items.

(c) Grade Band 3–4 is a transitional stage that gave insight into the formal instruction effect on participants' strategies as they worked on the reallocation problems.

(d) Grade Band 5–7 informed the study in terms of older participants'' strategy selection and abilities as they worked on the reallocation problems

Following section reports the findings from the first and second sets of interviews for each grade band ending with a comparison across the grade bands and a discussion of the overall results.

Grade Band K–2

This section introduces findings related to the effects of manipulatives usage on the participants' strategies as they worked on the reallocation items. It also examines the participants' abilities to solve the items by comparing the results from the first sets of interviews, in which the participants were not presented with manipulatives, with the second sets of interviews, in which they were.

In the first set of interviews, I worked with one kindergarten-aged and one first-grade-aged participant. They solved the paper-pencil assessment items without using manipulative materials. Either of the two participants could solve the reallocation items. San, the kindergartener, was presented with this item: Four pirates fairly shared 12 coins and each pirate got 3 coins. Then one of the pirates needed to go home and left his coins to the other pirates. How could the other pirates share his coins and still end up with fair shares?

San first tried to fairly share 12 coins among four pirates but she could not. Then I read the question to her again.

Interviewer: How many coins does each pirate have initially?

San: I do not know.

Interviewer: What happens in the question?

San: (silence)

Interviewer: What happens when one of the pirates leaves the group?

San: (silence)

Interviewer: What happens when this pirate [who left the group] gave his coins to his friends?

San: (silence) I don’t know. (She tried to draw coins but could not model the situation given in the item; she drew 3 coins and four pirates.)

The other students were either not interested in the problem-solving process or could not solve the paper-pencil assessment item. This lack of responsiveness led me to work with younger students again in a second set of interviews, but to provide concrete materials so that they could manipulate to model the situation. In the second set of interview, I worked with the same kindergartener, one first grader, and three second graders. In each interview, the students were allowed to use manipulatives as they worked on the items. San was presented with same item as related previously. Because her reading ability was weak, I read the question to her step by step. First, I read the first part of the problem: Four pirates fairly shared 12 coins and each pirate received three coins. She took 12 coins and fairly shared them among the four pirates. She represented each pirate with a pencil. She then formed four groups of coins and each group included three coins. Then, I read the second part of the question: Then one of the pirates needed to go home and left his coins to the other pirates. San took away one of the pencils and the three coins in its group. I read the last part of the question: How could the other pirates share his coins and still end up with fair shares? She gave one coin to each remaining pirate. Then the interaction continued as follows:

Interviewer: How many coins does each pirate have now?

San: (counted the number of coins in each group) four coins.

Interviewer: Do they have a fair share or not?

San: Yes.

Interviewer: How do you know they have a fair share?

San: (counted the number of coins in each group again) Since they all have four coins.

The same participant was presented with different reallocation items that included the following item cases: a reallocation departure task, a reallocation arrival task and a reallocating uneven shares task. For the reallocation departure task, which was the pirate item already discussed, she used a reallocation strategy as she solved the item. For the reallocation arrival task, I said, "Three pirates fairly shared 15 coins and each got five coins." She then took 15 coins and fairly shared them among three pirates by using composite units: She gave two coins at a time to each pirate and then dealt out single coins. She formed three stacks of five coins. Then I continued to the second step of the item, "Two more pirates want to join the group and want to have fair shares. How will they all still end up with fair shares and use all the coins?" First, she represented two additional pirates with crayons and then took 1 coin from the existing pirates' shares and gave them to each new pirate until new fair shares were created.

San did not use the reallocation strategy for the reallocating uneven shares task. In this item case, she used a fair-sharing collection strategy: dealing by ones. She put the collection of coins back together and gave one coin to each receiver (systematically dealing by ones) until she exhausted the collection.

The first-grade participant, Heidi, was presented with six reallocation items including three reallocation departure tasks, one reallocation arrival task, and two reallocation uneven shares tasks. For the departure reallocation items, presented with or without pictorial representations, she used a reallocation strategy to generate new fair shares. Even if a picture was included in the departure reallocation item, she also chose to use manipulative materials. For the arrival reallocation items, she used manipulative materials to support her application of a reallocation strategy. For example, she was given the shells item that three children shared 12 shells fairly. The item stem presented a picture of each friend's share in an array format (see Table 2 in Chapter 3 for an example of the array format). The item then read that one more child joined the group and asked how all the children could still end up with fair shares, using all the shells. Our conversation follows:

Interviewer: (Read question aloud to Heidi) So, what happens?

Heidi: There is another friend coming. (She started to take the manipulative from the box.) Each one of them have four. (She formed three groups and each contains four coins, as in Figure 15, Picture A.) Then another friend came.

Interviewer: So how can all have . . . ?

Heidi: Same amount. (She took 1 coin from each group.) Each friend can give him 1. (She formed four groups and each contained four coins as in Figure 15, Picture B.)

Interviewer: When you give one coin from each friend—(Heidi started to talk.)

Heidi: They can have same amount.

Picture A

Picture B

Figure 8. Heidi forms three groups of four coins(15A), takes one coin from each group, and her models the final four groups of three coins (15B).

All the reallocation uneven shares item stems included either organized or unorganized layouts of the initial shares pictures. Heidi qualitatively described her strategy for solving them but could not quantify what she described by using a reallocation strategy. For the chips item, she was presented with pictures of four children's unequal shares and the information that "Peter has three chips, Mary has six chips, Tim has four chips and Sharon has seven chips." The question asked how the friends could fairly share the chips and how many chips each one would receive in their fair shares. Heidi's qualitative description of her mathematical strategy was "She has a lot of chips (pointed out Sharon's chips), and Mary will share these two together (pointed out Mary and Tim's chips) they share these two together (pointed out Peter and Sharon's chips). Then they can each grab a different chip."

When she took the manipulative materials, she formed two groups that she described qualitatively. Then she joined the groups. When I said that each friend would like to have the same amount of chips, she did not fairly share the whole collection among the four friends. Instead, she referred to the groups in her previous strategy, “they eating the same amount; they have the same amount.” Her strategy implicated both reallocation and collection strategy reasoning. Because she first determined two pairs of friends in each group, one of them had a lesser amount of chips and the other had a greater amount of chips. This partition implied a reallocation strategy of preserving the least amount of chips in each share and reallocating from the friends that had more chips to the share. But then she pointed out the two groups she formed from the initial share pictures on the item stem and said, “they eating the same amount; they have the same amount.” That might imply that her strategy was to deal collections systematically.

The second-grade participants—Berr, Senn, and Suda—were presented with items from each case and allowed to use manipulatives as they worked on the problems. Before they used the manipulative materials, I asked them to first try to solve the items without using manipulatives. Afterward, they could use manipulatives. Senn used manipulative materials and employed a reallocation strategy on a reallocation departure task, the crayon item, with no picture. It asked,

> Five children fairly shared 40 crayons, and each child had 8 crayons in their fair shares. Then one of the children, Becca, left the group. How many of Becca’s crayons should each remaining child get so everyone has a fair share of the crayons?

Initially, she thought for a while after I asked “How would you figure out your answer?” She could not figure out how to solve the problem. Then she used the manipulative materials and said, “Each child has eight crayons” right away. Then she represented each child’s initial share by forming five stacks that included eight manipulatives as shown in Figure 9.

Figure 9. Senn represents each child’s initial fair share.

Then Senn said, “Becca leaves the group.” She started to take manipulative materials from the first stack as shown in Figure 10.

Figure 10. Senn takes away Becca's share.

Then she fairly shared Becca's crayons among the remaining four friends by systematically dealing by ones and forming four stacks, each containing 10 manipulative materials. Figure 11 shows each child's final shares.

Figure 11. Senn's representations of the final shares after Becca has left the group.

The conversation between Senn and I continued as follows:

> Interviewer: Can you explain to me what you did right now? Becca leaves the group . . .
>
> Senn: I gave them two crayons to each person, so they can have the same.

Berr did not use manipulative materials if the item stem included a picture of the initial share or if she began by drawing a picture of the initial share for each child in array format. For the crayon item, she drew the initial share and fairly shared Becca's crayons among the remaining children by crossing 1 crayon out from her initial share drawing at a time.

The chips item was a reallocation uneven shares item and its stem included a picture of the initial unequal shares. Three of the second graders, Senn and Suda, could not solve the problem without using manipulatives. With them, they could reallocate from the share that included more chips to the shares that had fewer chips until they generated fair shares. Senn and Suda first tried reallocating chips by using the initial shares, but they could not track their reallocation moves. Then they used manipulatives to form each child's initial share and reallocated chips from Sharon's share to Peter's share and then from Mary's share to Tim's share. In each redistribution of chips move, they counted each child's share to make sure they created fair shares. Berr, rather, used the picture given in the item stem to employ a reallocation strategy until she generated fair shares. Her reallocation strategy was the same as Senn and Suda's, but she did not use the manipulative materials. Instead, she manipulated the chips based on the initial shares picture shown in Figure 12.

