ABSTRACT

MCGOVERN, EMILY CATHERINE. Combinatorial Techniques for Module Categories over Positive Characteristic Web Categories . (Under the direction of Corey Jones).

In this dissertation, we construct and study module categories over web categories in positive characteristic fields. These categories appear in the study of fundamental representations of Lie algebras and quantum groups, and were introduced by Kuperberg in [Kup96]. We are particularly interested in a modification of the $\mathfrak{sl}_n$ spiders of [CKM14] in positive characteristic that appears in [BEAEO20]. Our main goal is to generalize the fiber functors from these categories constructed by C. Jones in [Jon21].

We equip the category of vector bundles over the vertices of a locally finite $\tilde{A}_{n-1}$ building Δ with the structure of a module category over a category of type A_n webs in positive characteristic. This module category is a q-analogue of the $Rep(SL_n)$ action on vector bundles over the sl_n weight lattice. We show our module categories are equivariant with respect to symmetries of the building, and when a group G acts simply transitively on the vertices of Δ this recovers the fiber functors constructed in [Jon21].

We also begin an extension of this theory to type C. In particular, we construct a graph related to the degenerate building in type $\tilde{C}_2$. We then show that a slight modification of the $\mathfrak{sp}_4$ category presented in [Bod22] into the graph planar algebra ([Jon99]) of this graph. We hope that this embedding can be extended to $n > 2$ using the web categories of [BERT21].

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Combinatorial Techniques for Module Categories over Positive Characteristic Web Categories

by
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DEDICATION

To my late grandfather, Dr. Victor Dillon. Thanks for the tough love, wisdom and unconditional support. If I touch half as many people with this math degree as you did with your orthodontics, it will be worth it.

BIOGRAPHY

Emily McGovern was born and grew up in Frederick County, Maryland. She was homeschooled until she left for college in 2015, and because of this was able to explore her natural aptitude and enjoyment for math at her own pace. Through middle and high school Emily participated in MathCounts, and other similar math competitions, which further developed her passion for problem solving and introduced her to the world of discrete mathematics. She dual-enrolled in Frederick Community College to complete many college level math classes while still in high school, and discovered a love for proof writing through her linear algebra class.

Emily started school at University of Maryland, Baltimore County in 2015. It was at UMBC where she was first exposed to abstract algebra, combinatorics and other mathematical topics that would push her to pursue graduate work in mathematics. She spent her (almost non-existent) free time working as a teaching assistant for the computer science department, and through this job discovered a love for teaching. While in college, Emily had her first taste of research through participating in the SURF program at the National Institute of Standards and Technology, and also did an REU at Missouri State University. She graduated Summa Cum Laude in May 2019 with double degrees in Mathematics and Computer Science.

In September 2019, Emily started the graduate program at North Carolina State University. She earned her Masters of Science in mathematics in May 2021. In Spring 2021, Emily started research with her advisor Corey Jones. Through her time at NC State, she has participated in the Algebra and Combinatorics Seminar, Teaching and Learning Seminar, and Pure Mathematics Graduate Seminar. She has taught various math courses as recitation leader and instructor of record. Emily has most enjoyed being able to travel to various conferences, present her research and network in her field. After graduating with her PhD, Emily will be starting a postdoctoral position at University of Oregon, and hopes to find a permanent job at a small, undergraduate focused institution.

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As I gather my thoughts to write this, I know that I would have not made it to this day if not for the grace of God. To Him be all the glory. Writing my acknowledgements overwhelms me because if I were to fully detail the impact of the people in my life on my journey in mathematics, I'd have enough content for entire thesis chapter. I guess the following is the abstract for that imaginary chapter.

I know for certain that I would not have not gotten to this point without the guidance and support of my advisor, Corey Jones. To Corey, thank you for helping me come up with research ideas, for introducing me to your professional network and for helping me get through the toughest weeks of grad school. I will always appreciate the time you spent to help me succeed. Thanks for trusting me to be your first graduate student, I hope I made you proud.

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To my parents, Mark and Stephanie, thank you for your undying support and unconditional love. As I get older, I only get more appreciative of the sacrifices you made so I could pursue my passions. Thanks for always picking up the phone and listening to my worries about school, life and whatever else. Thanks to my siblings, Abby, Kelly, and Curt, who love and support me as only siblings can.

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Thanks also to all who have been influential in my journey as a mathematician by providing support and advice over the years, especially Mark and Christine Pritt, Val Lochmann, Les Reid and Adam Telatovich. A special shoutout to Dr. Keith Pardue, who challenged me to grow in mathematical maturity through his classes, and Dr. Katherine Gibson, who has greatly influenced my philosophy as a teacher. To my math friends at UMBC, especially Sean Owen, Puja Trivedi and Taylor Webb, thanks for keeping me sane through long homework sessions.

My time at NC State has in part been so great because of the friends that I have made in the math department, and I just want to mention a few of those people here - although I am appreciative of all. First, thank you to Jessica Stevens for being my steadying force through graduate school during a pandemic. I very well may have dropped out without our lunch time Zooms and Friday night ice creams. Thanks also to Corey's research group (Sean, Kylan, Alex, Noah and Ian), our Thursday meetings have become a highlight of my week. Finally, thank you to Spencer Daugherty for helping me through qual classes, job applications and everything in between.

I'd be remiss not to thank my friends in Raleigh who've made this city feel like home. I won't name anyone, for fear I'll forget some, but you know who you are. Thanks for always providing entertainment and friendship to help me get my mind off of math every once in a while. I will always remember ultimate frisbee Sundays, game night shenanigans, book clubs and Bible studies. It's because of you that the thought of leaving Raleigh hurts so much.

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CHAPTER

1

INTRODUCTION

The goal of this dissertation is to construct a class of nonstandard module categories for web categories (also known as spiders). Our constructions will generalize the fiber functors of [Jon21]. We will use a categorification of a combinatorial structure called an affine building to give these module categories their nonstandard nature. We will show that over certain positive characteristic fields, these buildings locally satisfy the relations of the appropriate web categories. Additionally, given a group action on a building, we can construct another module category using the process of equivariantization.

We will give a complete treatment of this construction in type A. In type C, we will give an introductory result and discuss the process of generalizing it. In the rest of this chapter, we will give context to the problem we are tackling by discussing the history of web categories and buildings. Then, in section 1.2, we will outline the rest of the dissertation and state our main results.

1.1 History and Context

Web categories, or spiders, are diagrammatic presentations of certain representation theories, particularly those of Lie algebras. The first instance of these is the A_1 spider, capturing the representation theory of $\mathfrak{sl}_2$. This spider is the well-known Temperley-Lieb algebra $TL_n(\delta)$, introduced in [TL71]. This algebra of transfer matrices is generated by $e_1, ..., e_{n-1}$, subject to the relations

$$e_i^2 = \delta e_i, \quad e_j e_i e_j = e_i, \text{ for } |i-j| = 1, \text{ and } e_i e_j = e_j e_i \text{ for } |i-j| > 1,$$

which are explicitly stated in [Bax82]. V. Jones showed that $TL_n(\delta)$ is finite dimensional over any field in [Jon83]. The first diagrammatic presentation of this algebra was given by Kauffman in [Kau90]. For more information on the conception and early history of the Temperley-Lieb algebra, see [DG23].

