# THE USE OF ANALYTIC CONTINUATION IN SOLUTIONS OF PLANE THERMOELASTIC PROBLEMS ON DOUBLY CONNECTED REGIONS

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ABSTRACT

The method of analytic continuation is applied to the plane thermoelastic problems on doubly-connected regions, which are the steady temperature fields without any heat generation at the inner points. Starting from Muskhelishvili's basic equations of two-dimensional thermoelasticity and following after Buchwald's and Davies' approach, one of two complex potentials is continued analytically into the inverse region across one of the boundaries which is a circle or can be mapped on to a circle. Then the conditions on the circular boundary are satisfied exactly and it is only necesarry to consider the conditions on the outer boundary. The procedures are illustrated by solutions for the elliptic regions with a confocal elliptic hole and polygonal regions with a circular hole at the center. In each region the boundary conditions are considered for two cases: a) the case of both boundaries free and b) the case of the inner boundary free and the outer boundary fixed. In the case of a), the results of present paper are compared with the former results, obtained using the point-matching method for real stress functions and the mechanical analog procedure, by authors and others.

## 1. INTRODUCTION

Recently many researches were done about the plane thermoelastic problems, however the analyses for the multiply-connected regions, which necessitate the consideration for the single-valuedness of the displacements and rotation, are not so many. One of the authors and Takeuti [1] considered a general method for plane thermoelastic problems in multiply-connected regions, partitioning the Airy stress function into a principal function and three biharmonic measures and using Michell's [2] conditions for the single-valuedness of displacements and rotation, and treated examples for polygonal regions with a circular hole with the aid of polar coordinates and point-matching method. This reseach has been continued and extended to more general cases by Takeuti, Nakanishi and the authors [3], [4], [5], [6]. On the other hand, one of the authors, S.Sumi, Matsumoto, Okamoto and Andoh [7] , [8] , [9] and S.Sumi and Ogiwara [10] announced general mechanical analog procedures, being based upon the above-mentioned general method for plane thermoelastic problems in multiply-connected regions and Southwell's [11] analogy, and carried out several analog experiments for the doubly-connected regions. However in these researches, Airy stress functions are always used in the forms of real functions and these procedures are suitable for the stress analyses in the case of stress boundary value problems. However, in the case of simultaneous stress and displacement analyses and in the case of displacement boundary value problems or mixed boundary value problems, it seems that the treatments with the aid of complex potentials by Muskhelishvili [12] are advantageous.

In the present paper, starting from Muskhelishvili's basic equations of two-dimensional thermoelasticity, the effective application of analytic continuation to the plane thermoelastic problems on finite doubly-connected regions, one of the boundaries of which is a circle or can be mapped on to a circle, is treated. The present analyses follow after Buchwald's and Davies' [13] approaches for the isothermal plane problems of elasticity. One of two complex potentials is continued analytically into the inverse region across one of the boundaries which is a circle or can be mapped on to a circle. Then the conditions on this boundary are satisfied identically. Therefore the present treatment is reduced to obtain one of two complex potentials, which can be determined only by the conditions on the outer boundary, using the Laurent series expansion.

The method is illustrated by numerical examples to the elliptic region having a confocal elliptic hole and to the polygonal region having a circular hole at the center. In each region the boundary conditions are considered for two cases:

a) the case of both boundaries free from the surface tractions,
b) the case of the inner boundary free and the outer boundary fixed.

For the elliptic region with a confocal elliptic hole, mapping the region on to the annulus, the coefficients of the Laurent series are obtained by comparison of coefficients and the perturbation method, and the solutions may be obtained exactly.

For the polygonal region with a circular hole, only the boundary conditions for the inner circle are satisfied identically and those for the outer polygonal boundary may be satisfied approximately using the point-matching method.

For the square regions with a circular hole and free from the surface tractions, the result of present paper is compared with the former results obtained by using the point-matching method for real stress functions and the mechanical analog procedure, by one of the authors and others.

The accuracy of the point-matching method is estimated by the residual surface tractions on the outer boundary for the case of a) and by the displacements on the boundary for the case of b).

## 2. BASIC EQUATIONS

The basic differential equation for the plane thermoelasticity is given by

$$\nabla^4\chi + \alpha_e E_e \nabla^2\tau = 0 \tag{1}$$

where

$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$

$\chi$ = Airy stress function

$\alpha_e = \alpha$ for plane stress, $\alpha(1+\nu)$ for plane strain

$E_e = E$ for plane stress, $E/(1-\nu^2)$ for plane strain

$\alpha$ = coefficient of linear thermal expansion

$E$ = Young's modulus, $\nu$ = Poisson's ratio

$\tau$ = temperature

In the following, it will be assumed that the cases of steady temperature fields without heat generation in the regions are considered. Then the two-dimensional temperature function $\tau=\tau(x,y)$ must satisfy

$$\nabla^2\tau = 0 \tag{2}$$

Let $h(x,y)$ be the harmonic function, conjugate to $\tau(x,y)$, then the equation

$$f(z) = \tau + ih \tag{3}$$

will represent a function of complex variable $z=x+iy$. Furthermore put

$$u_\tau(x,y)+iv_\tau(x,y) = \alpha_e\int f(z)dz = \alpha_e\int(\tau+ih)dz \tag{4}$$

Obviously

$$\frac{\partial u_\tau}{\partial x} = \frac{\partial v_\tau}{\partial y} = \alpha_e\tau \tag{5}$$