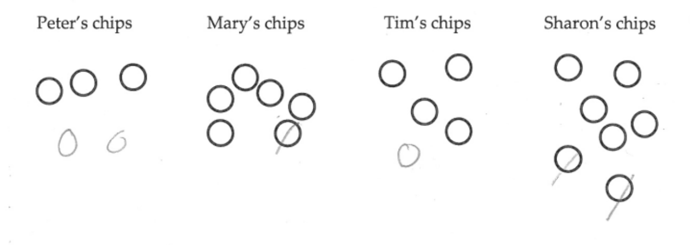


Figure 12. Berr's reallocation strategy to generate fair shares (Chips item)

Table 7 shows the overall response frequency for Grade Band K–2 with respect to each strategy (i.e., reallocation or collection) used on the reallocation items accounting for different attributes such as item stems with or without a picture and the participant's preferences to use manipulatives or not.

Table 7

Grades K–2 Responses to Reallocation Items by Successful Strategy and Item Attributes

	Reallocation Strategy				**Collections Strategy**			
	Manipulatives Available		*Manipulatives Not Available*		*Manipulatives Available*		*Manipulatives Not Available*	
Task Classes	*With Picture*	*Without Picture*	*With Picture*	*Without Picture*	*With Picture*	*Without Picture*	*With Picture*	*Without Picture*
Reallocation departure task	3	6	4	1[a]	1			
Reallocation arrival task	2	4	1	1[b]	2			1
Reallocation uneven shares	3		1					

Note. [a]Berr drew an initial share picture of the item description when the item did not include a pictorial representation of the initial shares. [b]Senn drew an initial share picture of the item description when the item did not include a pictorial representation of the initial shares.

Table 7 shows that, overall, across all three item class types, 26 of the responses were examples of reallocation and four of the responses were examples of putting all the collection back into one collection and fairly sharing it among new number of the people. This brings the questions what might be the possible reason for this difference in participants' strategies? One can also see from the table that participants were able to solve the problems either with pictures or with manipulatives or with access to both. There were only two successful reallocation strategy with no pictures and no manipulatives [7]across all three item class types items where as this number increased to ten if manipulatives were available in the no picture condition. In the picture condition, overall provision of manipulatives produces the same level of success in using reallocation strategy for the departure task but for uneven shares task, more students were successful with manipulatives and the pictures than with the picture alone. For arrival task, students were more successful while employing reallocation strategy with manipulatives than without manipulatives.

Grade Band 3–4

The first set of interviews involved one third-grade and one fourth-grade participant solving paper-pencil assessment items without using manipulative materials. the second set of interviews included one third-grade and one fourth-grade participant. The third grader used manipulative materials as she engaged with the reallocation items so findings will not be presented from her data. Since, in this grade band my aim was to see whether there is a transition phase between concrete representations (K-2) to formal representation of mathematics (5-7). The fourth grader received structured and multiple method item. In structured items she was presented with the items that included both description of reallocation and collection strategy and in multiple method items she was asked to produce two solution ways to the presented item: collection and reallocation strategy.

During the first set of interviews, the third grader did not respond to the reallocation items. The fourth grader Nina used a reallocation strategy when she was presented with four reallocation departure task items. She also used a reallocation strategy on a reallocation

[7] Although item stem did not include picture of initial share, students draw their own picture of initial shares as they worked out their answers.

arrival task (shell item, 3 to 4 sharers). She used straight division facts as she worked on the two reallocation uneven shares items. While solving the reallocation uneven shares item (chips item, 5 to 4 receivers) she found the total number of the chips by adding each child's share together ($3+6+4+7=20$). Then she used division facts to find the fair shares for each child. She divided 20 chips by the number of the children, which was four ($20 \div 4 = 5$). She stated her final answer as "Everybody gets five chips." This solution process revealed that Nina did not use a collection strategy of dealing by ones or composite units. She used division operation to find each child's share and did not use a reallocation strategy, either.

In the second set of interviews, the fourth-grade student, Aise, was given regular, multiple methods, and structured reallocation items. She was given one structured reallocation departure task item, and she preferred to use the reallocation method to generate the new share, which was "Tyson says that they should keep all the marbles they already have and just share Alec's marbles amongst them." When I asked the participant why she did not pick the other method, which was "Danny says they should put all of the marbles back in a pile in the middle and share them evenly again," she said, "It is just gonna be hard if you put them all back together again and split them back out. You could just like divide them." From this statement, I inferred that she used the idea of "split them back out" to mean what I have coded as a "collection" strategy preferring to use division operations if she were not to select the first method of reallocation. For the rest of the multiple methods items, unpictured, reallocation departure task items, she employed reallocation strategies as her first method. For her second method, she described the division facts but could not carry out division algorithm, so first she estimated the nearest amount of the collection by building up from known facts and then she gave the amount of items in her composite units to each remaining receiver. Second she found the leftover pieces and dealt them to the remaining people. The strategy she used will be examined in more detail in an upcoming section.

For the reallocation-arrival-task, multiple-method items, I presented her with two items that had the same question stem. One of them, however, included a picture of initial share and the other did not. The first item was

> Alex is going to have a sleepover with five of his friends. His mother gave him six bags of peanut butter candies. Each boy received a bag that contained seven peanut butter candies. One of his friends, Mark, brought his younger brother with him to the sleepover unexpectedly. Show or describe two different ways the boys could share their candy with Mark's brother and still receive a fair share of all the peanut butter candies. What is the number of peanut butter candies in each person's fair share? (Peanut Butter Candies Item)

First, Aise was presented with the item stem that did not contain a picture. After she worked on some other items, I gave her the same item that did not include picture of the initial share. I did not give those items successively because I did not want her to recall her previous answer to the same question. For the first one, she tried to produce two different strategies but could not complete the division operation with straight calculations to determine that each friend's share as six ($42 \div 7 = 6$). Instead, first she determined that the total number of the people was seven by adding Mark's little brother to the count. Then she calculated up to total number of the peanut butter candies by adding $7+7+7+7+7+7=42$. She then described how she used division facts to find each friend's share of the 42 candies, "You need to split them between seven people." She wrote 42 divided by seven but still could not produce the answer. She found another way to solve the problem by building up from known facts and stated, "We may first do five for each" and gave five candies to each person. Then she determined that the number of leftover candies was seven by subtracting 35 candies from 42 candies. Then she fairly shared the remaining seven candies among the seven friends by dealing by ones. At the end of the solution process, she explained her strategy as "I just want to estimate, and if there is leftover, I just add." Figure 13 shows her pictorial representation of her solution strategy.

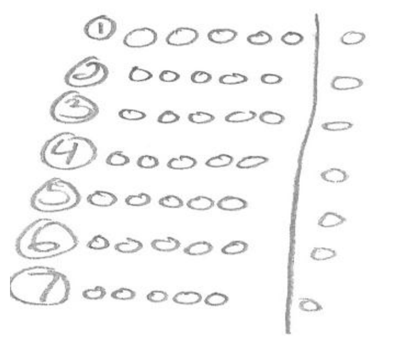


Figure 13. Fourth grader Aise's estimation and collection strategy.

For the same reallocation arrival item (peanut butter candies item, 6 to 7 receivers) that included picture of initial shares in the item-stem. Aise used both division operations and reallocation to find each child's fair share. Our conservation described how she used both methods:

Interviewer: What is the question asking for?

Aise: They want you to divide (She started to count the peanut butter candies in a bag) six candy bags and seven pieces of candy for each of them into another candy bag for Mark's little brother . . . So you could do . . . (silence as she counted all the candies represented in the picture) all these equal 42, and you need to divide 42 into seven—(used her fingers to figure out the answer)—six. Each boy will have six pieces of peanut butter candy, including Mark's little brother.

Interviewer: Is there any other way that you solve this problem?

Aise: So you could do first, you could take one out of each bag. (She drew 1 candy near to each bag.) Each of them has six left and one, two, three, four, five, six (points out each candy taken from the existing six bags). There is six candy bags, one piece of candy came out of each candy bag. Those 6 leftover, you could like make your own candy bag into six pieces. (She drew a new candy bag with six candies in it).

With this solution she was demonstrated that she is able to use both a division strategy-aided by way to solve that division problem for unknown facts and a reallocation strategy.

For the reallocation uneven shares item without a picture, Aise employed two strategies as she did with the reallocation arrival task items. First, her estimation and dealing strategy was evident when she drew a picture. Then she used division facts. For the same reallocation uneven shares item with a picture, she used her first strategy of picture drawing.

For her second strategy, she employed a reallocation strategy by establishing the least amount in each share and distributing the rest of the items from the share that had more items to the share that had the fewest items. The presented item stem included the description of each child's initial share: seven friends as one child had nine, two children had seven, three children had five and one child had four candies. Initially, she gave four candies to each child, and then she distributed the remaining candies among seven children and found the each child's share as six candies.

Grade Band 5–7

For the first set of interviews, I worked with seven sixth-grade and four seventh-grade participants. They did paper-pencil assessment items without using manipulative materials. During the second set of interviews, I worked with two fifth-grade students. They received structured and multiple method items.

In the first sets of interviews, six out of seven 6th grade students used collection strategy and division facts as they were working three different reallocation arrival tasks. Two of those questions were used with 1st, 3rd and 4th graders. For instance, the shell item (12 shells initially share among three friends than one more friend joins the group) was a linking item that 1st grade Heidi and 4th grade Nina employed reallocation strategy. On the other hand, 6th grade student Andre used collection strategy to solve the problem. He first gathered all collection of shells together and then used division to find each friend's share (12÷4= 3). Moreover, he stated that "this is commutative property of multiplication since three times four is 12 and one person joins to group four times three is 12". For reallocation departure task, 12 out of 19 6th graders response declared a reallocation strategy instead of using division facts or a collection strategy. Table 8 shows the frequency of students' responses falling under each strategy with respect to different item attributes; for given item class types.