The study of spiders in general was initiated by Kuperberg in his paper [Kup96]. In it, he outlined the importance of spiders to multi-linear invariant theory, a field which had regained interest after the introduction and popularization of quantum groups. Kuperberg defined spiders as spherical categories that are abstractions of some representation theory, with the operations of tensor product, cyclic permutation/rotation and contraction. Such a representation theory can be described with planar graphs. He then completely defined the notion of spiders for the rank two Lie algebras (that is, A_2, B_2 and G_2) by giving generators and relations for these spiders and showed that they are isomorphic to the respective representation categories. In addition, he defined spiders for the quantum deformations for these Lie algebras. Kuperberg also conjectured that analogous spiders exist for all simple Lie algebras and their quantum groups, which established the research program of determining these spiders for higher rank representation theories.

Over the first decade of research to find these spiders, several partial results were established. Kuperberg's student Kim took on the task of generalize Jones-Wenzl projectors, a special class of idempontents in $TL_n(\delta)$ (see [Jon87], [Wen87]), to other Lie algebras. He developed a theory for similar idempotents, called clasps in the general case in $U_q(sl(3, \mathbb{C}))$. Along with this, he conjectured a set of web relations on $U_q(sl(4, \mathbb{C}))$ (these conjectures and results can be found in [Kim03] and [Kim07]). Later, Westbury ([Wes08]) provided a diagrammatic description for the invariant tensors of $\mathfrak{so}(7) = B_3$. Finally, in his PhD thesis, Morrison ([Mor07]) took the first steps to providing a description for webs in every type A Lie algebra by constructing diagrammatics and defining a functor from the diagrammatics to the subcategory of fundamental representations of $U_q(\mathfrak{sl}_n)$. Additionally, Morrison developed methodology to identify elements of the kernel of this functor, but could not completely determine the kernel of this functor, thus leaving the problem open.

The breakthrough results in type A came in the paper [CKM14], which provided a complete description of $Rep(SL_n)$ as a pivotal tensor category with generators and relations. Building off the functor described in [Mor07], objects are tensor powers of fundamental representations. The addition of a set of relations on all webs with less than 4 boundary edges defining the kernel of the functor allowed the authors to construct the desired monoidal equivalence. The main tool in proving this result was skew Howe duality, a technique where relations in $Rep(SL_n)$ are realized as truncation of relations $U(\mathfrak{gl}_m)$ for a sufficiently large m. The authors also deformed the SL_n spider by q to get a diagrammatic presentation of the category of $U_q(\mathfrak{sl}_n)$-modules. In addition, they noted that these webs can be used to compute quantum knot invariants by taking advantage of the braided monoidal structure of $Rep(SL_n)$, providing alternate techniques to find the results of [MOY98] on closed webs.

Further work in type A has included Elias's work in [Eli15] to extend the results in [CKM14] to webs over $\mathbb{Z}[q, q^{-1}]$ (the latter defined webs over $\mathbb{Q}(q)$ by constructing a $\mathbb{Z}[q, q^{-1}]$-basis for the appropriate morphism spaces. This paper also worked towards recovering $Rep(SL_n)$ from $Fund(SL_n)$, proving several results about the structure of clasps on SL_n. Finally, Elias made a connection between webs and tilting modules, stating that $\mathfrak{sl}_n$ webs are an integral form for tilting modules over the quantum group.

In [Big18], Bigelow provided a new definition for the SL_n spider and showed that it contains the spider of [CKM14]. He noted that the previous set of relations may seem somewhat arbitrary to a reader, and motivated his definition using the combinatorics of the root system in type A. Bigelow's webs (which he calls cobwebs), are built of bivalent vertices labeled by roots $\alpha_{i,j}$ and tags on strings and are extended to webs with virtual crossings. Bigelow conjectures that this new definition is related to the graph planar algebra (see [Jon00] and later discussion) with paths in the root lattice of $\mathfrak{sl}_n$.

The results of quantum deformation on webs is known and often developed at the same time as the original theory of webs as in [Kup96] and [CKM14]. The webs we have previously discussed are defined over characteristic 0 fields (almost always $\mathbb{C}$). In [BEAEO20], the authors looked at a new type of deformation for webs, namely, defining the category over a positive characteristic field. The goal of this paper was to describe the semisimplification of titling modules of GL_n over a field of characteristic p. The authors did this for $0 < p < n$ by using the p-adic decomposition of n to decompose the semisimplification of $Tilt(GL_n)$, extending results of [GK92], [GM94] and [EO22]. An important tool in the proof of this results is the Schur category [BEAEO20, Definition 4.2], the diagrammatic presentation ([BEAEO20, Theorem 4.10] of which is a category called $PolyWeb(GL_n)$, which we will interpret as the type A web category in positive characteristic. The authors gave credence to this interpretation by constructing a full monoidal functor from the Schur category to $Tilt(GL_n)$.

In [Jon21, Section 3.2], C. Jones continued the development of webs in positive characteristic by defining a category $\mathrm{Web}(\mathrm{SL}_n^-)$ that is a monoidal quotient of [BEAEO20]'s $PolyWeb(GL_n)$ and showing that it is equivalent to $Tilt(SL_n)$ over an algebraically closed field. The main purpose of the paper was to define a class of fiber functors from $\mathrm{Web}(\mathrm{SL}_n^-)$ using Cartwright's ([Car95]) triangle presentations (We will discuss the details of this construction in later sections as it is key to the main results of this dissertation). These fiber functors also generate positive characteristic solutions to the Yang-Baxter equation ([Jon21, Section 4.1]).

After the landmark results of [CKM14], work has been done to find the corresponding web categories for other Lie types. The most attention has been paid to type C, corresponding to the Lie algebra $\mathfrak{sp}_{2n}$. The first attempt at specifying these categories was by [ST19]. The authors strategy in the paper was to use an analogue of Howe duality in other types, and were able to construct a framework for webs that answered questions about what this analogue is in types BCD. However, their approach made some concessions. In particular, the quantization of the enveloping algebra used to construct the dualities is not the standard one.

In [RT22], the authors defined a set of a relations for the category $\mathrm{Web}(\mathrm{SL}_6)$ and conjectured that it is equivalent to the fundamental representations of sp_6, constructing a full, essentially surjective functor to the fundamental representation category. The case of $\mathfrak{sp}_4$ was completed in [Bod22], which uses the type C analogue of the light ladders introduced in [Eli15] to find a diagrammatic presentation for type B_2 slightly different than the original B_2 spider defined by Kuperberg. Finally, [BERT21] provided a complete description of spiders in type C, showing that their webs are equivalent to the subcategory of fundamental representations of $\mathfrak{sp}_{2n}$. They prove this using Brauer-Schur-Weyl duality, a type C analogue of Schur-Weyl duality in which they student a certain full subcategory of $FundRep(\mathfrak{sp}_{2n})$. While this is certainly the most complete extension of [CKM14] to other types, [SW22] gave an diagrammatic presentation of the invariant tensors of exceptional type F_4.