As $\tau(x,y)$ is a single-valued harmonic function, the general form of $(u_\tau+iv_\tau)$ may be written in the case of doubly-connected regions

$$u_\tau+iv_\tau = L^* z\log z + M^*\log z + (u_\tau^*+iv_\tau^*) \tag{6}$$

where

$L^*$ = real constant, $M^*$ = complex constant

$u_\tau^*+iv_\tau^*$ = holomorphic function of $z$

On the other hand, The Airy stress function $\chi$ is given by

$$2\chi = \bar{z}\varphi(z)+z\bar{\varphi}(\bar{z})+\psi(z)+\bar{\psi}(\bar{z}) \tag{7}$$

where $\varphi(z)$ and $\psi(z)$ are analytic functions and the bars over these functions express the complex conjugate in the sense of Muskhelishvili. Furthermore the stress and displacement components are given by

$$\sigma_{xx}+\sigma_{yy} = 4\,\mathrm{Re}\{\varphi'(z)\} \tag{8}$$

$$\sigma_{yy}-\sigma_{xx}+2i\sigma_{xy} = 2\{\bar{z}\varphi''(z)+\psi''(z)\} \tag{9}$$

$$u+iv = \{(1+\nu_e)/E_e\}\{\varkappa\varphi(z)-z\bar{\varphi}'(\bar{z})-\bar{\psi}'(\bar{z})\}+(u_\tau+iv_\tau) \tag{10}$$

where

$\nu_e = \nu$ for plane stress, $\nu/(1-\nu)$ for plane strain

$\varkappa = (3-\nu_e)/(1+\nu_e)$

For the pure thermal stress problems due to only the temperature field and without surface tractions, the boundary conditions, in the case of a doubly-connected region as shown in Fig.1, are given by

$$\varphi(z) + z\,\bar{\varphi}'(\bar{z}) + \bar{\psi}'(\bar{z}) = C_j \qquad (j=1,2) \tag{11}$$

where

$j=1$ for the inner boundary, and 2 for the outer boundary

$C_j$ = complex constant

Furthermore if the displacement components are assigned on the boundary j, eq.(10) is used as the boundary conditions.

The general forms of $\varphi(z)$ and $\psi(z)$ that assure the single-valuedness of stress components are given by

$$\varphi(z) = Lz \log z + M \log z + \varphi^*(z) \tag{12}$$

$$\psi(z) = Nz \log z + K \log z + \psi^*(z) \tag{13}$$

where

$\varphi^*(z)$, $\psi^*(z)$ = holomorphic functions

$L$ = real constant, $M, N, K$ = complex constants

However, $\varphi(z)$ and $\psi(z)$, given by eqs.(12) and (13), do not always assure the single-valuedness of rotation and displacements and the equilibrium of forces and moment along an arbitrary circuit as shown by the dotted line C in Fig.1. The increments of the rotation $\omega_z$ and the displacements $(u,v)$, resultant forces $(X,Y)$ and moment $M_O$ along the closed circuit C, become

$$\left[\omega_z\right]_A^A = 2\pi\left[4L+E_eL^*\right]/E_e \tag{14}$$

$$\left[(u+\omega_z y)+i(v-\omega_z x)\right]_A^A = 2\pi i\left[(3-\nu_e)M+(1+\nu_e)\bar{N}+E_eM^*\right]/E_e \tag{15}$$

$$X+iY = 2\pi\left[-M+\bar{N}\right] \tag{16}$$

$$M_O = 2\pi\,\mathcal{I}m\{K\} \tag{17}$$

where A is a point on C and $\mathcal{I}m\{\ \}$ means the imaginary part of $\{\ \}$.

Assume that the analytic function

$$z = \omega(\zeta) \qquad \zeta = \xi+i\eta = \rho e^{i\theta} \tag{18}$$

maps given doubly-connected region on to an annulus $a<|\zeta|<b$, then $\varphi(z)$ and $\psi(z)$ become

$$\Phi(\zeta) = \varphi(\omega(\zeta)) = \varphi(z), \qquad \Psi(\zeta) = \psi(\omega(\zeta)) = \psi(z) \tag{19}$$

in $\zeta$ plane. The stress and displacement components in a curvilinear system, corresponding to the concentric circles and radial lines in $\zeta$ plane, are

$$\sigma_{\rho\rho}+\sigma_{\theta\theta} = 4\,\mathcal{R}e\{\Phi'(\zeta)/\omega'(\zeta)\} \tag{20}$$

$$\sigma_{\theta\theta}-\sigma_{\rho\rho}+2i\sigma_{\rho\theta} = \{2\zeta^2/\rho^2\bar{\omega}'(\bar{\zeta})\}\Big[\bar{\omega}(\bar{\zeta})\left[\{\Phi''(\zeta)/\omega'(\zeta)\}-\{\Phi'(\zeta)\omega''(\zeta)/(\omega'(\zeta))^2\}\right] + \left[\{\Psi''(\zeta)/\omega'(\zeta)\}-\{\Psi'(\zeta)\omega''(\zeta)/(\omega(\zeta))^2\}\right]\Big] \tag{21}$$

Let

$$\begin{aligned} T &= T(\xi,\eta) = \tau(x,y), & H &= H(\xi,\eta) = h(x,y) \\ U_T &= U_T(\xi,\eta) = u(x,y) & V_T &= V_T(\xi,\eta) = v(x,y) \end{aligned} \tag{22}$$

One obtains from eq.(4)

$$U_T+iV_T = \alpha_e\int(T+iH)\,\omega'(\zeta)\,d\zeta \tag{23}$$

From eq.(10), displacements may be expressed by

$$u+iv = \{(1+\nu_e)/E_e\}\left[\varkappa\Phi(\zeta)-\{\omega(\zeta)\bar{\Phi}'(\bar{\zeta})/\bar{\omega}'(\bar{\zeta})\}-\{\bar{\Psi}'(\bar{\zeta})/\bar{\omega}'(\bar{\zeta})\}\right]+(U_T+iV_T) \tag{24}$$

Substituting eq.(19) into eq.(11), one obtains

$$\Phi(\zeta)+\{\omega(\zeta)\bar{\Phi}'(\bar{\zeta})/\bar{\omega}'(\bar{\zeta})\}+\{\bar{\Psi}'(\bar{\zeta})/\bar{\omega}'(\bar{\zeta})\} = C_j \quad (j=1,2) \tag{25}$$

as the boundary conditions for the free boundaries, where $C_1$ may be assumed to be zero.