Table 8

Grade 6 Responses to Reallocation Items by Successful Strategy and Item Attributes

	FREQUENCY of USED STRATEGIES					
	Reallocation			Collections (Usage of Multiplication and Division Facts)		
Item attributes → ↓	*With Picture*		*Without Picture*	*With Picture*		*Without Picture*
	Organized Array	*Randomly Distributed*		*Organized Array*	*Randomly Distributed*	
Reallocation departure task	10		3	1		6
Reallocation arrival task				4		4
Reallocation uneven shares				4	3	2

Table 8 shows that, overall, across all three item class types items, 13 of the responses were examples of reallocation and 26 of the responses were examples of putting all the collection back into one collection and using division facts to find the each person's new shares. This suggests that 6th graders mainly used division facts to figure out new shares except reallocation departure task class type items. All 13 responses were examples of reallocation strategy in the case of departure. One can also see from the table that participants were able to solve the problems either with pictures or without pictures. Overall provision of both with picture and without picture condition produced same level of success in using collection strategy: 12 responses for each. It is evident that no matter whether picture in item stem organized or randomly distributed 6th graders used collection strategy in reallocation arrival and uneven shares items.

When I examined the 7th grade students' solution strategies, I found following

1. They usually employed reallocation strategies when presented with a reallocation departure task
2. They usually employed collection strategies for reallocation remediating uneven shares task.
3. They either used reallocation or collection strategies for reallocation arrival tasks where the item included the picture of initial share in array format.

Table 9 shows the 7th graders answer frequency distribution to each reallocation items.

Table 9

Grade 7 Responses to Reallocation Items by Successful Strategy and Item Attributes

	FREQUENCY of USED STRATEGIES					
Item → **attributes** ↓	Reallocation			Collections (Usage of Multiplication and Division Facts)		
	With Picture		*Without Picture*	*With Picture*		*Without Picture*
	Organized Array	*Randomly Distributed*		*Organized Array*	*Randomly Distributed*	
Reallocation departure task	7	1	1			1
Reallocation arrival task	1			2		
Reallocation uneven shares	1/1	1		6/1	1	1

This table does not include the incorrect answer of the students. Since only one seven grade produced an incorrect answer for a reallocation uneven shares items and one sixth grade student produced an incorrect answer for a reallocation arrival item. In their responses both of the students stated the correct division fact but could not produce the correct results to find the new far shares. Table 9 shows that, overall, across all three item class types items, 12 of the responses were examples of reallocation and 11 of the responses were examples of putting all the collection back into one collection and using division facts to find the each person's new shares. Nine out of 12 responses were examples of reallocation strategy in the case of departure. This suggests that 7th graders typically used division facts to figure out new shares except reallocation departure task class type items. One can also see from the table that participants were able to solve the problems either with pictures or without pictures. It is evident that no matter whether picture in item stem organized or randomly distributed 7th graders typically used collection strategy in reallocation arrival and uneven shares items. For reallocation arrival task in two out of three responses they used division facts and for reallocation uneven shares task in eight out of ten responses, they used division facts. The red color coded response indicated that one 7th grade student Kay produced two different strategies as she worked on following item.

> Ted, Betty, Hank, Pat, Mike and Ellen have all collected shells of the same size and shape to use for a science project. The six friends each collected the number of shells shown in the problem below. Use the diagram above to show how each child can receive a fair share of the shells. How many shells will each child's fair share contain? Explain your answer.

The diagram can be seen in Figure 14. In this diagram each children's share represented in an array format which also helped participant, Kay, to work on the given diagram and visually compare the each share after each redistribution move she made.

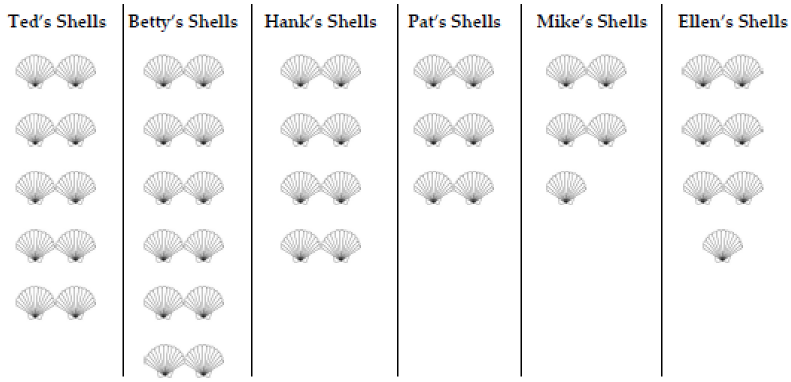


Figure 14. Initial allocation of shells.

After she read the question, she described her solution strategies as follows,

> Kay: Since they each did not get the same amount of shells collected. What I am gonna have to do, combine all the shells into one big total of the shells and divide that how many friends there are which I can see six. So I would just count them up and add them is for me the waste time, counting is easier. [Then she counted by ones to find the total amount of shells as 48 shells] So 48 shells total and then since there is six friends, I divide that by six, they all get even number of shells. And that equals eight. So the answer will be eight
>
> Interviewer: Can you use the picture to show how everybody get eight? What will be the best way to do that?

Kay: So, since Ted has [Started to count Ted's shells by ones] 10 shells. [she counted each children's share separately and wrote how many shells they had below the picture of each children's shares] One of the way you could do it, this takes a lot longer, is distribute the one that have more [pointed the Betty's share with her pencil] to the one that have less, eventually come out the same amount.

Interviewer: What you mean by that?

Kay: So, say this has 12 [pointed Betty's share] and this has five [pointed Mike's share] You can distribute these shells [pointed extra shells in Betty's share] to the lowest number [pointed Mike's share, five shells]. So you all get same number. Since, [she paused] I am not going to use my previous answer [pointed her answer eight shells from her previous solution]. What I would do was, I would take three of these [from Betty's share] and give to this [Mike's share]. [Then she crossed out three shells from Betty's share]. When you subtract three from twelve you will get nine [wrote this under Betty' share] so there is nine shells left there. But that is still more than eight. So I see this has seven [pointed Ellen's share] I subtract one from there [Betty's share, 9-1=8 shells] and then add one to that [Ellen's share, 7+1=8 shells]. So there is eight in this and this [pointed Mike's and Ellen's share respectively than pointed Betty's share and Hank's share which already had eight shells in it]. So I had ten and six [pointed Ted's and Pat's shares respectively]. And I know six plus two eight and ten minus two eight. So I am gonna subtract two of these [crossed out two shells from Ted's share] and add to that [drew two shells into Pat's share]. So you will end up with eight.

In all across the interviews, this was the only response; a student explicitly used the term "distribution" to describe her solution strategy. First she realized every people had minimum five shells in their collections. Then she started to remediate each share by distributing from the largest share to the smallest share.

The 5th grade participant, Nez, was given multiple methods, and structured reallocation items. She presented with three multiple method items and three structured items that covered all reallocation task classes respectively. In multiple methods items except for the reallocation departure task, she could not able to produce two different methods. She used collection strategy for task involving arrival and uneven shares. She put all the collection back together and divided the total number of the collection by the number of the people. In the reallocation departure task, she used reallocation strategy. Initially, she drew the picture of the initial share, and then reallocated the person's extra share among the existing people. She used dealing by ones strategy as she fairly shared the extra share. At the end she represented new shares with picture. When I asked is there any other way? She could not produce another way to solve the problem.

For the structured items, for all reallocation task classes she made sense of the described two methods but declared which one was easier for her and which one she would prefer. For reallocation arrival and remediating uneven shares tasks, she declared that she preferred the method which described reassembling all collection and equipartitioning again among the existing number of people (collection strategy). But for reallocation departure task items, she said "reallocation strategy is an effective and easy way to solve the problem".

Overall findings on reallocation items can be stated as, across all three item class types, students (K-7) typically showed successful strategy in departure case. For arrival and uneven shares case, they typically showed a successful collection strategy or division strategy. Younger students (K-2) could not solve the presented paper-pencil items showed a successful reallocation strategy when they used manipulative materials. 3rd and 4th grade participants showed a transition stage in terms of their strategies. They tried to used both reallocation and division facts to solve the problems. Students from grade band 5-7 strictly used division facts as they solved the problems. On the other hand, finding on multiple structure items indicated that students realized both collection and reallocation strategies yielded equivalent results. Also, they stated their preferences in terms of strategy: they would choose to use reallocation strategy only on departure task since it was more efficient way for them in this type of tasks.

Analysis of Participant's Responses to Covariation Items

First, this section will provide a detailed description of the mathematical strategies used or mathematical behaviors shown by the participants during the covariation items by using the video-coding scheme developed during the analysis. Second, this section will examine the factors affecting the participants' strategies and provide examples of each finding from the transcription of their responses and written work. Finally, this chapter will report the findings on the performance differences between younger participants and older participants based on two grade bands: 2–4 and 6–7.The kindergarten and first-grade participants were not given the covariation items.