The other major tool we use in this dissertation is a building. We will define these structures in Section 2.2, but we now present a brief history of their conception. For a more thorough treatment of buildings both historically and mathematically, see [AB10], [Bro89] and [Ron09].

Buildings were pioneered by Tits as a geometric way of understanding semisimple complex and p-adic Lie groups (and later algebraic groups). In [Bru54], Bruhat used the existing notions of a Weyl group ($W = N/T$ for T a maximal torus and N its normalizer) and Borel subgroup B of a complex Lie group and defined the Bruhat decomposition of a Lie group. This approach was later refined by Chevalley ([Che55] [Che05]) by finding a unified approach to proving the Bruhat decomposition for all types. From this work and his work on geometries in the 1960s, Tits was able to develop the initial axioms for buildings, which appear in [Tit62].

Tits continued to develop this theory throughout the 1970s (see [Tit95] for the initial definition). In [Tit74], he gave a full account of the theory to that point, taking a simplicial approach to buildings. This initial idea of a building was to view it as simplicial complex with apartments made of chambers. This book ([Tit74]) carefully detailed the structure of spherical buildings, that is, those buildings coming from semi-simple Lie groups. For several Lie groups (A_n, C_n, D_n, E_n, F_4) it gave explicit descriptions of these buildings as geometries.

Further development of the field went in two directions. The first of these was a re-imagining of the original definition of buildings. Tits developed an alternate paradigm for buildings, viewing them as chamber systems and forgetting the system of apartments ([Tit92]). This led to the introduction of twin buildings - whose key pieces are chambers and a Weyl-group valued distance function. A third approach to buildings is the study of the geometric realization of a building as a metric space, which leads to some nice properties. This is a natural approach for spherical and affine buildings (which have an Euclidean metric on apartments), but was show to work in general by M. Davis in [Dav98].

The second direction for the development of buildings was to study different types of groups and the resulting buildings. This originated from the work of the authors of [IM65] with p-adic groups, but was also studied for groups produced by Kac-Moody Lie algebras, resulting in Kac-Moody buildings, by [MT72]. A p-adic groups can be used to construct an affine building. Unlike spherical buildings, affine buildings are infinite simplicial complexes. Bruhat and Tits developed the general theory of affine buildings in [BT72].

Affine buildings have many interesting applications. Among these, MacDonald used them in [Mac71] to study spherical functions on p-adic groups. Borel and Serre used affine buildings to prove results about arthimetic groups in [BS76] and [Ser03]. Finally, Quillen used them in his work on K-groups of a curve, later recorded by Grayson in [Gra82]. The later chapters of this dissertation will introduce another interesting application of affine buildings.

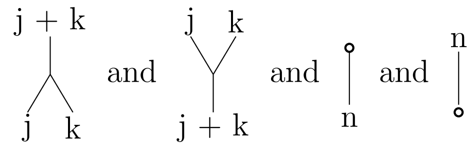
The last tool that we wish to provide context for are graph planar algebras. Planar algebras were introduced by V. Jones in [Jon99] as a tool in subfactor theory. Specially, planar algebras (defined abstractly as invariant tensor powers of a bimodule) aided in calculating subfactors by providing another axiomization of the standard invariant of finite index subfactors. Graph planar algebra were defined in [Jon00], and associate to a planar algebra a bipartite graph, creating a model using statistical mechanical sums defined by a labelled planar tangle. These graphs are usually finite, but [GJS10] constructed planar algebras for infinite graphs. This paper also noted that graph planar algebras give connections between subfactors, random matrices, and free probability theory. We will use graph planar algebras as a way to categorify the data found in an affine building in this dissertation.

1.2 Contents and Selected Results

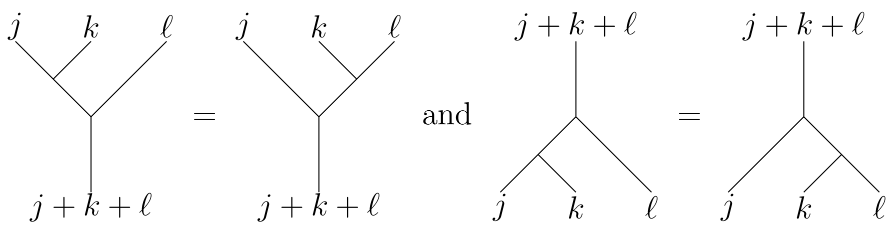
We will now briefly outline the contents of this dissertation and provide an overview of the results to follow. In Chapter 2, we rigorously develop the underlying theory of the structures key to our results. We separate the background material into two sections. In Section 2.1, we introduce categorical background needed to understand module categories. This includes definitions of monoidal categories, rigid monoidal categories, Abelian categories and tensor categories, with examples and important facts included throughout the section. The main goal of the section is to justify our understanding of module categories as a categorification of R-modules for a ring, and it concludes with the definition of both a module category and a module functor.

Section 2.2 treats the combinatorial structures needed to understand the theory of affine buildings. We start with the basics of Coxeter systems and simplicial complexes, before joining these concepts by defining a Coxeter complex. While we make all definitions in general, we are specifically interested in types A_{n-1} and $\tilde{A}_{n-1}$, and provide examples using these systems throughout. Finally, we define buildings, and provide detailed examples of both finite and affine buildings in Type A.

In Chapter 3, we build a class of $\mathrm{Web}(\mathrm{SL}_n^-)$-module categories using the structure of type $\tilde{A}_{n-1}$ buildings. These module categories generalize the fiber functors of [Jon21] (we show explicitly why this is the case in Example 3.41). The category $\mathrm{Web}(\mathrm{SL}_n^-)$ ([BEAEO20], [Jon21]) is the rigid monoidal category with objects finite sequences of integers in $\{1, ..., n\}$ and morphisms generated by



subject to the following relations:



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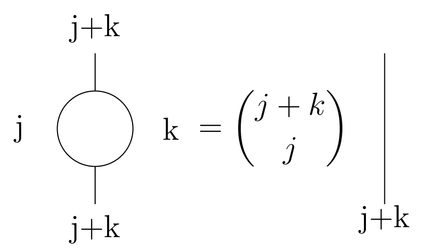

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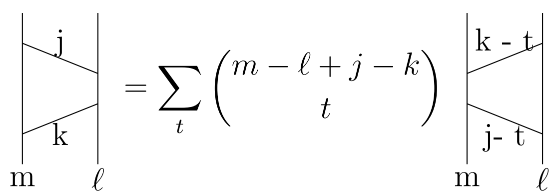

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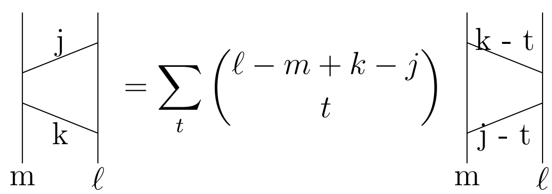

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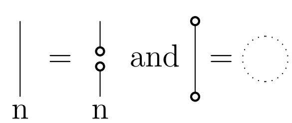

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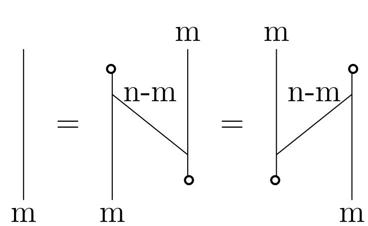

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Our module category construction happens in two steps. In the first, we take advantage of the local structure of buildings (in particular, a result of Tits in [Tit74] that the link of a vertex in a type $\tilde{A}_{n-1}$ building is a finite projective geometry). We use the 1-skeleton of a locally finite $\tilde{A}_{n-1}$ building Δ to build a graph Γ_Δ. We then construct the graph planar algebra ([Jon99]) of Γ_Δ and define images for the $\mathrm{Web}(\mathrm{SL}_n^-)$ morphisms in $G(\Delta)$.