Considering the multi-valuedness of $\Phi(\zeta)$, $\Psi(\zeta)$ and $U_T+iV_T$, one obtains

$$\Phi(\zeta) = L\omega(\zeta)\log\zeta + M\log\zeta + \Phi^*(\zeta) \tag{26}$$

$$\Psi(\zeta) = N\omega(\zeta)\log\zeta + K\log\zeta + \Psi^*(\zeta) \tag{27}$$

$$U_T+iV_T = L^*\omega(\zeta)\log\zeta + M^*\log\zeta + (U_T^*+iV_T^*) \tag{28}$$

where $\Phi^*(\zeta)$, $\Psi^*(\zeta)$ and $U_T^*+iV_T^*$ are holomorphic functions of $\zeta$.

Throughout this paper, one of the boundaries $|\zeta|=\rho=a$ is assumed to be free from the surface tractions.

Let the inverse region of the annulus S: $a<|\zeta|<b$ in the circle $|\zeta|=a$ be $S^*$, i.e. $a^2/b<|\zeta^*|<a$, as shown in Fig.2. As $\Phi(\zeta)$ and $\Psi(\zeta)$ are the analytic functions of $\zeta$, denoting a point of $S^*$ by $\zeta^*$, the analytic continuation of $\Phi(\zeta)$ into $S^*$ is defined by

$$\Phi(\zeta^*) = -\{\omega(\zeta^*)/\bar{\omega}'(a^2/\zeta^*)\}\bar{\Phi}'(a^2/\zeta^*)-\bar{\Psi}'(a^2/\zeta^*)/\bar{\omega}'(a^2/\zeta^*) \tag{29}$$

Hence

$$\bar{\Psi}'(a^2/\zeta^*) = -\omega(\zeta^*)\bar{\Phi}'(a^2/\zeta^*)-\bar{\omega}'(a^2/\zeta^*)\Phi(\zeta^*) \tag{30}$$

As $\bar{\zeta}=a^2/\zeta^*$ from the definition of inversion, substituting this relation into eq.(30), one obtains

$$\bar{\Psi}'(\bar{\zeta}) = -\omega(a^2/\bar{\zeta})\bar{\Phi}'(\bar{\zeta})-\bar{\omega}'(\bar{\zeta})\Phi(a^2/\bar{\zeta}) \tag{31}$$

Substituting eq.(31) into eq.(25), one obtains

$$\Phi(\zeta)-\Phi(a^2/\bar{\zeta})+\bar{\Phi}'(\bar{\zeta})\{\omega(\zeta)-\omega(a^2/\bar{\zeta})\}/\bar{\omega}'(\bar{\zeta}) = C_j \quad (j=1,2\ ;\ C_1=0) \tag{32}$$

as the boundary conditions for free boundaries. $\Phi(\zeta)$, thus obtained, is analytic throughout the region $S+S^*$ and the boundary conditions (32) are satisfied identically for $|\zeta|=a$. Therefore we have only to obtain $\Phi(\zeta)$, analytic for $a^2/b<|\zeta|<b$ and satisfying only the boundary conditions for $|\zeta|=b$. For the case, in which the boundary $|\zeta|=a$ is free and the displacement components are given on $|\zeta|=b$, the conditions for $|\zeta|=a$ are same as eq.(32) and as mentioned above it is satisfied identically. For the latter case, the boundary conditions for $|\zeta|=b$ become

$$u+iv = \{(1+\nu_e)/E_e\}\left[\varkappa\Phi(\zeta)+\Phi(a^2/\bar{\zeta})+\bar{\Phi}'(\bar{\zeta})\{\omega(a^2/\bar{\zeta})-\omega(\zeta)\}/\bar{\omega}'(\bar{\zeta})\right]+(U_T+iV_T) \tag{33}$$

where $u+iv$ is assigned along the outer boundary $|\zeta|=b$. As a particular case, when the boundary $|\zeta|=b$ is fixed, we may assume $u+iv=0$ in eq.(33).

## 3. AN ELLIPTIC REGION WITH A CONFOCAL ELLIPTIC HOLE

The analytic function

$$z = \omega(\zeta) = R\{\zeta+(m/\zeta)\} \tag{34}$$

maps the annulus $a<|\zeta|<b$ in $\zeta$ plane on to the elliptic region with a confocal elliptic hole, where R and m are real constants. When the inner and outer boundaries in z plane are mapped on to the concentric circles $\rho=a$ and $\rho=b$ in $\zeta$ plane, m may be assumed to satisfy $b^2>a^2>m\geq 0$. The correspondence between z plane and $\zeta$ plane is as shown in Fig.3.

When the temperatures at the inner and outer boundaries are $T_a$ (=constant) and 0, respectively, the steady temperature without any heat generation in the region is given by

$$T = B_0+\Gamma \log \rho, \qquad B_0 = T_a \log b/\log(b/a), \qquad \Gamma = -Ta/\log(b/a) \tag{35}$$

Let $B_0'$ be a real constant, then

$$T+iH = (B_0+iB_0')+\Gamma \log \zeta \tag{36}$$

Hence, one obtains

$$U_T+iV_T = \alpha_e \int (T+iH)\omega'(\zeta)d\zeta = \alpha_e\{\Gamma\omega(\zeta)\log \zeta + (B_0-\Gamma)R\zeta+(B_0+\Gamma)Rm\zeta^{-1} + iB_0'\omega(\zeta)+C_0+iC_0'\} \tag{37}$$

where $C_0$ and $C_0'$ are real constants. $B_0'$, $C_0$ and $C_0'$ are concerned in the rigid body displacements and may be assumed to be zero. Comparing eqs.(28) and (37), one obtains

$$L^* = \alpha_e\Gamma, \qquad M^* = 0 \tag{38}$$

From the physical viewpoint, assuming eqs.(14), (15), (16) and (17) to be zero, one obtains

$$L = -E_eL^*/4 = -\alpha_eE_e\Gamma/4, \qquad M = N = 0 \qquad \mathcal{I}m\{K\} = 0 \tag{39}$$

As the boundary conditions, the two cases are treated, i.e.

a) the case of both boundaries free from the surface tractions,

b) the case of the inner boundary free and the outer boundary fixed.