The participants who were presented with covariation items were three second graders, one third grader, one fourth grader, five sixth graders, and four seventh graders. The analysis of the responses yielded five different mathematical strategies which were structured into a set of levels:

Level 1: The participants were unable to produce a correct mathematical strategy and solution.

Level 2: The participants showed an additive misconception or tried to figure out the number rules.

Level 3: The participants used an additive strategy.

Level 4: The participant used a splitting strategy and then multiplicative reasoning.

Level 5: The participant recognized equivalence either within a ratio or between ratios and preserved the ratio while covarying the number of the collections or whole to be shared and the number of receivers.

Figures 15 through 18 show the distribution of mathematical behaviors and related strategies described in each level, from the least sophisticated to the most sophisticated. The vertical axis indicates the frequency of a particular participant's responses that fell under a particular level description in each grade per covariation item. Each color represented a particular participant and participant's responses in the graph. For instance, in grade two there were three participants, and their responses were colored respectively, Suda's responses represented with a blue color, Berr' responses represented with red color and Senn's responses represented with a green color.

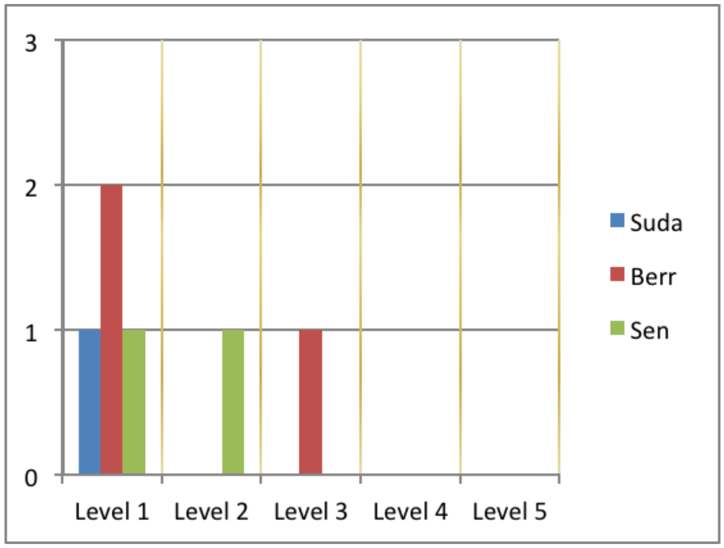


Figure 15. Frequency of participants' responses that fall under particular levels for given covariation items. 2nd grade participants: Suda, Berr and Senn.

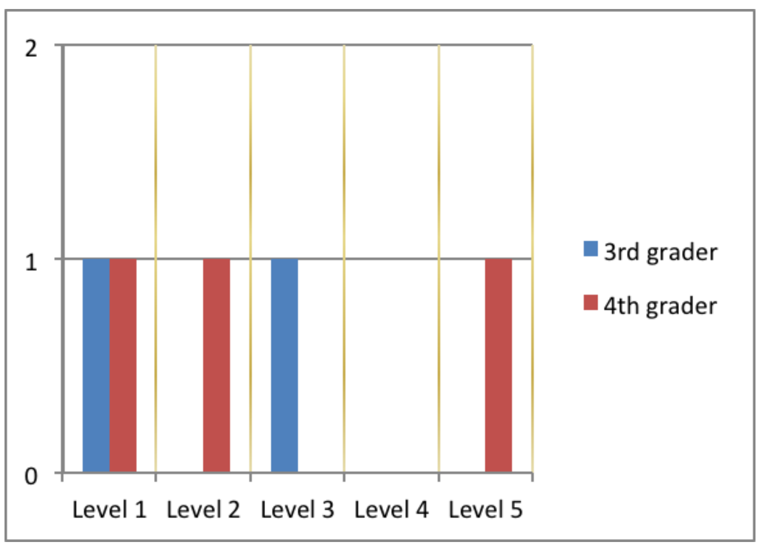


Figure 16. Frequency of participants' responses that fall under particular levels for given covariation items, 3rd and 4th grade participants.

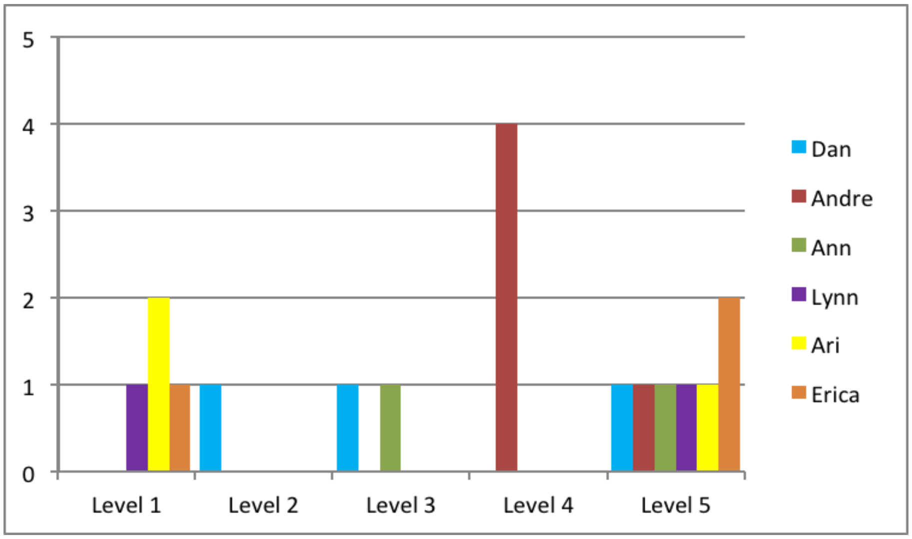


Figure 17. Frequency of participants’ responses that fall under particular levels for given covariation items. 6th grade participants: Dan, Andre, Ann, Lynn, Ari,and Erica.

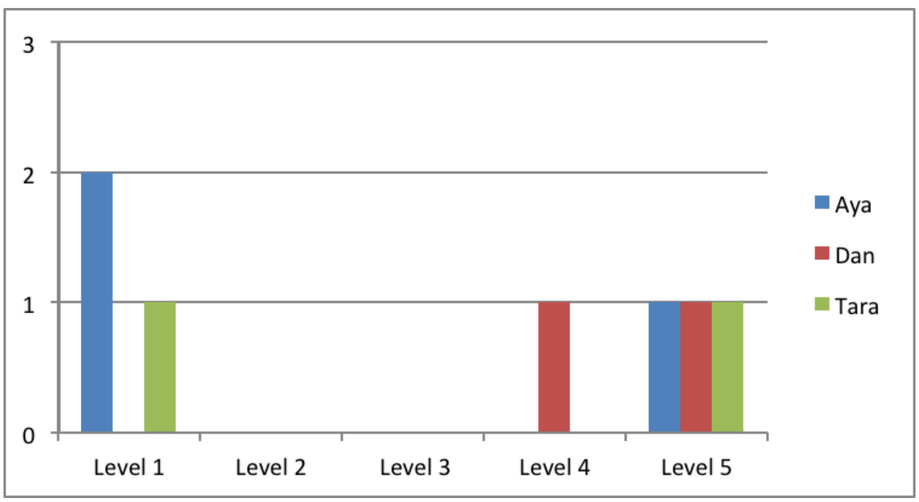


Figure 18. Frequency of participants’ responses that fall under particular levels for given covariation items. 7th grade participants: Dan, Aya, and Tara. Data was unavailable for one participant, Kay.

Grade Band 2-4

As Figures 15 and 16 shows that for the most part, the younger participants (grades 2–3) were unable to solve the covariation items. Only one young participant, a second grader, answered two of the covariation items correctly. None of the second grade participants could solve the rest of them.

The most challenging issue for the second grade participants was to fairly share multiple whole among multiple people, which is Proficiency Level 12 of the ELT. For example, second grader Suda struggled with sharing six carrots among four rabbits. She gave one carrot to each rabbit but could not figure out how to fairly share the remaining two carrots among the four rabbits. She could not adjust amount of shares for receivers by using same factor because she could not figure out the share per rabbit and she could not recognize that the size of the share remained the same (equivalence of ratio or covariation).

As Figures 15 and 16 show, one second grader, one fourth grader (Level 2) showed an additive misconception when solving the three covariation items. For instance, fourth-grade participant Nina was presented with this item:

> Matt knows that 6 carrots will feed 4 rabbits if they are shared fairly. Predict the number of carrots needed for each number of rabbits listed in the table, so each rabbit will get the same share of carrots. Explain how you figure out your answer. (Rabbit Item)

The item stem included a table (Figure 19).

Number of rabbits	Number of carrots
2	
4	6
8	

Figure 19. Table presented to Nina.

Nina initially gave one carrot to each rabbit without paying attention to the information on the table that stated that six carrots feed four rabbits, a ratio of 4:6 between rabbits and carrots. However, she then stated that six carrots can feed four rabbits and revised her initial share per rabbit to two carrots. Her justification for her answer showed an additive misconception that failed to yield a correct answer. Using additive reasoning, she captured the incorrect relationship (pattern) between the number of rabbits and number of carrots. She stated her solution process as follows:

> Nina: two plus two will be four. (She writes her answer under the number of carrots for two rabbits.) Then you add two more and you get six. (She points to the number of carrots for four rabbits with her pencil.) You have eight and it should be 16. (She writes her answer under the number of carrots for eight rabbits.)
>
> Interviewer: Can you explain your reasoning? How do you find 16?