An important invariant of a building Δ of type $\tilde{A}_{n-1}$ is its order q (the order of the finite projective geometry describing the links of its vertices). We first look at the degenerate case, where $q = 1$. In this case, the building Δ is simply the Coxeter complex Σ of type $\tilde{A}_{n-1}$. In Section 3.3.2, we first show that the 1-skeleton of the Coxeter complex is equivalent as a graph to both the Cayley graph of the $\mathfrak{sl}_n$ weight lattice and the triangle presentation (see [Car95] and [Jon21]) of order 1 (also called the degenerate triangle presentation). More accurately, we first show in Proposition 3.18 that the weight lattice and triangle presentation are isomorphic, and that the triangle presentation and Coxeter complex are isomorphic in Theorem 3.19. The most important result of this section is Theorem 3.21.

Theorem 3.21. *If Γ is the sl_n weight lattice, then there exists a functor $\mathrm{Web}(SL_n^-) \to G(\Gamma)$ as defined in section 3.3.1 and [Jon21, Section 4], where the relevant categories may be defined over any field $\mathbb{k}$.*

This result is important since it is the only known case where this functor exists for a field $\mathbb{k}$ of any characteristic (In both [Jon21] and our later results, we must put restrictions on the characteristic). In Section 3.3.3, we extend Theorem 3.21 from Coxeter complexes to buildings. To properly set up this functor, we recall that $\mathrm{Web}(\mathrm{SL}_n^-)$ is defined over some field $\mathbb{k}$ with characteristic p. We choose this field so that $p \geq n+1$, and take a $\tilde{A}_{n-1}$ building Δ where the order q of Δ satisfies $q \equiv 1 \mod p$. This choice allows us to reduce q-integers to normal integers, and is the key to proving that the embedding of $\mathrm{Web}(\mathrm{SL}_n^-)$ in the graph planar algebra $G(\Delta)$ defined in Section 3.3.1 is a monoidal functor. We then prove the second important result of the chapter, Theorem 3.24.

Theorem 3.24. *Suppose Δ is an $\tilde{A}_{n-1}$ building of order q and $G(\Delta)$ is the graph planar algebra of Δ over a field $\mathbb{k}$ of characteristic $p \geq n-1$ where $q \equiv 1 \mod p$. Then the maps we defined as the images of the $\mathrm{Web}(SL_n^-)$ maps in the graph planar algebra satisfy the $\mathrm{Web}(SL_n^-)$ relations. Therefore these maps define a functor from $\mathrm{Web}(SL_n^-)$ to the graph planar algebra of Δ over $\mathbb{k}$.*

With this result in hand, we can turn to the second step of defining $\mathrm{Web}(\mathrm{SL}_n^-)$. In this step, we use the action of a group G on a $\tilde{A}_{n-1}$ building Δ to construct a class of module categories for each building. To do this, we first define the category $Vec(\Delta)$ of Δ-graded vector spaces, whose object are tuples of vector spaces indexed by the vertices of Δ. We define a special class of functors $F_m, m \in \{0, ..., n-1\}$ and show that natural transformations between compositions of these functors can be interpreted as morphisms in $G(\Delta)$. From this, we construct a functor from $G(\Delta)$ to $End(Vec(\Delta))$, which allows us to extend Theorem 3.24 to a monoidal embedding $\mathrm{Web}(\mathrm{SL}_n^-) \to End(Vec(\Delta))$, thus giving $Vec(\Delta)$ the structure of a $\mathrm{Web}(\mathrm{SL}_n^-)$-module category.

The final step is to choose a group G acting on Δ and re-define the F_m functors in $Vec(\Delta)^G$, the G-equivariantization of $Vec(\Delta)$. We then show that the construction of a $\mathrm{Web}(\mathrm{SL}_n^-)$ module category above is compatible with the G-action in a way which allows us to extend the module category structure to $Vec(\Delta)^G$, giving us Theorem 3.40.

Theorem 3.40. *For a fixed n, if $\Bbbk$ is a field of characteristic $p \geq n-1$ and Δ is an $\tilde{A}_{n-1}$ building of order $q \equiv 1 \mod p$, $Vec(\Delta)^G$ has the structure of a $\mathrm{Web}(SL_n^-)$-module category, where the action is by the equivalence in Theorem 3.31 pre-composed with the functor defined in Theorem 3.24.*

Nothing in our construction of these module categories suggests that our results should be limited to type A, and so our hope is that we can find similar module categories over web categories in other types. Chapter 4 of this dissertation includes the initial results of attempting to build a module category for a Type C web category by finding an embedding for [Bod22]'s C_2 webs into some graph planar algebra. Our first step to defining this embedding is noting that the $\tilde{C}_2$ Coxeter complex is isomorphic to the $\mathfrak{sp}_4$ weight lattice. Because of this, we can use $\mathfrak{sp}_4$ representation theory to label the edges of the Coxeter complex.

Our main concern here is the lack of symmetry in this simplicial complex. To work around this, instead of constructing a graph planar algebra on the $\tilde{C}_2$ Coxeter complex, we create a new graph we call the partial completion of the $\tilde{C}_2$ Coxeter complex. We define the graph planar algebra on this graph exactly as we do in Chapter 3.

To define an embedding functor η into $G(\Gamma_{\overline{\Gamma}_{\tilde{C}_2}})$, we first define $\mathrm{Web}(\mathrm{SP}_4^-)$ as the category we get when we take $q = -1$ in Bodish's relations (note that if we are able to replicate this process for buildings, we can define the web category over a field of characteristic p where $q \equiv -1 \mod p$). One major difference in this embedding functor is that we must fix an orientation for this graph, as our definition of the images of the web generators in $G(\Gamma_{\overline{\Gamma}_{\tilde{C}_2}})$ depend on this orientation. This construction allows us to prove the following theorem.

Theorem 4.12. *The functor η as defined in Section 4.3 is a monoidal functor and thus provide an embedding of* $\text{Web}(SP_4^-)$ *into* $G(\bar{\Gamma}_{\tilde{C}_2})$.

While this is certainly a weaker result than we have achieved in type A, it gives us some hope that an more general embedding functor exists. The main issues in type C are that the labeling of the vertices in the Coxeter complex does not match up with the labeling of the weights in the $\mathfrak{sp}_4$ weight lattice in a canonical way (in type A, we were able to show that these both correspond to subsets on $\{1, ..., n\}$). The final section of Chapter 4 discusses in depth the failings we have found in type C and what could be done to fix them.