However, on account of limited space, these cases will be treated en bloc. Lumping eqs.(32) and (33) under one equation, one obtains

$$\lambda\Phi(\zeta)+\Phi(a^2/\bar{\zeta})+\bar{\Phi}'(\bar{\zeta})\{\omega(a^2/\bar{\zeta})-\omega(\zeta)\}/\bar{\omega}'(\bar{\zeta}) = -\delta_1C_2-\delta_2(U_T+iV_T)E_e/(1+\nu_e) \tag{40}$$

for the outer boundary, where

$$\begin{aligned} &\zeta = b\exp(i\theta) \\ &\lambda = -1, \quad \delta_1 = 1, \quad \delta_2 = 0 \quad \text{for case a)} \\ &\lambda = \varkappa, \quad \delta_1 = 0, \quad \delta_2 = 1 \quad \text{for case b)} \end{aligned} \tag{41}$$

The boundary conditions for the inner boundary are satisfied identically, as shown in the previous section.

Substituting the second of eqs.(39) into eq.(26) and expanding $\Phi^*(\zeta)$ into the Laurent series, one obtains

$$\Phi(\zeta) = L\omega(\zeta)\log \zeta + \sum_{n=-\infty}^{\infty}A_n\zeta^n \tag{42}$$

Owing to the symmetrical character of the present problems with respect to the real axis and the origin, $A_n$ must be real and n must be odd. Substituting eqs.(34) and (42) into eq.(40), one obtains

$$\sum_{n=\pm1,\pm3,\cdots}\Big[(-\lambda b^{n}-a^{2n}b^{-n})A_n - (n-2)(b^{-n+2}-a^{2}b^{-n})A_{-n+2} + m(a^{2n-4}b^{-n}+\lambda b^{n-4})A_{n-2} + mn(a^{-2}b^{-n}-b^{-n-2})A_{-n}\Big]\sigma^{n}$$
$$= \sum_{n=\pm1,\pm3,\cdots} b^{-n}q_n\sigma^{n} + \delta_1 C_2(1-mb^{-2}\sigma^{2}) + \delta_2\{\omega(\zeta)\bar{\omega}'(\bar{\zeta})/R\}\big[(\lambda+1)L+\alpha_e E_e\Gamma/(1+\nu_e)\big]\log\sigma \tag{43}$$

where

$$\sigma = \exp(i\theta)$$
$$q_1 = RL\big[\{(\lambda-1)b^{2}\log b + 2a^{2}\log a - b^{2} + a^{2}\} + m^{2}\{-(\lambda-1)b^{-2}\log b - 2a^{-2}\log a - b^{-2} + a^{-2}\}\big] + \delta_2\{\alpha_e E_e R/(1+\nu_e)\}$$
$$\times\big[\{\Gamma b^{2}\log b + (B_0-\Gamma)b^{2}\} + m^{2}\{-\Gamma b^{-2}\log b - (B_0+\Gamma)b^{-2}\}\big]$$
$$q_3 = mRL\{-(\lambda-1)b^{2}\log b - 2a^{2}\log a - b^{2} + a^{2}\} \tag{44}$$
$$+ \delta_2\{m\alpha_e E_e R/(1+\nu_e)\}\{-\Gamma b^{2}\log b - (B_0-\Gamma)b^{2}\}$$
$$q_{-1} = mRL\{(\lambda-1)b^{-2}\log b + 2a^{-2}\log a - b^{-2} + a^{-2}\}$$
$$+ \delta_2\{m\alpha_e E_e R/(1+\nu_e)\}\{\Gamma b^{-2}\log b + (B_0+\Gamma)b^{-2}\}$$

The other $q_n$ are all zero. As mentioned above, n is odd, the coefficients of $\sigma^2$ must vanish in eq.(43), i.e.

$$C_2 = 0 \tag{45}$$

Hence, comparing the coefficient of $\sigma^n$ on the both sides of eq.(43), one obtains

$$(-\lambda b^{n}-a^{2n}b^{-n})A_n - (n-2)(b^{-n+2}-a^{2}b^{-n})A_{-n+2} + m(a^{2n-4}b^{-n}+\lambda b^{n-4})A_{n-2} + mn(a^{-2}b^{-n}-b^{-n-2})A_{-n} = b^{-n}q_n \tag{46}$$

where the coefficient of log σ in eq.(43) vanishes owing to the uniqueness of rotation (see eq.(39)). In order to determine $A_n$ in eq.(46), the perturbation method is used. Assume the power series expansion for $q_n$ and $A_n$ of the form

$$q_n = \sum_{i=0}^{\infty} q_n^{(i)}m^{i}, \qquad A_n = \sum_{i=0}^{\infty} A_n^{(i)}m^{i} \tag{47}$$

where $q_n^{(i)}$ and $A_n^{(i)}$ are independent of m. Comparing eqs.(44) with the first of eqs.(47), one obtains

$$q_1^{(0)} = RL\{(\lambda-1)b^{2}\log b + 2a^{2}\log a - b^{2} + a^{2}\}$$
$$+\{\delta_2\alpha_e E_e R/(1+\nu_e)\}\{\Gamma b^{2}\log b + (B_0-\Gamma)b^{2}\}$$
$$q_3^{(1)} = RL\{-(\lambda-1)b^{2}\log b - 2a^{2}\log a - b^{2} + a^{2}\}$$
$$+\{\delta_2\alpha_e E_e R/(1+\nu_e)\}\{-\Gamma b^{2}\log b - (B_0-\Gamma)b^{2}\} \tag{48}$$
$$q_{-1}^{(1)} = RL\{(\lambda-1)b^{-2}\log b + 2a^{-2}\log a - b^{-2} + a^{-2}\}$$
$$+\{\delta_2\alpha_e E_e R/(1+\nu_e)\}\{\Gamma b^{-2}\log b + (B_0+\Gamma)b^{-2}\}$$
$$q_1^{(2)} = RL\{-(\lambda-1)b^{-2}\log b - 2a^{-2}\log a - b^{-2} + a^{-2}\}$$

$$+\left\{\delta_2\ \alpha_e E_e R/(1+\nu_e)\right\}\left\{-\Pi b^{-2}\log b - (B_0+\Gamma)b^{-2}\right\}$$