Nina: You can do two plus two, four because there are two rabbits and they probably want two carrots.

Interviewer: Why do they want two carrots? Maybe they want more carrots?

Nina: No, they probably want two carrots. And then four rabbits add two more here. (She points out six carrots in the table.) And eight. (She erased her initial answer of 16.) Then they probably eat ten carrots.

Nina's explanation of her work showed some evidence of doubling and then shifting into additive strategy. Since she initially gave two carrots per rabbit and concluded two rabbits eat four. Then she wrote her initial answer for eight rabbits as 16 carrots. But the question of "Why do they want two carrots" might lead her to shift her strategy into an additive pattern. So she concluded six carrots for four rabbits (four plus two) and ten carrots for eight rabbits.

On a different covariation item, Nina showed some evidence of recognizing ratio equivalences (Level 5) between given quantities and could preserve this ratio while covarying when given two quantities on a different covariation item. That item stem did not include a table representation:

It was time for breakfast at Gracie's slumber party. Gracie knew that 1 box of cereal shared fairly among three friends. As they were about to eat, six more friends came over. How many boxes of cereal will she need so that everybody gets the same fair share? (Cereal Item)

She stated that one box would feed three of the friends. She may have recognized the within-state ratio (Noelting, 1980), *a*:*b*, in which *a* represents the number of the boxes and *b* represents the number of friends, 1:3 in this case. She then preserved this ratio for the three new friends and repeated the action for the next three additional friends. Thus, she concluded that because six more people had come over, Gracie would need two more boxes of cereal.

Although Nina presented a Level 5 strategy on this item, this item was easier than the previous item. This item yielded a unit ratio 1:3 and numbers of additional people were two times larger than the existing number of people. Doubling and halving is easier for younger children, since the children progressed to making equal parts through dichotomous, trichotomous, or both methods of division (Piaget et al. 1960). Moreover in grades 1 and 2 students have lots of experience with doubling and halving.

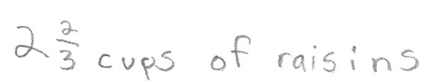
As shown in Figures 15 and 16, one second grader, one third grader used a correct additive strategy (Level 3) as they worked on each of the covariation items. Using this strategy, they found one person's share and then continually added this share to each person's fair share as the number of receivers differed. This additive strategy works because the presented covariation item yielded a unit ratio (where it is per receiver). Then they found this unit ratio as a ratio unit and added this ratio repeatedly as a form of covariation. For instance, third grader Ryan's strategy on the rabbit item was to draw four rabbits and 6 carrots and initially give 1 carrot to each rabbit. Then he split the remaining 2 carrots into two and gave half of a carrot to each rabbit. From this, he concluded that each rabbit's fair share was 1 and 1/2 carrots. Then he said, "If there are two rabbits . . . (Silence)," and he added 1/2 a carrot and found that two rabbits got 3 carrots. He used the same additive strategy to figure out the numbers of carrots needed for eight rabbits. When I asked him, "How do you know this is the correct amount of carrots?" he replied, "One rabbit can eat 1 and 1/2 carrots. Two rabbits can eat 3 carrots, each 1 and 1/2 carrots."

Grade Band 6-7

As Figures 15 through 18 shows seventh-grade participants' three responses and sixth grade participants' seven responses fell under a Level 5 behavior. For instance, 6th grade participants presented with the item

> The recipe calls for 2/3 of a cup of raisins for every batch of cookies that feeds 15 people. If Sue-Ming and her mother are going to bake enough cookies to feed all 60 children in her grade, how many cups of raisins should she use so the cookies will have the amount of raisins the recipe calls for? (Recipe Item)

This participant, Dan, determined that the factor of increase in the number of the people was four. Then he used the same factor to find the number of cups of raisins needed. He may have recognized that the *between ratio* of *a*2:*a*1(Noelting, 1980), where *a*1 represents the initial number of the people and *a*2 represents the new number of the people, was 60:15. He explained his strategy as "If two-thirds cups of raisins will feed 15 people, 60 divided by 15, I get four. Multiply four times two-thirds. (Silence)". He wrote his answer as shown in the following:



As shown in Figure 17, just Andre among 6th graders and Dan among 7th graders used a splitting strategy and multiplicative reasoning (Level 4) to answer a covariation item wherein one of the quantities varied. One of the sixth-grade participants, Andre, used this strategy to figure out the number of carrots needed for each number of rabbits in the rabbit item. From the table, he knew that four rabbits fairly shared six carrots. To find a fair share for one rabbit, he first gave each rabbit one carrot. Then he fairly shared the remaining two carrots among four rabbits, giving 1/2 a carrot to each rabbit. He concluded that one and half carrot would be given per rabbit. This showed that he figured out a unit ratio (1: 1.5) as his unit ratio. He then multiplied each fair share by the number of the rabbits: two rabbits would get 3 carrots (2 x 1.5) and eight rabbits would get 12 carrots (8 x 1.5). During the last part of the interview, Andre realized the equivalence of between ratios as he worked on the item, and he used covariation to find two given quantities (the number of the rabbits and the number of the carrots) with the same factor (Level 5):

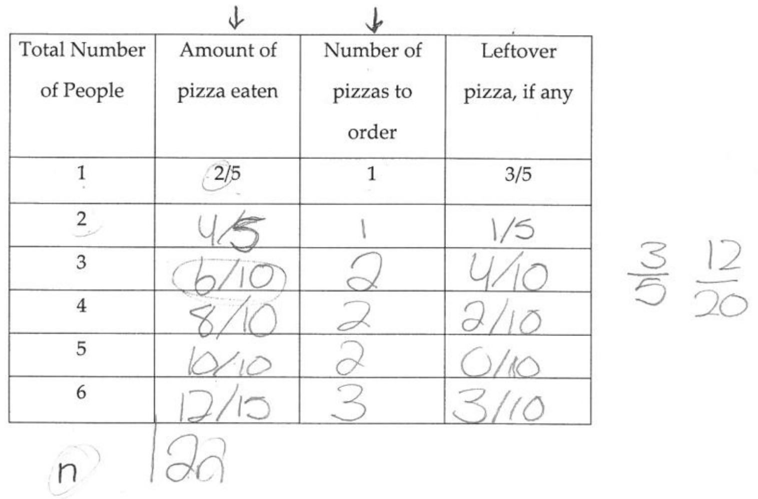
> Interviewer: Can you see any pattern on this table?
>
> Andre: Yes, two times two equal four, two times three is six.
>
> Interviewer: Why do you think this pattern works out?
>
> Andre: The number of rabbits times two, the number of carrots times two.

As I examined the participants' responses, I realized one of the 7th grader, Kay was presented with just one covariation item and her strategy could not be coded under existing levels. She exhibited a common misconception while working on the item:

Fill in the table below to show the exact amount of pizza eaten by up to 6 people. How many whole pizzas should be ordered to have enough, and how much pizza would be left over, if any? (*The second part of the item was)* Explain in your own words how you found the total amount of pizzas the kids would eat with the chart above. (Pizza Item)

The item's table is seen in Figure 20. Kay transferred her prior knowledge of operations with whole numbers into operations with rational numbers, a common problem discussed in the literature review. In order to determine new people share she repeatedly added the one person share ($2/5^{th}$ of pizza) to figure out the new number of people's share. As she was working on this she displayed a common misconception while adding fractions. She added denominators to find the common denominator. For instance, when she found the amount of pizza for three people she added $2/5^{th}$ of a whole pizza (amount of pizza per person) to $4/5^{th}$ of a whole pizza (amount of pizza for two people) and wrote her answer as $6/10^{th}$ of a whole pizza

Figure 20 shows her final answer to the item.



Total Number of People	Amount of pizza eaten	Number of pizzas to order	Leftover pizza, if any
1	2/5	1	3/5
2	4/5	1	1/5
3	6/10	2	4/10
4	8/10	2	2/10
5	10/10	2	0/10
6	12/15	3	3/10

Figure 20. Kay’s final answer for the pizza covariation item.

Another important finding in grade band 6-7 is, some older students could not fairly share multiple wholes among particular numbers of receivers, but they could perceive the constant ratio within two quantities. One of this study’s sixth-grade participants, Ari, provided a good example of that concept. Although she could not determine the unit fractions in the cereal item where each friend’s share was one-third of a cereal box, she preserved the composite unit: 1 box of cereal per three friends. She found that the total number of cereal boxes needed to produce fair shares for six more friends was 3, or 1 box of cereal per three friends.

Finally, all these findings pointed the differences between older participants (grades 6–7) and younger participants (grades 2–4) performance on covariation items. Older participants outperformed younger participants on each of the covariation items. The previously mentioned reasons why younger participants likely performed poorly on the items—not receiving official instruction on ratio, lack of experience with multiplication and division facts, and so on—will be discussed in the next chapter. Figure 14 shows the frequency and percentage distribution by grade bands of the participants' responses to the covariation items that fell under particular levels. Each color represents a particular strategy (yellow = Level 5, purple = Level 4, green = Level 3, red = Level 2, and blue = Level 1). Because the number of items presented to each grade and the number of the participants in each grade band differed, each grade band shows the dispersion of results in terms of percentages and frequencies of the responses.