CHAPTER

2

PRELIMINARIES

2.1 Tensor Categories and Module Categories

One of the major structures we will make use of throughout this dissertation is that of a module category for a tensor category. This concept is a categorification of the concept of modules for a ring. Just as module theory relies heavily on the theory of rings, in order to build up a theory of module categories, we must first develop the theory of tensor categories. The following will introduce the major players on the categorical side of this dissertation. We briefly recall that the structure of a category is a set of objects, and given two objects, a set of morphisms on these objects for which composition is well-defined and identity morphisms exist. A functor is a map between categories that is well defined on morphisms and respects composition and identities.

We also recall the basic structure of a ring for our convenience, so we can see what concepts we must categorify to build a theory of tensor categories that is a true categorification of the theory of rings. Recall that a ring is a set R along with two operations $+$ and $*$ such that $(R, +)$ is an Abelian group and $(R, *)$ is a monoid, where multiplication distributes over addition.

We discuss the necessary categorical background in the sections below. Our treatment is mostly due to [EGNO15] and [HV19] and specific references will be given throughout.

2.1.1 Monoidal Categories

The first concept we will categorify is that of a monoid, that is, a set with a associative binary relation and identity . To do this, we must define some concept of a binary relation on our category, specifically on its set of objects.

Definition 2.1. ([EGNO15, Definition 2.2.8], [HV19, Definiton 1.1]) A monoidal category is a category M along with a operation $\otimes : ob(M) \times ob(M) \to ob(M)$, where $\otimes(X, Y)$ is written as $X \otimes Y$ for $X, Y \in ob(M)$, a unit object $\mathbf{1} \in ob(M)$ and choice of natural isomorphisms $\alpha : (- \otimes -) \otimes - \to - \otimes (- \otimes -)$, $\ell : - \to - \otimes \mathbf{1}$ and $\rho : - \to \mathbf{1} \otimes -$ that satisfy the following coherence axioms:

$$\begin{array}{ccccc} & & (((A \otimes B) \otimes C) \otimes D) & & \\ & \swarrow^{\alpha_{A,B,C} \otimes id_D} & & \searrow^{\alpha_{A \otimes B,C,D}} & \\ (A \otimes (B \otimes C)) \otimes D & & & & (A \otimes B) \otimes (C \otimes D) \\ \downarrow \alpha_{A,B \otimes C,D} & & & & \downarrow \alpha_{A,B,C \otimes D} \\ A \otimes ((B \otimes C) \otimes D)) & & \xrightarrow[id_A \otimes \alpha_{B,C,D}]{} & & A \otimes (B \otimes (C \otimes D)) \end{array} \tag{2.1}$$

$$\begin{array}{ccc} (A \otimes \mathbf{1}) \otimes B & \xrightarrow{\alpha_{A,\mathbf{1},B}} & A \otimes (\mathbf{1} \otimes B) \\ & \searrow_{\ell_A^{-1} \otimes id_B} \quad \swarrow_{id_A \otimes \rho_B^{-1}} & \\ & A \otimes B & \end{array} \tag{2.2}$$

The pentagon and triangle axioms tell us that we can identify all distributions of parentheses in a tensor product of 3 objects to each other in a well-defined way. This a property we wish to extend to any length tensor product. The following theorem gives this to us.

Theorem 2.2. *([HV19, Theorem 1.2]) If $\mathcal{M}$ is a monoidal category with associator α^M and unitors ℓ^M and ρ^M. Then any well-typed equation in α^M, ℓ^M, ρ^M and there inverses holds.*

As stated, this important result allows us to identify all possible bracketings of a tensor product $A_1 \otimes ... \otimes A_n$. Because of this, we can move parentheses and add or remove the unit object without much thought when working in monoidal categories.

A natural next question is how the tensor product in a monoidal category affects morphisms. We can build morphisms $f \otimes g : A \otimes B \to C \otimes D$ from morphisms in the category, and these morphism satisfy the following.

Theorem 2.3. *([HV19, Theorem 1.7]) If $f_1 : A_1 \to B_1$, $g_1 : B_1 \to C_1$, $f_2 : A_2 \to B_2$, $g_2 : B_2 \to C_2$, then*

$$(g_1 \circ f_1) \otimes (g_2 \circ f_2) = (g_1 \otimes g_2) \circ (f_1 \otimes f_2)$$

One can look at the properties assigned to the unit object in $\mathcal{M}$ and wonder if there is more than one object in $\mathcal{M}$ that we can equip with unitors to satisfy these properties. The answer is no, not meaningfully.

Proposition 2.4. *([EGNO15, Proposition 2.2.6]) The unit object of a monoidal category is unique up to isomorphism.*

The natural isomorphism α gives us a notion of associativity of this multiplication on objects. Similarly, ℓ and ρ give us a notion of multiplication by an identity object. If α, ℓ, ρ are all simply the identity natural transformation, we call M a **strict** monoidal category ([HV19, Defintion 1.25]).

We now give several examples of strict and non-strict monoidal categories.

Example 2.5. ([EGNO15, Example 2.3.3]) Consider the category Vec where objects are finite dimensional vector spaces over a field $\Bbbk$ and morphisms are linear transformations We can give Vec a monoidal product by taking the monoidal product as the standard tensor product on vectors spaces and $\Bbbk$ as the monoidal unit. We can choose the canonical isomorphism $(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$ and $\Bbbk \otimes V \simeq V \simeq V \otimes \Bbbk$ for our associators and unitors. This construction gives Vec the structure of a non-strict monoidal category.

Example 2.6. ([EGNO15, Example 2.3.12])If $\mathcal{C}$ is any category, the category $End(\mathcal{C})$ of endofunctors on $\mathcal{C}$ (functors $F : \mathcal{C} \to \mathcal{C}$) is a monoidal category, with $F \otimes G = F \circ G$ and unit given by the identity functor. Since composition of functors is associative directly, this is an example of a strict monoidal category.

Example 2.7. ([EGNO15, Example 2.3.6/8]) For a field $\Bbbk$ and a group G, let Vec_G be the category of G-graded vector spaces, that is, the category whose objects are

$$V = \bigoplus_{g \in G} V_g$$

and morphsims are direct sums of component morphisms. Take δ_g to be the object with $\Bbbk$ in the g component and 0 in all other components. We can write every object in Vec_G as a direct sum of these objects, so they form a sort of "basis" for Vec_G. Now, we can define a tensor product on Vec_G as

$$(V \otimes W)_g = \bigoplus_{h_1 h_2 = g} V_{h_1} \otimes V_{h_2}$$

Under this product, see $\delta_g \otimes \delta_h \simeq \delta_{gh}$. We can also choose δ_1 as the monoidal unit. As defined, this category has trivial associators, but we can give it additional structure by choosing a 3-cocycle ω with values in some Abelian group A. Recall that $\omega : G \times G \times G \to A$ is a 3-cocyle if

$$\omega(gh, x, y)\omega(g, h, xy) = \omega(g, h, x)\omega(g, hx, y)\omega(h, x, y)$$

for all $g, h, x, y \in G$. Now we can define an associator α by $\alpha_{\delta_g,\delta_h,\delta_k} = \omega(g, h, k) id_{\delta_{ghk}}$ and extending to all of Vec_G. The properties of a 3-cocycle ensure that this choice satisfies the pentagon axiom. The unitors are extensions of $\ell_{\delta_g} = \omega^{-1}(1, 1, g) id_{\delta_g}$ and $\rho_{\delta_g} = \omega(g, 1, 1) id_{\delta_g}$. If we choose these associators and unitors, we call this new monoidal category Vec_G^{ω}.