The other $q_n^{(i)}$ are all zero. Substituting eq.(47) into eq.(46), one obtains

$$\begin{aligned}&(-\lambda b^{n}-a^{2n}b^{-n})A_n^{(i)} - (n-2)(b^{-n+2}-a^2b^{-n})A_{-n+2}^{(i)}\\ &= b^{-n}q_n^{(i)} - (a^{2n-4}b^{-n}+\lambda b^{n-4})A_{n-2}^{(i-1)} - n(a^{-2}b^{-n}-b^{-n-2})A_{-n}^{(i-1)}\end{aligned} \tag{49}$$

First, for i=0 the only non-zero term on the right hand side of eq.(49) is $b^{-1}q_1^{(0)}$, and one obtains

$$A_1^{(0)} = q_1^{(0)}/\left\{(b^2-a^2)-(\lambda b^2+a^2)\right\},\qquad A_n^{(0)} = 0 \quad (n\neq 1) \tag{50}$$

In order to derive the recurrence formula for $A_n^{(i)}$ $(i=1,2,\cdots)$, replace n in eq.(49) by -n+2, then one obtains

$$\begin{aligned}&n(b^{n}-a^2b^{n-2})A_n^{(i)} - (\lambda b^{-n+2}+a^{-2n+4}b^{n-2})A_{-n+2}^{(i)}\\ &= b^{n-2}q_{-n+2}^{(i)} - (a^{-2n}b^{n-2}+\lambda b^{-n-2})A_{-n}^{(i-1)} - (n-2)(b^{n-4}-a^{-2}b^{n-2})A_{n-2}^{(i-1)}\end{aligned} \tag{51}$$

Eliminating $A_{-n+2}$ from eqs.(49) and (51), one obtains the recurrence formula for $A_n^{(i)}$ as follows:

$$A_n^{(i)} = \left\{E_1q_n^{(i)} + E_2q_{-n+2}^{(i)} + E_3A_{n-2}^{(i-1)} + E_4A_{-n}^{(i-1)}\right\}/E \tag{52}$$

where

$$\begin{aligned}E_1 &= -\lambda b^{-2n+4}-a^{-2n+4}, \qquad\qquad E_2 = (n-2)(b^2-a^2)\\ E_3 &= (\lambda^2+\lambda a^{2n-4}b^{-2n+4}+\lambda a^{-2n+4}b^{2n-4}+1) + (n-2)^2(a/b-b/a)^2\\ E_4 &= n(-\lambda b^{-2n+2}+a^{-2n+2}+\lambda a^{-2}b^{-2n+4}-a^{-2n+4}b^{-2})\\ &\quad + (n-2)(-\lambda b^{-2n+2}+a^{-2n+2}-a^{-2n}b^{2}+\lambda a^{2}b^{-2n})\\ E &= (\lambda^2b^4+\lambda a^{2n}b^{-2n+4}+\lambda a^{-2n+4}b^{2n}+a^4) + n(n-2)(b^2-a^2)^2\end{aligned} \tag{53}$$

Obtaining $A_n^{(i)}$ from eqs.(50) and (52), $A_n$ and accordingly $\Phi(\zeta)$ are determined completely.

In the present treatment, as $\Psi(\zeta)$ is represented by $\Phi(\zeta)$ using eq.(31), it is unnecessary to obtain $\Psi(\zeta)$ concretely. The stress components expressed only by $\Phi(\zeta)$ become

$$\begin{aligned}\sigma_{\rho\rho}+\sigma_{\theta\theta} &= 4\,\mathrm{Re}\left\{\Phi'(\zeta)/\omega'(\zeta)\right\}\\ \sigma_{\theta\theta}-\sigma_{\rho\rho}+2i\sigma_{\rho\theta} &= \left[2\zeta^2/\rho^2\bar{\omega}'(\bar{\zeta})\{\omega'(\zeta)\}^2\right]\Big[\left\{\bar{\omega}(\bar{\zeta})\omega'(\zeta)-\bar{\omega}(a^2/\zeta)\omega'(\zeta)\right\}\Phi''(\zeta)\\ &\quad + \left\{(a^2/\zeta^2)\omega'(\zeta)\bar{\omega}'(a^2/\zeta)-\bar{\omega}(\bar{\zeta})\omega''(\zeta)+\bar{\omega}(a^2/\zeta)\omega''(\zeta)\right\}\Phi'(\zeta)\\ &\quad + (a^2/\zeta^2)\{\omega'(\zeta)\}^2\,\bar{\Phi}'(a^2/\zeta)\Big]\end{aligned} \tag{54}$$

Therefore, obtaining the analytically continued $\Phi(\zeta)$ and substituting it into eqs.(54), one obtains the stress components.

The numerical calculations were carried out for the cases of b=1, m=0.05 $\nu$=1/3 and a=0.4, 0.5 and 0.6. Figs.4 and 5 show the stress distributions on x-axis and boundaries, respectively, for the case of both boundaries free (case a). Figs.6 and 7 show those for the case of the inner boundary free and the outer boundary fixed (case b). For these two cases, a) and b), it is assumed that the stress conditions are plane strain conditions.