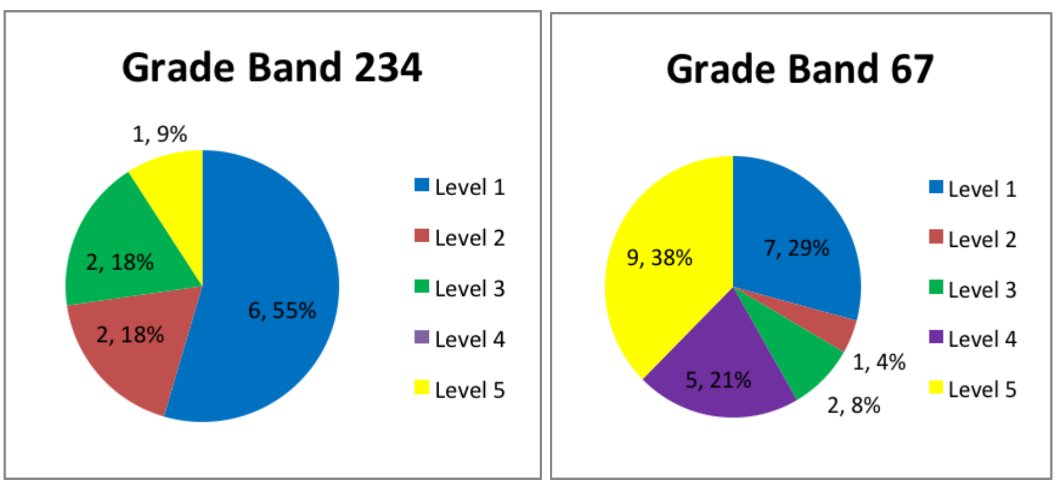


Figure 21. The frequency and percentage distribution participants' responses to the covariation items

In grades 2 through 4, only 9% of the overall participants' responses fell under Level 5. In grades 6 and 7, 38% of the overall participant responses felt under Level 5. Only 29% of the overall participant responses failed to yield a correct mathematical answer (Level 1) for grades 6 and 7, but 55% of the overall participant responses from Grade Band 2–4 failed to yield a correct mathematical answer.

Table 10 provides the total numbers of participants who were unable to solve the items (Level 1) and who showed additive misconceptions (Level 2) as well as the total number for overall responses to six covariation items by grade band. Because the same questions were given to different participants, the sums in each grade band indicate greater values than the numbers of participants in each grade band. For example, in Grade Band 2–4, the rabbit item was given to all participants, but more than one student was unable to solve it so the numbers do not match up.

Table 10

Participants Incorrect Response, Levels 1 and 2, Frequency per Item

Grade Band	Cereal Item	Rabbit Item	Pizza Item	Cupcake Item	Recipe Item	Hamster Item
2–4	1	4				3
6–7	1	1	2	1		3
Total Number of Responses for Grade Band 2–4	4	4				3
Total Number of Responses for Grade Band 6–7	4	8	3	2	1	6

Note. Empty cells indicate items that were not given to the corresponding grades.

From this table representation, we can infer that older participants performed better than the younger students on the covariation items. The rabbit item was given to all participants, although one second-grade and one seventh-grade participant were unable to complete it within the interview time and a second seventh-grade participant was not given it at all. In Grade Band 6–7, only one of the nine participants who were given the item was unable to produce the correct mathematical solution and justification. On the other hand, four out of the five students in Grand Band 2–4 who received the item were unable to produce the correct mathematical solution and justification for it. None of the younger participants could produce the correct mathematical solution for the hamster item as well. In Grade Band 6–7, 50% of the participants produced the correct mathematical solution.

CHAPTER 5: RESULTS AND DISCUSSION

In this chapter, I will examine my initial conjectures for the three main research questions of the study. Then, I will discuss how the findings inform ELT reallocation and covariation levels. Finally, I will discuss the study limitations and propose future research suggestions.

Research Question 1

What are the observed differences and similarities among participants' strategies in earlier grades (K–4) and higher grades (5–7) while solving reallocation items and sharing collections items based on evidence from clinical interviews? Is there any systematic difference observed among participants' strategies at different grade levels when solving reallocation items? If yes, what might be the reasons for these differences? (item class types, item-stem format, item type)

<u>Conjectures:</u> *Younger participants (K–4) will tend to use reallocation and older participants (5–7) will usually use multiplication and division facts. The possible reason for this difference in strategies is that older participants will have received more formal instruction on multiplication and division than younger participants. Item attributes (i.e., item-stem formats, item types, item classes) may affect participants' strategies while dealing with the reallocation items in clinical interviews. Participants tend to use reallocation on reallocation strategy departure task classes items while tends to use collection strategy on reallocation arrival and reallocation uneven share task classes items. If the item stem includes an organized picture of the initial shares for all task classes, participants more likely to use reallocation strategy. Since picture of the initial shares help them to act on the*

shares as they reallocate. Multiple method and structured item types will elicit both reallocation and collection strategy as participants worked with presented item. Since in structured items both collection and reallocation strategies were described in item stem and asked for choosing the correct strategy (ies). Multiple method items were asked for two different ways to solve presented reallocation item.

This study's findings showed that younger participants from grades K–2 could use reallocation strategies to solve the reallocation items if they were allowed to use manipulatives or item stem include picture of initial shares. Their strategy was first to model the content presented in the item stem with manipulatives to determine the initial fair shares of each person; then they reallocated from each share to create new fair shares. To make sure their shares were fair, they used counting strategies to justify them.

For participants' responses from grades 3–4, the findings indicated a transition phase from using reallocation strategies to depending on multiplication and division facts. The fourth-graders used strictly a reallocation strategy when they were presented with reallocation departure items. But only one student use reallocation strategy reallocation arrival item. On the other hand they used division facts when presented with reallocation uneven shares items. This finding differs from the K–2 grade results, because the younger participants used reallocation skills only or put the entire collection back together and dealt it out again among the new numbers of people.

Another fourth-grade-student Aise showed clear evidence of this transition between strategies as she solved the reallocation items. When she tried to use division facts to determine each person's new share, she could not find the answer. Then she moved onto a strategy of establishing a composite unit of objects for each person rather than using multiplication facts. Then she subtracted the composite unit amount from the total collection amount and used division to figure out how much more she needed to give each person. Then she added her initial composite unit amount and the redistributed object amount to find each person's share. Her strategy entailed combining composite dealing from sharing collections, multiplication to find how much she fairly shared, then either reallocation to deal the rest of the collection or division to figure out how much more she needed to give to each person. This is interesting because reallocation was a bridge between splitting and reassembly to division and multiplication for her. In numerical form, she split 42 objects among six people. One person joined the group, and she gave 5 objects to each person, then said "5 times 7 is 35," and then, "42 minus 35 is 7, so I need to give 1 more object to each person." She used this strategy consistently for all multiple method reallocation items she was presented with.

The findings from grades 5–7 indicated that participants typically used multiplication and division facts as they worked on the given reallocation items, except for reallocation departure task class items and the items that asked for multiple methods or described multiple ways of reaching solutions. In the first set of interviews, sixth-grade participants typically used reallocation strategies (n = 12 out of 19 responses). For the items that did not include an organized picture of initial shares, they used collection strategies and division and multiplication facts (n = 7 out of 19 responses). For remediating uneven shares, six of the seven participants first determined the total number of the collection and used a division fact to create new fair shares. For reallocation arrival task class items, six of the seven sixth-grade participants used multiplication and division facts. The seventh-grade participants typically used multiplication and division facts except on reallocation departure task class items.

For the structured items, participants from the fourth and fifth grades made sense of the two described methods and understood why they worked but declared which ones were easier to use personally and which ones they preferred for all reallocation task classes. For reallocation arrival and remediating uneven shares task classes items, they declared they that both of the method are correct and they preferred the method that described reassembling the collection and equipartitioning it again among the existing number of people, a collection strategy. But for reallocation departure task items, although they recognized both methods are correct and yield the equivalence result, they would prefer a reallocation strategy because it was more effective and easier to use than a collection strategy.

Moreover, the findings revealed that a specific sets both reallocation departure and arrival task classes items, in the case of one person leaving or arriving, served as an important base for understanding the commutative property of multiplication. These specific cases involve the cases if the number of the objects and number of the people in the items were relatively primes. For instance, only one participant, Andre, explicitly stated the property name used for the case item when one person arrives. He stated, “3 [people] times 4 [objects per person] is 12 [objects], and 4 [people] times 3 [objects per person] is 12 [objects].” Other participants made similar statements to check their answers, and they realized that in both cases, the collection amount remained the same but the number of people and their fair shares changed.

Based on these results, I conclude that item-stem formats and item-task classes (reallocation departure, reallocation arrival and reallocation uneven shares) had effects on participants' strategies. If participants were presented with pictorial representations of fairly shared objects arranged in arrays, then younger participants (grades 2–4) used reallocation to adjust each share so an arriving person received a fair share. Also, some of the participants from grades 2–7 used this strategy if they were presented with pictorial representations of unequal shares arranged in arrays. In contrast, if an item without a picture or an unorganized pictorial representation (a cluster) of each share was given, then the participants tended to directly reassemble all items and fairly share them among the new number of people. Participants in all grades typically used reallocation strategies in reallocation departure tasks. On the other hand, for reallocation arrival tasks, participants typically reassembled the items and used division or the fair-sharing strategy to fairly share them. Additionally, item types affected the variety of 5th grade participants' strategies for solving reallocation problems. Although they typically used multiplication and division facts in solving the open-ended items and could not produce a reallocation strategy for the multiple-method items, when presented with structured item types, they gradually realized that the reallocation strategy worked for other reallocation task classes.