These are just some of the examples of monoidal categories, and we will see several more in later chapters of the dissertation. Given a notion of monoidal category, it is natural to also have a notion of a functor between these categories. Of course, we can simply take any functor without giving thought to the extra structure we have imposed, but we can also make the following definition

Definition 2.8. ([EGNO15, Definition 2.4.1] [HV19, Definition 1.27]) A monoidal functor from $(C_1, \otimes_1)$ to $(C_2, \otimes_2)$ is a functor $F : C_1 \to C_2$ and a natural transformation $J : F(-) \otimes F(-) \to F(- \otimes -)$ such that the following diagram commutes.

$$\begin{array}{ccc} (F(O_1) \otimes F(O_2)) \otimes F(O_3) & \xrightarrow{\alpha^2_{F(O_1),F(O_2),F(O_3)}} & F(O_1) \otimes (F(O_2) \otimes F(O_3)) \\ \downarrow{\scriptstyle J_{O_1,O_2} \otimes id_{F(O_3)}} & & \downarrow{\scriptstyle id_{F(O_1)} \otimes J_{O_2,O_3}} \\ F(O_1 \otimes O_2) \otimes F(O_3) & & F(O_1) \otimes F(O_2 \otimes O_3) \\ \downarrow{\scriptstyle J_{O_1 \otimes O_2, O_3}} & & \downarrow{\scriptstyle J_{O_1, O_2 \otimes O_3}} \\ F((O_1 \otimes O_2) \otimes O_3) & \xrightarrow[\alpha^1_{O_1,O_2,O_3}]{} & F(O_1 \otimes (O_2 \otimes O_3)) \end{array} \tag{2.3}$$

If the functor F is an equivalence of categories, then (F, J) is a monoidal equivalence.

We previously discussed the notion of a strict monoidal category, and now note that working in strict categories can be less complicated, since we don't have to worry about the action of associators. This makes the following theorem a rather useful tool.

Theorem 2.9 (MacLane Strictness Theorem)**.** *([EGNO15, Theorem 2.8.5] [HV19, Theorem 1.25]) Any monoidal category is monoidally equivalent to a strict monoidal category.*

Now, we discuss several additional structures that we can place on a monoidal category. The first is a partial categorification of the concept of inverse objects in a monoid.

Definition 2.10. ([EGNO15, Definition 2.10.1]) Given a monoidal category $\mathcal{M}$ and an object $M \in ob(\mathcal{M})$, a left dual for M is an object $M* \in ob(\mathcal{M})$ and morphisms $ev^{\ell}_M : X * \otimes X \to \mathbb{1}$ and $coev^{\ell}_M : \mathbb{1} \to X \otimes X*$ such that the following compositions of morphisms evaluate to the identity.

$$\begin{gathered} X \xrightarrow{(coev^{\ell}_x \otimes id_x)\circ\rho_x} (X \otimes X^*) \otimes X \xrightarrow{\alpha_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{\ell^{-1}_X \circ (id_x \otimes ev^{\ell}_X)} X \\ X^* \xrightarrow{(id_{X^*} \otimes coev^{\ell}_X)\circ\ell_{X^*}} X^* \otimes (X \otimes X^*) \xrightarrow{\alpha_{X^*,X,X^*}} (X^* \otimes X) \otimes X^* \xrightarrow{\rho^{-1}_{X^*} \circ (ev^{\ell}_X \otimes id_{X^*})} X^* \end{gathered} \tag{2.4}$$

A right dual for M is an object $*M \in ob(\mathcal{M})$ and morphisms $ev^{\rho}_M : X \otimes *X \to \mathbb{1}$ and $coev^{\rho}_M : \mathbb{1} \to *X \otimes X$ satisfying analogous axioms.

Example 2.11. Suppose $\mathcal{M} = Vec$, the category of finite dimensional vector spaces over a field $\mathbb{k}$. Now, for any vector space V, take V^* to be the usual dual vectors space (which we think of as linear functionals $f : V \to \mathbb{k}$). Fix a basis $v_1, .., v_n$ of V and call its dual basis $f_1, ..., f_n$. Now we can define $ev^{\ell}_V(f \otimes v) = f(v)$ and $coev^{\ell}_V(1) = \sum_{i=1}^n v_i \otimes f_i$. This satisfies the required axioms to be a left dual in the monoidal sense. We will show the first composition to be the identity below, by performing it on a generic vector $v = \sum_{i=1}^n c_i v_i$.

$$\sum_{i=1}^n c_i v_i \mapsto \sum_{j=1}^n v_j \otimes f_j \otimes \sum_{i=1}^n c_i v_i \mapsto \sum_{i=1}^n \sum_{j=1}^n v_j \otimes f_j \otimes c_i v_i \mapsto \sum_{i=1}^n \sum_{j=1}^n v_j \otimes f_j(c_i v_i)$$

$$= \sum_{i=1}^n v_i \otimes c_i f_i(v_i) = \sum_{i=1}^n v_i \otimes c_i \mapsto \sum_{i=1}^n c_i v_i$$

where first and second equalities follow from the definition of a dual basis. Showing the other composition is also an identity follows similarly.

As with the monoidal unit, there is nothing in the definition of a left or right dual that requires uniqueness of the dual object. However, it turns out that this is the case.

Proposition 2.12. *([EGNO15, Proposition 2.10.5], [HV19, Theorem 3.4]) Left and right duals are unique up to isomorphism.*

Now, we notice that in our example of Vec, every object has a both a left and right dual (in this case, the left and right duals are the same object, but this is not true for all monoidal categories). This observation motivates the following definiton

Definition 2.13. ([EGNO15, Definition 2.10.11]) An object in a monoidal category $\mathcal{M}$ is rigid if it has both a left dual and a right dual. $\mathcal{M}$ itself is called rigid if every object in $\mathcal{M}$ is rigid.

Example 2.14. ([EGNO15, Example 2.10.14]) Vec_G^ω is rigid. To see this, take $\delta_g^* =^* \delta_g = \delta_g^{-1}$ and extend to the entire category, while taking $ev_{\delta_g} = \omega(g, g^{-1}, g) id_{\delta_1}$.

The second piece of structure we have partially categorifies commutativity for a monoidal product.