## 4. A POLYGONAL REGION WITH A CONCENTRIC CIRCULAR HOLE

Consider the polygonal regions with a circular hole at the center as shown in Fig.8. Let the number of sides of a polygon p, the radii of the circular hole and the inscribed circle of the outer polygonal boundary be a and b respectively. Futhermore, the temperatures of the inner and outer boundaries are assumed to be $T_a$ (=constant) and 0 respectively. Then the temperature distribution in the region has the same symmetrical character as the outer polygonal boundary. Hence, we may consider the region, enclosed by the radial lines $\theta=0$, $\pi/p$ and the inner and outer boundaries, as the principal region. Owing to the above-mentioned symmetrical characters, the temperature $\tau$ may be written

$$\tau = a_0 + \gamma \log r + \sum_{n=p,2p,\cdots} (a_n r^n + b_n r^{-n}) \cos n\theta \tag{55}$$

where r and $\theta$ are the polar coordinates as shown in Fig.8. From the boundary condition for the inner circular boundary, one obtains

$$a_0 = T_a - \gamma \log a, \qquad a_n = -b_n a^{-2n} \quad (n=p,\ 2p,\cdots) \tag{56}$$

Hence, substituting eq.(56) into eq.(55), one obtains

$$\tau = T_a + \gamma \log(r/a) + \sum_{n=p,2p,\cdots} b_n r^n (r^{-2n} - a^{-2n}) \cos n\theta \tag{57}$$

In order to determine the coefficients $\gamma$ and $b_n$ the point-matching method is used. Replacing $\sum^{\infty}$ by $\sum^{mp}$ approximately, eq.(57) contains m+1 unknown coefficients and considering the boundary conditions at m+1 points on the outer boundary, we obtain m+1 equations for m+1 unknown constants. Solving these m+1 equations, the coefficients $\gamma$, $b_n$ and the approximate formula for the temperature $\tau$ may be determined. Then from eq.(4)

$$\begin{aligned} u_\tau + iv_\tau = \alpha_e \int (\tau + ih)dz = \alpha_e \Big[ & \gamma z \log(z/a) + (T_a - \gamma) z + i a_0' z + c_0 + i c_0' \\ & + \sum_{n=p,2p,\cdots} \{ b_n\ z^{-n+1}/(-n+1) - a^{-2n} z^{n+1}/(n+1) \} \Big] \end{aligned} \tag{58}$$

where $a_0'$, $c_0$ and $c_0'$ are real constants. As these constants are concerning to the rigid body displacements, we may assume for them to be zero. Comparing eqs.(6) and (58) to each other, one obtains

$$L^* = \alpha_e \gamma \qquad M^* = 0 \tag{59}$$

Next, as in the previous section, let us treat the problems for two kinds of boundary conditions, i.e. a) the case of both boundaries free and b) the case of the inner boundary free and the outer boundary fixed en bloc. As the inner circular boundary is free, $\varphi(z)$ may be continued analytically into the inverse region of S and the boundary conditions on r=a are satisfied identically. The boundary conditions for the outer polygonal boundary may be obtained substituting $z=\omega(\zeta)=\zeta$ into eq.(40) as follows:

$$\lambda\varphi(z) + \varphi(a^2/\bar{z}) + \{(a^2/\bar{z}) - z\}\bar{\varphi}'(\bar{z}) = -\delta_1 C_2 - \delta_2 E_e (u_\tau + iv_\tau)/(1+\nu_e) \tag{60}$$

The single-valuedness of rotation and displacements and the equilibrium of forces and moment for the inner boundary are assured by assuming eqs.(14), (15), (16) and (17) to be equal to zero. Using $M^*=0$ from the second eqs.(59), one obtains

$$L = -E_e L^*/4 = -\alpha_e E_e \gamma/4, \qquad M = N = 0, \qquad \mathcal{I}m\{K\} = 0 \tag{61}$$

From the symmetrical character of the present problem, $\varphi(z)$ is written

$$\varphi(z) = Lz \log z + \sum_{n=0,\pm p,\pm 2p,\cdots} \alpha_{n+1} z^{n+1} \tag{62}$$

where L and $\alpha_{n+1}$ $(n=0,\pm p,\pm 2p,\cdots)$ are real constants.

Substituting eq.(62) into eq.(60), one obtains

$$\sum_{n=0,\pm p,\pm 2p,\cdots} \alpha_{n+1}\left\{(-\lambda r^{2n+2}-a^{2n+2})/\bar{z}^{n+1} + (n+1)(r^2-a^2)\bar{z}^{n-1}\right\}$$
$$= (L/\bar{z})\left\{\lambda r^2 \log z - r^2 \log \bar{z} + 2a^2 \log a - r^2 + a^2\right\}$$
$$+ \delta_1 C_2 + \delta_2\left\{E_e/(1+\nu_e)\right\}(u_\tau + iv_\tau) \tag{63}$$

where $z=r\exp(i\theta)$ and $\delta_1$, $\delta_2$ and $\lambda$ are the same as previous section.

Putting $\delta_1=1$, $\delta_2=0$ and $\lambda=1$ for the case of the outer boundary free, from the consideration of $\theta=0$ and $\pi/p$, one obtains

$$C_2 = 0 \tag{64}$$

In order to determine $\alpha_{n+1}$ from eq.(63), the point-matching method is used. Hereafter, let us replace $\sum_{n=0,\pm p,\pm 2p,\cdots}$ in eq.(63) by $\sum_{n=-Np,\cdots,-p,0,p,\cdots(N-1)p}$ then, eq.(63) contains 2N unknown coefficients $\alpha_{n+1}$. Considering eq.(63) for N+1 points on the outer boundary with argument $\theta=n\pi/Np$ $(n=0,1,2,\cdots,N)$ and taking the real and imaginary parts of eq.(63), 2N equations are obtained since the imaginary part of eq.(63) is satisfied identically for $\theta=0$, $\pi/p$. Solving these 2N equations for 2N unknown coefficients $\alpha_{n+1}$, $\varphi(z)$ is determined and the stress and displacement components may be obtained by eqs.(54) and (33) for $z=\omega(\zeta)=\zeta$.