As a result, participants in each grade band used similar mathematical strategies as they worked on the reallocation items. Participants from grades K–2 used the fair sharing and reallocation strategies by modeling problems with manipulatives and justifying their shares by counting. Participants from grades 3–4 showed a transition stage in strategies, from reallocation to usage of multiplication and division facts or from fair sharing to multiplication and division facts. Participants from grades 5–7 typical used multiplication and division facts but also reallocation for departure task class items. In addition, item-task classes, item-stem format, item types, and manipulatives usage affected participants' strategies and performances as they worked with the items. All these results confirm and deeply informed my initial conjectures except for the grades 3–4 findings that showed a transition stage that I did not state in my conjectures.

How Does This Inform the ELT?

Reallocation arrival task class items will not be included in the reallocation level description in the ELT since in this study participants across grades did not use a reallocation strategy. Reallocation should inform future multiplication and division LTs that the DELTA team will construct because reallocation builds a bridge to multiplication and division from fair sharing and reassembly. To elicit a wide range of mathematical strategies in an assessment environment, item development is a key. Having different types of items is important for making them accessible to different ability groups and grade levels. Also, it is important to see how item structure interacts with the mathematics the item conveys to participants.

Research Question 2

What is the effect of manipulative use on younger participants' (grades K–2) strategies while solving reallocation items? Is there any difference between participants' abilities to solve reallocation items when they do and do not have manipulatives?

Conjectures: *Young participants who cannot solve reallocation problems with a paper-pencil assessment may be able to solve the same reallocation problems with the use of manipulatives.*

When I compared findings on the participants' abilities to use the reallocation strategy on given items in both sets of interviews, I concluded that manipulative materials helped younger participants reallocate to generate fair shares. In the first set of interviews, none of the participants from grades K–2 could employ the reallocation strategy as they worked on a paper-pencil assessment item. But in the second set of interviews, the same kindergarten and first-grade participants used the reallocation strategy on the same and additional different items as they solved them correctly. For the reallocation of uneven shares item, both participants could not use the reallocation strategy even when they were allowed to use manipulatives.

In the second set of interviews, three participants from the second grade also typically used the reallocation strategy. They were first asked to try to solve the items without using manipulatives, and only one of the participants, Berr, could solve them when the item stem included a picture of the initial share. Then, they were asked to use the manipulative materials to solve the same reallocation items. Both of the second-graders were able to use the reallocation strategy to solve the given items with manipulatives. Table 7 (see Chapter 4) shows that 18 out of 30 participants' (grades K–2) responses indicated use of a reallocation strategy as they worked with manipulatives. Of those responses, eight conveyed a reallocation strategy without manipulative usage, but in five of those eight responses, the participants worked on a reallocation item that had an organized picture of the initial share.

For these grades, only 4 of 30 responses conveyed a collection strategy. The findings from both sets of interviews confirm my initial conjectures and indicate that there is progressive development in students' abilities to use a reallocation strategy when presented with manipulative materials and in their performances on reallocation items.

How Does This Inform the ELT?

To validate the ELT, the DELTA team administered paper-pencil assessment items in field tests, and findings from the tests indicated that reallocation items were hard for students, especially younger ones. Findings from this study indicated that paper-pencil assessment items may not make mathematics accessible for all students, especially younger ones, so the usage of concrete materials—manipulatives— make the items accessible to different ability groups.

Research Question 3

What are participants' performance levels and mathematical strategies in solving covariation items? Is there any systematic difference observed among their performances at different grade levels while dealing with covariation items?

Conjectures:.

The older participants are more likely to perform better on covariation items. For covariation items, when the item stem includes a table representation, participants will search for a pattern in the given table to solve the problem without having a correct justification. When the item stem does not include a table, participants will use a splitting strategy. First, they try to find the each person share and then use this amount to find the amount of share for existing amount of people.

The findings from this study indicated that younger participants (grades 2–4) and older participants (grades 6–7) both used five different mathematical strategies as they worked on the covariation items. Those mathematical strategies were categorized from least sophisticated to most sophisticated, and each strategy was ordered by levels. The least sophisticated level, Level 1, stated that participants could not produce a correct mathematical strategy. In Level 2, participants showed an additive misconception as they tried to covary both pairs of quantities. In Level 3, participants used an additive strategy in which they found each person's share and continued with one person's fair share until they found the new fair share for the existing amount of people. In Level 4, participants used a splitting strategy and multiplicative reasoning in which they found each person's share by using equipartitioning then multiplied that share by the number of the people to find the number of objects for each person. In Level 5, participants recognized within or between ratio units within the pair of given quantities (number of objects and number of people) and preserved this ratio as they covaried both quantities.

In grades 2–4, only 9% of the overall participants' responses showed a mathematical strategy that fell under Level 5, whereas in grades 6–7, 38% of the overall participant responses fell under Level 5. In the same grade band, only 29% of the overall participant responses failed to yield a correct mathematical answer (Level 1), whereas 55% of the overall participant responses from grades 2–4 failed to yield a correct mathematical answer. The findings confirmed my initial conjectures that older participants would outperform the younger participants and that they would use more complex strategies.

The reasons why young participants may have been unable to solve the covariation items are

(a) a lack of proficiency in ELT levels,

(b) not being exposed to formal instruction yet, and

(c) not being able to read the table representations in the item stems.

Lacking Proficiency in the ELT

Younger participants had difficulty fairly sharing the multiple wholes among multiple people, which is Level 12 of the ELT. They could not find each receiver's share, and so they could not divide those shares among the new numbers of existing receivers. None of the younger participants (2-4) produce correct answer when they presented with the items that requires fairly sharing multiple wholes. For instance, the Rabbit problem was presented, and the younger participants did not show a correct mathematical strategy and answer. On the other hand, only one out of four older participants did not show a correct mathematical strategy and answer. In this item, the table presented 6 carrots fairly shared among four rabbits and asked how many carrots were needed for two rabbits.

Not Being Exposed to Formal Instruction Yet

The second-grade participants had newly entered the second grade, and they had not yet received formal instruction on multiplication and division. The Common Core Mathematics Standards (2011) for multiplication and division are started in the third grade. The third-grade participants had likewise just become third-graders, and when I talked with their classroom teachers, they had not yet introduced multiplication and division into their classrooms. As a result, the participants from those grades had a difficult time perceiving the multiplicative relation between given pairs of quantities, either in tables or item stems.

Moreover, because they worked in the additive world in those grades rather than in the splitting and multiplication world, only two participants (18% of the second-, third-, and fourth-graders) tried to perceive multiplicative the relation between the number of objects to be fairly shared and the number of receivers. On the other hand, both sixth- and seventh-grade participants had more experience with multiplication, division, and fractions and the advanced sixth-grade participants in algebra classes and seventh-grade participants formally introduced the concept of ratios. As a result, participants from the sixth and seventh grades outperformed the younger participants on the covariation items.

Not Able to Read the Table Representation in the Item Stems

Younger participants had difficulty reading the information on the tables. They could not relate the quantities given in the table to the problem. As a result, they could not figure out the fair-sharing situation or how to covary both quantities to preserve the size of share per person.

Participants from the sixth and seventh grades read the tables and recognized the relation between different quantities as *x* to *y*. But only two out of ten realized the relation between same quantities as "*x* increases by this factor so *y* has to be increased by this amount" while they worked on the covariation items. One of the possible reasons for some of the participants' difficulties is that they learned through instruction to add the same things (the same unit), not different things, so while they concentrated on the same quantities, they tried to explain the relation in terms of additive relations. As a result, they presented an additive misconception (4% of overall responses in grades 6–7 and 18% in grades 2–4) while they worked on the covariation items. But when they tried to perceive the relation among different quantities, they tended to use multiplicative reasoning to explain the relation. So it was more common that the participants stated the relation in terms of ratios as they worked on covariation items, especially the items that included tables. These findings support my second conjecture that table representations in item stems likely affected participants' strategies for two reasons. First, it is hard to read information from tables for younger participants. Second, participants sought additive relations between the quantities given in the table.

How Does This Inform the ELT?

In equipartitioning, the students begin by developing a means to share a collection (x) fairly among (n) people. Then they assert that x divided by n produces c items per person. In this sense, they may say later that x divided by n equals c per person or that x divided by n equals c. One is a division fact with an associated multiplication, and the other is a fundamental ratio relation that declares that x is to n as c is to 1. This is the first time equivalence is asserted. Covariation builds on this to fill in the intermediate relations or to perhaps use a co-splitting strategy to reach that solution. It only produces integral values building back up or splitting down. In my study, I examined both situations: what happens once students are presented with a multiple-whole problem producing a ratio with a fractional value and one with an integral value. The younger participants (grades 2–3) had difficulty with the first case because they did not have proficiency at ELT Level 12 and did not have prior knowledge of multiplication, division, and fractions. On the other hand, older participants who had more experience with those mathematics content areas could recognize the unit ratio and covaried both quantities using either multiplication facts or by preserving the ratio. Splitting and equipartitioning multiple wholes is an important base for covariation levels of the ELT.