Definition 2.15. ([HV19, Definition 1.17/20]) A braided monoidal category $\mathcal{M}$ is a monoidal category with an additional natural transformation defined component-wise $b_{AB} : A \otimes B \to B \otimes A$ so that the following diagrams commute:

$$\begin{array}{ccc} C_1 \otimes (C_2 \otimes C_3) & \xrightarrow{id_{C_1} \otimes \beta_{C_2,C_3}} & C_1 \otimes (C_3 \otimes C_2) \\ {\scriptstyle \alpha^{-1}_{C_1,C_2,C_3}} \downarrow & & \downarrow {\scriptstyle \alpha^{-1}_{C_1,C_3,C_2}} \\ (C_1 \otimes C_2) \otimes C_3 & & (C_1 \otimes C_3) \otimes C_2 \\ {\scriptstyle \beta_{C_1 \otimes C_2, C_3}} \downarrow & & \downarrow {\scriptstyle \beta_{C_1,C_3} \otimes id_{C_2}} \\ C_3 \otimes (C_1 \otimes C_2) & \xrightarrow{\alpha^{-1}_{C_3,C_1,C_2}} & (C_3 \otimes C_1) \otimes C_2 \end{array} \tag{2.5}$$

$$\begin{array}{ccc} C_1 \otimes (C_2 \otimes C_3) & \xrightarrow{\beta_{C_1, C_2 \otimes C_3}} & (C_2 \otimes C_3) \otimes C_1 \\ {\scriptstyle \alpha^{-1}_{C_1,C_2,C_3}} \downarrow & & \downarrow {\scriptstyle \alpha_{C_2,C_3,C_1}} \\ (C_1 \otimes C_2) \otimes C_3 & & C_2 \otimes (C_1 \otimes C_3) \\ {\scriptstyle \beta_{C_1,C_2} \otimes id_{C_3}} \downarrow & & \downarrow {\scriptstyle id_{C_2} \otimes \beta_{C_1,C_3}} \\ (C_2 \otimes C_1) \otimes C_3 & \xrightarrow{\alpha_{C_2,C_1,C_3}} & C_2 \otimes (C_1 \otimes C_3) \end{array} \tag{2.6}$$

If furthermore, $b_{BA} \circ b_{AB} = id_{A \otimes B}$, then the braiding is called a symmetry and $\mathcal{M}$ is called symmetric monoidal category

Example 2.16. For $\mathcal{M} = Vec$, extending the map $v \otimes w \mapsto w \otimes v$ gives us a braiding.

2.1.2 Tensor Categories

A monoidal category will serve as the categorification of the multiplicative monoid for a ring. We now need to find a similar categorification for the additive group of a ring. Our first step to do this is by defining a category which has a notion of addition of morphisms.

Definition 2.17. ([EGNO15, Deinition 1.2.1/2]) An additive category $\mathcal{C}$ is a category with the following additional axioms:

1. $Hom_{\mathcal{C}}(A, B)$ can be equipped with the structure of an Abelian group (written additively). Composition of morphisms is biadditive (that is, $f \circ (g_1 + g_2) = f \circ g_1 + f \circ g_2$ and $(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g$.

2. There exists an object $0 \in Ob(\mathcal{C})$ such that $Hom_{\mathcal{C}}(0, 0) = 0$.

3. Direct sums of objects exist: Given $X, Y \in Ob(\mathcal{C})$, there is some object Z with maps $i_1 : X \to Z$, $p_1 : Z \to X$, $i_2 = Y \to Z$, $p_2 = Z \to Y$, where $p_1 i_1 = id_X$, $p_2 i_2 = id_Y$ and $i_1 p_1 + i_2 p_2 = id_Z$.

If additionally, $Hom_{\mathcal{C}}(X, Y)$ are $\Bbbk$-vector spaces and composition of morphisms in $\Bbbk$-linear, we say that $\mathcal{C}$ is $\Bbbk$-linear.

Definition 2.18. ([EGNO15, Defintion 1.2.3]) A functor F is additive (or $\Bbbk$-linear) if the maps $Hom_{\mathcal{C}}(X, Y) \to Hom_{F(\mathcal{C})}(F(X), F(Y))$ are group homomorphisms (or, respectively, $\Bbbk$-linear transformations).

Remark 2.19. ([EGNO15, Proposition 1.2.4]) An additive functor F distributes across direct sums. That is,

$$F(X) \oplus F(Y) \simeq F(X \oplus Y)$$

.

Definition 2.20. ([EGNO15, Section 1.3]) Let $\mathcal{C}$ be a category and $f : X_1 \to X_2$ be a morphism in $\mathcal{C}$.

- A kernel of f is an object K along with a morphism $\kappa : K \to X_1$ satisfying $f \circ k = 0$ such that if $g : M \to X_1$ so that $f \circ g = 0$, there is an unique morphism $m : M \to K$ so that $g = \kappa \circ m$.

- A cokernel of f is an object C along with a morphism $c : X_2 \to C$ satisfying $c \circ f = 0$ such that if $h : X_2 \to D$ so that $h \circ f = 0$, there is an unique morphism $d : D \to C$ so that $h = d \circ c$.

Kernels and cokernels, when they exist, are unique up to isomorphism. A monomorphism in an additive category is a morphism whose kernel is 0, and an epimorphism is a morphism whose cokernel is 0.

Definition 2.21. ([EGNO15, Definition 1.3.1]) An Abelian category $\mathcal{A}$ is an additive category such that for every morpism $f : X_1 \to X_2$ in $\mathcal{A}$, there is a sequence:

$$K \xrightarrow{k} X_1 \xrightarrow{i_1} I \xrightarrow{i_2} X_2 \xrightarrow{c} C$$

such that:

1. $i_2 \circ i_1 = f$
2. (K, k) is a kernel of i_1 and (C, c) is a cokernel of i_2.
3. (X_1, i_1) is a cokernel of k and (X_2, i_2) is a kernel of C.

Note that this sequence, when it exists, is unqiue up to unique isomorphism. In an Abelian category, we can define several relations on objects.

Definition 2.22. ([EGNO15, Section 1.3]) If X_1 is an object in an Abelian category $\mathcal{A}$

- A subobject of X_1 is an object X_2 and a monomorphism from $X_2 \to X_1$. (we will use subset notation $X_1 \subset X_2$ to represent the subobject relation). In this setting, we call the cokernel of this monomorphism X_1/X_2.
- A quotient of X_1 is an object X_2 and an epimorphism $X_1 \to X_2$.

([EGNO15, Section 1.3]) A simple object X is one where the only subobjects are 0 and X.

Definition 2.23. ([EGNO15, Definiton 1.5.3]) An object A in an Abelian category $\mathcal{A}$ is of finite length if there exists a sequence

$$0 \subset F_1 \subset F_2 \subset ... \subset F_{n-1} \subset F_n = A$$

where F_{i+1}/F_i is simple for all i. In this case, any filtration of A must have n objects and we call n the length of A.

Definition 2.24. ([EGNO15, Section 1.5]) A $\Bbbk$-linear category $\mathcal{C}$ is locally finite if $Hom_{\mathcal{C}}(X, Y)$ is finite dimensional for all objects X and Y, and all objects have finite length.

We now have all of the concepts that we need to define a tensor category.

Definition 2.25. *If $\mathcal{C}$ is a locally finite $\Bbbk$-linear, Abelian, rigid monoidal category, $\mathcal{C}$ is a multi-tensor category if the monoidal product is bilinear on morphisms. In addition, if $End_{\mathcal{C}}(\mathbb{1}) \simeq \Bbbk$, then $\mathcal{C}$ is a tensor category.*

2.1.3 Module Categories

We finally have the context need to present the categorifcation of R-modules. We will show that some important notions in the theory of modules can be categorified as well. For this section, let $(\mathcal{C}, \otimes, \alpha, \ell, \rho)$ be a monoidal category.