The numerical calculations were carried out for the cases of p=3, 4, 6 and 8. In these calculations, the temperature functions $\tau$ were obtained, adopting m=9 and solving 10 simultaneous equations for $\gamma$ and $b_n$. The complex potentials $\varphi(z)$ were obtained, adopting N=14 and solving 28 simultaneous equations for $\alpha_{n+1}$. The typical examples of stress distribution are shown for p=4 in Figs.9, 10 and 11. Fig.9 shows the principal stresses on the x-axis for the case of both boundaries free and Fig.10 shows those for the case of the inner boundary free and the outer boundary fixed. Fig.11 shows the distribution of tangential stress along the circular boundary. Fig.12 shows the maximum stress for the number of sides of the polygon. Furthermore for comparison, the results from the former research by one of the authors and others for the case of both boundaries free and a=0.25, 0.5 are shown in Fig.9, and we can see the present results agree with the former ones fairly well.

Lastly, for the estimation of accuracy of the point-matching method, the values of $\sigma_{xx}$ and $\sigma_{xy}$ for the case a) and u, v for the case b), which must vanish along the outer boundary, are shown in Tables 1 and 2. From these tables, it may be seen that the accuracy of the present calculations is satisfactory for practical applications.

## 5. CONCLUSION

As another method to the former research, the method of analytic continuation proposed by Buchwald and Davies for isothermal two-dimensional elasticity is applied to the thermoelastic problems. Using this method, the plane thermoelastic problems of doubly-connected regions may be treated only by the Laurent series for one of two complex potentials. The boundary conditions on one of the boundaries may be satisfied without expanding the other complex potential into the Laurent series. Furthermore, the use of complex functions enables us to treat the stress boundary conditions and displacement boundary conditions en bloc.

The numerical examples were shown for the elliptic region with a confocal elliptic hole and for the polygonal region with a circular hole. In the latter example, the results of the present paper were compared with the former results and it was assured that the both results agree fairly well with each other. Lastly, the accuracy for the point-matching method was estimated by the normal and tangential stresses and the displacements along the outer polygonal boundary for the case a) and b) respectively, and it was assured that the results of the calculation were satisfactory for practical applications.

## ACKNOWLEDGMENT

The authors wish to express their gratitude to Mr. T. Katayama for his suggestion in computer programing.

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Table 1

$\sigma_{xx}$ and $\sigma_{xy}$ along outer boundary (p=4)

| y | $\sigma_{xx}/\alpha ET_a$ | | | $\sigma_{xy}/\alpha ET_a$ | | |
|---|---|---|---|---|---|---|
| | a=0.25 | a=0.5 | a=0.75 | a=0.25 | a=0.5 | a=0.75 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.25 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.5 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.75 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0.001827 | 0.001050 | 0.000257 | 0.000052 | -0.001393 | -0.001740 |

(Absolute values smaller than $5X10^{-7}$ entered as 0)

Table 2

u and v along outer boundary (p=4)

| y | $u/\alpha T_a$ | | | $v/\alpha T_a$ | | |
|---|---|---|---|---|---|---|
| | a=0.25 | a=0.5 | a=0.75 | a=0.25 | a=0.5 | a=0.75 |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.25 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.5 | 0 | 0 | 0 | 0 | 0 | 0 |
| 0.75 | 0.000001 | -0.000004 | -0.000003 | 0.000001 | -0.000003 | -0.000002 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 |

(Absolute values smaller than $5X10^{-7}$ entered as 0)

## List of figures

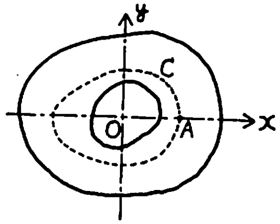



FIG. I

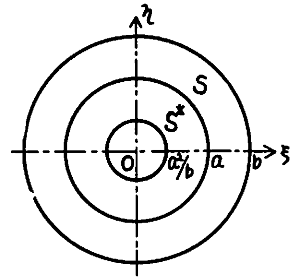



FIG. 2

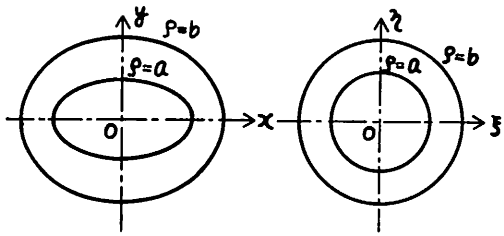
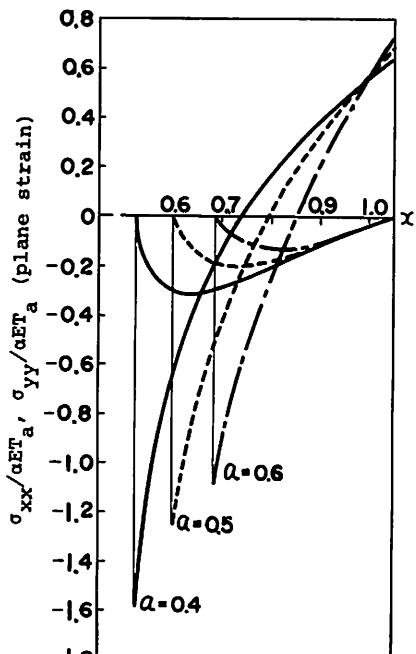



FIG. 3

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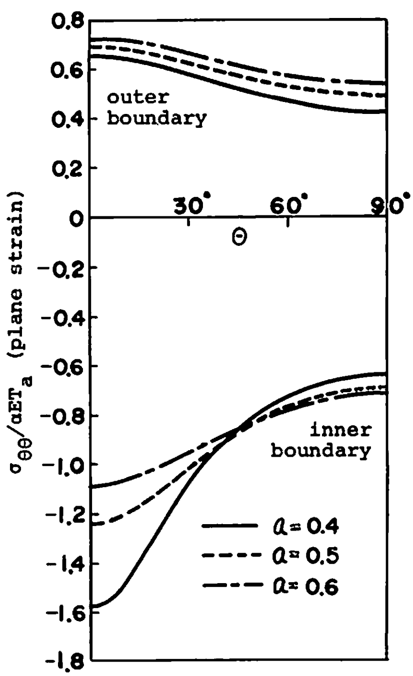