The overall results of this study are

(a) Younger participants demonstrated a significant change in their problem-solving strategies and abilities when they were presented with manipulatives.

(b) Younger and older participants demonstrated a significant or contrasting change in their problem-solving strategies while working on different item-stem formats and item classes.

(c) Older participants outperformed than younger participants on covariation items.

(d) Younger participants demonstrated reallocation strategies while older participants typically reassembled the whole collection and fairly shared it as they worked with reallocation items.

(e) Reallocation served as a bridge between splitting and reassembly and formal division and multiplication.

(f) The fair sharing of multiple wholes level of the ELT served as an important base for covariation.

(g) Participants used five different mathematical strategies as they solve covariation items which informs the outcome space of ELT level 14.

Limitations and Future Research Suggestions

In this study, the reallocation items introduced initial shares of receivers in the item stem. Sharing collections is Level 1 of the ELT, and reallocation is Level 9 of the ELT. I did not want to assess the participants' abilities on multiple levels in one item. Thus, the participants were not asked to figure out each receiver's original share; instead, they started from finding new fair shares. Because the participants did not actually do the original sharing to get the results for the presented items, the outcome of the problem solving and participants' strategy choices may be different than of they had solved the problem themselves. I hypothesize this because when I presented younger participants (K–2) with manipulative materials, even if the item itself stated the initial shares of each receiver, they modeled the initial share and then adjusted each share to get their final fair shares and they were more successful. Older participants (5–7) directly used the initial amount of each person's share as they worked out their answers. I did not check whether including both types of items, firstly including the original share amount for each person and secondly including the original share, would change the participants' strategies and the outcomes of their problem solving. Thus, the first limitation of the study is that I did not check all possible combinations of item attributes. In future research, paper-pencil field testing should include a different variety of items. Then statistical analysis can be used to test the main effects of item attributes and their interaction with participants' solution strategies.

The second limitation of this study is that some of the younger participants (K–2) had difficult times keeping track of their reallocation moves when the item stem did not include an outline of each person's share. One had a hard time recognizing which person's share had taken the item away from to reallocate the objects among the existing number of people. This observation is important for the design of future research. When the DELTA team presents reallocation tasks in a computer environment, they will need to leave outlines of the original chips. They should also design the computer tasks in which items and persons and objects are represented with differently on the screen. I could have done this by preparing paper pencil assessment items in which the pictures of initial shares were displayed separately for each person. I could have observed how that affected the interview results. In future research, items generated in computer environment will be useful for testing this type of items.

Moreover, using manipulatives in a computer environment will help students keep track of their modeling processes by using program features such as previews of previous moves. Also, they can reflect on their actions as they review their solution strategies. These are benefits because “computers allow students to store more than static configurations: They enable students to record sequences of their actions on manipulatives and later replay, change, and reflect on them at will” (Clements & Sarama, 2007). During this study, when participants were done with their modeling, they could not turn back and reflect on the process. They could only reflect on the end product. This is another important limitation: the use of paper-pencil assessment items with concrete manipulatives that participants had a hard time managing. When they were presented with large collections, the concrete manipulatives made it difficult for them to reflect on the modeling process of the problem’s content.

A successful design for building a LT requires an understanding of how students think and successive revisions and refinements on LT based on the empirical data about students’ thinking. This requires using empirical methods to identify students’ strategies, prior knowledge and thinking. Through clinical interviews, one could tap students' both verbal and non verbal works. The interview settings allowed researchers to use open ended questions that elicit more about phenomenon of interest of the study. This requires a huge effort in data analysis that a researcher should identify critical events, code the video data and triangulate the data. Usage of clinical interviews in my study was time consuming but yielded deep understanding about students' thinking that inform certain levels of ELT (Levels 9 and 14). In addition the results of the study inform the field how students from different grades think and reason on reallocation and covariation.

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APPENDIX

APPENDIX A

This appendix include the both reallocation and covariation items that were not functioned as we conjectured.

<table>
<tr><th colspan="3">ROW 9-Reallocation Items</th></tr>
<tr><td>Reallocation Person Arrives</td><td>Reallocation Person Leaves</td><td>Reallocation Uneven Shares</td></tr>
<tr>
<td>
Item 99 (15.9 % percent correct answer)

Four children were coloring a picture using a box of 40 crayons. They shared all the crayons in the box fairly so each child got 10 crayons. One more child arrived to color.

1. How many crayons should each child give the new child so that everyone has a fair share of the crayons? Show your work.

2. Describe each child’s share after the new child arrived by marking each description True or False.

T / F

_____ Each child has 2 more crayons

_____ Each child has 2 less crayons

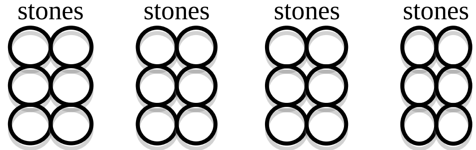
_____ Each child has 8 crayons.

_____ Each child has 12 crayons.

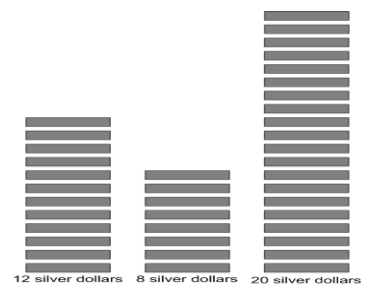
_____ You can’t tell.
</td>
<td>
Item 98 (70.1% percent correct answer)

Abdulla, Jamal, Michelle, and Sarah have collected stones of the same size and shape to make wind chimes for Earth Day. The four friends shared the stones as shown in the drawing below.

Abdulla has to go home. How could the 3 children share all of the stones fairly? Use the drawing below to explain your answer.

Abdulla’s stones | Jamal’s stones | Michelle’s stones | Sarah’s stones

</td>
<td>
Item 100 (35.3% percent correct answer)

A grandmother collected a set of silver dollars to give to her four grandchildren. She had them in three stacks. One stack contained 8 silver dollars. One stack contained 12 silver dollars. The last stack contained 20 silver dollars. How could each of her grandchildren receive a fair share of the silver dollars? Show your work.



</td>
</tr>
</table>

ROW 14- Covariation Items

Item 87 (3.2% correct answer)

Maria's ballet teacher promised her class of 10 children that they could each have some candies at the party after their recital. She bought fifty candies for them to share fairly. Only five children come to the party.

1. How many total candies does she need for the five children at the party so that they still get the fair share she planned for them?

2. How many times larger is the number of children in Maria's class than the number of children that came to the party?

3. How many times larger is the total number of candies that Maria would need for 10 children than the total number of candies she would need for 5 children?

4. How does the change in number of children in #2 compare to the change in total number of candies in #3?

Item 88 (6.9% correct answer)

Question 6 1130005_01_2_B07_88_6_Buncombe_SandHill-VenableElementary_167.PDF

Martha gave a package of four cupcakes to three boys to share fairly.

1. If six more boys come over, how many total cupcakes will Martha need so that each of the nine boys will get the same size share as the first three?

2. Explain your answer.

Item 89 (5.7% correct answer)

Question 5 1060001_01_3_B09_89_5_Buncombe_SandHill-VenableElementary_169.PDF

Randy is taking care of his friend's pet store and has to feed the goldfish.

1. If 4 goldfish eat a total of three sticks of food, how many sticks of food does he need for 20 goldfish? Draw pictures to explain your answer.

2. If 4 goldfish eat a total of three sticks of food, how many fish can he feed with 18 sticks of food? Draw pictures to explain your answer.

Item 90 (48.3% correct answer)

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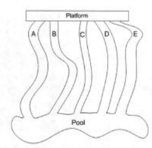
The teenager working at the top of the five waterslides directed each rider in turn to take a different waterslide, starting with A, in order: A, B, C, D, and E as shown in the drawing below.



After 20 total riders,

1. how many had ridden down slide E?

2. how many had ridden down slides D & E combined?

3. what share or part of all the riders had ridden down slide A?

C01 7 90

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4. (Fill in the blank):
the total number of riders who went down the water slides was _______ times as many as the number of riders who went down slide A.

5. A person keeps track of the 20 riders coming down the slides. Describe the number of riders he sees in relation to the number of slides he observes. Fill in the blanks.

Names of Slides Observed	Number of Slides Observed	Number of Riders
A	1	
A and B	2	
A, B, and C	3	
A, B, C, and D	4	
A, B, C, D, and E	5	20

Item 91 (52.6% correct answer)

Question 6 1820008_01_5_C02_91_7_Franklin_LongMillElementary_63.PDF

Verna knows that one pizza feeds 5 children if it is shared fairly. Predict the number of pizzas to buy for the number of children listed in the table below, so that each child will get the same share of pizza.

Number of children	Number of pizzas
5	1
10	
20	
35	

Item 95 (24.2 % correct answer)

	Question 4 On Day 1, a teacher asked two students to share a sheet of construction paper as shown in the drawing below.  Student 1 Student 2 On Day 2, the teacher gives the two students six times as many sheets of paper to share. 1. How does the size of each student's share change from Day 1 to Day 2? 2. If their share on Day 1 was one-half of a sheet of paper, how would you name their new share? 3. Draw a picture to explain your answer to #1. C01 5 95