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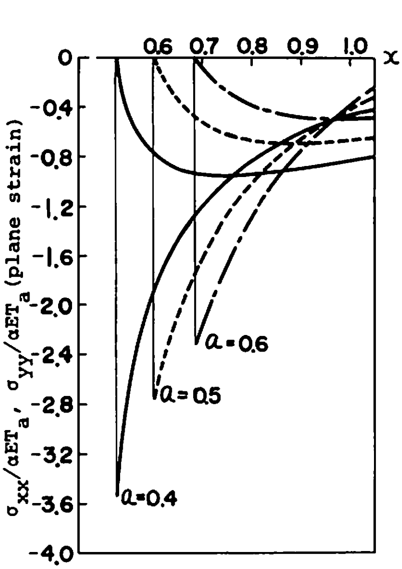


FIG. 6

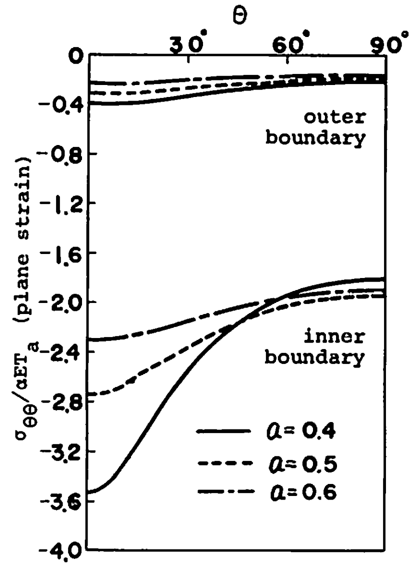


FIG. 7

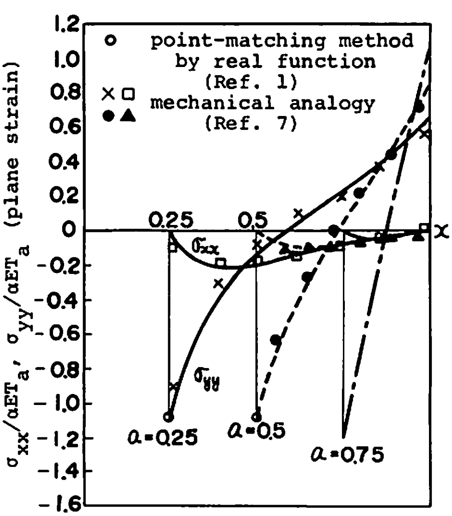


FIG. 9

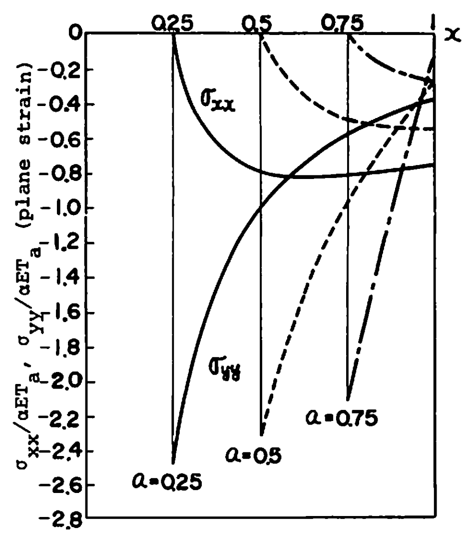


FIG. IO

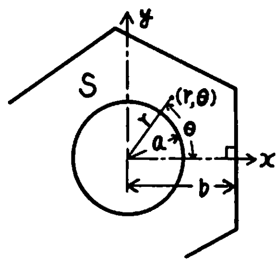


FIG. 8

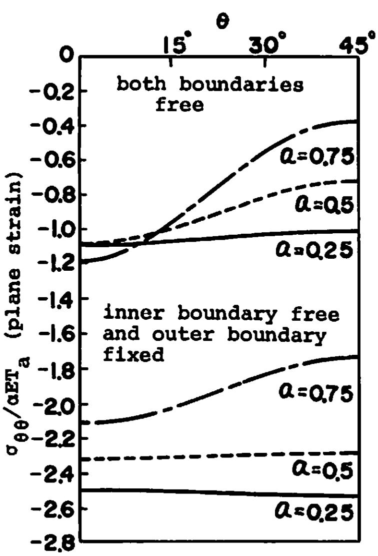


FIG. II

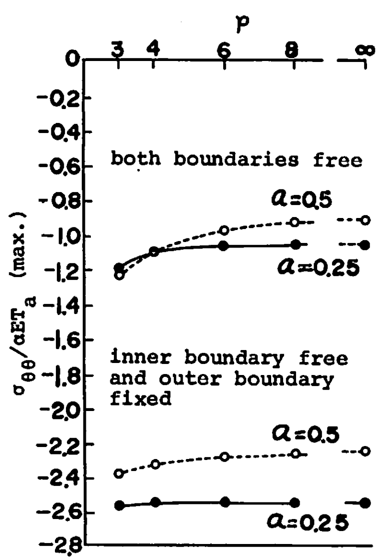


FIG. I2

## DISCUSSION

**Q**

A. KRAWIETZ, Germany

Analytic continuation is a fine method for treating such problems. But I think that the function $\omega(\zeta)$ is restricted to be rational. Can you deal with an arbitrary mapping function ?

**A**

T. SEKIYA, Japan

In the case of an arbitrary mapping function which is not rational, we are better to replace the function by an approximate mapping function which is rational.