Definition 2.26. ([EGNO15, Definition 7.1.1]) A left $\mathcal{C}$-module category is a category $\mathcal{M}$ along with a bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ which can be thought of as the action of $\mathcal{C}$ on $\mathcal{M}$ (we write $\otimes(C \times M) = C \otimes M$ for objects $C \in ob(\mathcal{C})$ and $M \in ob(\mathcal{M})$). For every choice of objects C_1 and C_2 in $\mathcal{C}$ and $M \in ob(\mathcal{M})$ we have an isomorphism

$$\mu_{C_1,C_2,M} : (C_1 \otimes C_2) \otimes M \to C_1 \otimes (C_2 \otimes M).$$

For every triple of object C_1, C_2, $C_3 \in ob(\mathcal{C})$ and object $M \in ob(\mathcal{M})$, the following diagram must commute:

$$\begin{array}{ccccc} & & ((C_1 \otimes C_2) \otimes C_3) \otimes M & & \\ & \swarrow^{\alpha_{C_1,C_2,C_3} \otimes id_M} & & \searrow^{\mu_{C_1 \otimes C_2, C_3, M}} & \\ (C_1 \otimes (C_2 \otimes C_3)) \otimes M & & & & (C_1 \otimes C_2) \otimes (C_3 \otimes M) \\ \downarrow \mu_{C_1, C_2 \otimes C_3, M} & & & & \downarrow \mu_{C_1, C_2, C_3 \otimes M} \\ C_1 \otimes ((C_2 \otimes C_3) \otimes M) & & \xrightarrow{id_{C_1} \otimes \mu_{C_2,C_3,M}} & & C_1 \otimes (C_2 \otimes (C_3 \otimes M)) \end{array} \tag{2.7}$$

It is analogous to define a right $\mathcal{C}$-module category but in this dissertation we will almost exclusively use left module categories.

Recall that in the theory of modules over a ring, we have a bijective correspondence between modules and representations of the ring. We are able to categorify this correspondence in the following way.

Proposition 2.27. *([EGNO15, Proposition 7.1.3]) Left $\mathcal{C}$-module category structures on a category $\mathcal{M}$ are in bijection with monoidal functors $F : \mathcal{C} \to End(\mathcal{M})$.*

The key to showing this is to make use of the natural isomorphism J that is part of the data of the monoidal functor (see Definition 2.8). If $F : \mathcal{C} \to End(\mathcal{M})$, then $J_{C_1 C_2} : F(C_1) \otimes F(C_2) = F(C_1) \circ F(C_2) \to F(C_1 \otimes C_2)$. We can define $C \otimes M$ as $F(C)(M))$. If we do this, then the inverse of J morphisms satisfy the relations for μ by definition.

As we have done previously, we can now use the structure that we have defined on a class of categories to put some additional structure on functors with these categories as sources and targets. We would like functors between $\mathcal{C}$-module categories to be compatible with the $\mathcal{C}$-action, as defined below.

Definition 2.28. ([EGNO15, Definition 7.2.1]) A module functor is a functor between $\mathcal{C}$-module categories $\mathcal{M}$ and $\mathcal{N}$, along with a natural isomorphism defining how the functor "distributes" along the respective $\mathcal{C}$-actions.

Remark 2.29. ([EGNO15, Section 7.2]) If $\mathcal{C}$ is a tensor category, we additionally require that $\mathcal{C}$-module categories are locally finite and Abelian, and that the $\mathcal{C}$-action is bilinear on morphisms and exact in the first variable.

We now present several basic examples of module categories.

Example 2.30. Any monoidal or tensor category is a module category over itself. Here the monoidal product is the $\mathcal{C}$-action and the data of μ is just the data of the associator α.

Example 2.31. If G is a group and $H \leq G$, then $Rep(H)$ is a $Rep(G)$ module category. We can act on a representation ρ_H of H by a representation ρ_G of G by restricting ρ_G to H and then taking the tensor product $\rho_G|_H \otimes \rho_H$.

2.2 Coxeter Combinatorics and Buildings

The second structure fundamental to this dissertation is that of affine and finite buildings. As discussed in Section 1.1, buildings are a combinatorial structure first introduced by Tits in the 1970s. Throughout the years, buildings have been used in many combinatorial, geometric and topological applications. In the following section, we detail the combinatorial approach to buildings, starting with foundational definitions, and developing examples throughout.

2.2.1 Coxeter Combinatorics Crash Course

The simplest way to come at the defintion of a building is as a simplicial complex with structure determined partially by a Coxeter group. For more detailed information on Coxeter groups, see [BB05].

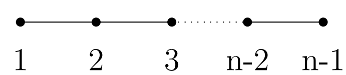
Definition 2.32. ([BB05, Section 1.1]) A Coxeter group W is a group that is determined by a choice $m_{ij} \in \{1, 2, ..., \infty\}$. Given this choice, we can present W as

$$W = \langle s_i \text{ for } i \in I | s_i^2 = 1 \text{ for all } i \in I, s_i s_j^{m_{ij}} = 1\rangle$$

Here, we can only choose $m_{ij} = 1$ if $i = j$. If we have such a set of generators $S = \{s_i | i \in I\}$, then we call (W, S) a Coxeter system.

If we have a Coxeter system (W, S), we can represent the information of the system as a graph, called a Coxeter diagram. This graph has one vertex for every generator s_i. We draw an edge between s_i and s_j when $m_{ij} > 2$, and label the edge with m_{ij} for $m_{ij} > 4$. We can also represent this data as a $|I| \times |I|$ matrix with entries m_{ij}.

Example 2.33. Perhaps the most well known example of a Coxeter group is the symmetric group S_n. Here we have a Coxeter system $W = S_n$ and $S = \{s_i = (i\ i+1) | 1 \leq i \leq n-1\}$. For $|i - j| > 1$, $m_{ij} = 2$, as in this case s_i and s_j will be disjoint (and therefore commuting) transpositions. If $i = j \pm 1$, then $s_i s_j$ is a 3-cycle ($s_i s_{i+1} = (i\ i+1\ i+2)$) and so $m_{ij} = 3$. When $i = j$, $m_{ij} = 1$, since all transpositions have order 2. The data we have just computed can be organized in a graph as follows, where we choose $n = 6$.

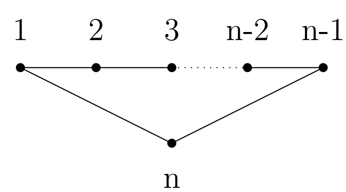


(2.8)

Figure 2.1: A_{n-1} Coxter diagram

The group A_n is an example of an irreducible Coxeter group, one whose Coxeter diagram is connected. Finite irreducible Coxeter groups are completely classified in [Cox35]. Along with this classification, there is a classification of affine Coxeter groups in [Car52]. The affine counterpart to S_n is defined below.

Example 2.34. The affine Coxeter group $\tilde{A}_{n-1}$ is Coxeter group with $S = \{s_i | 1 \leq i \leq n\}$. For $1 \leq i, j \leq n-1$, m_{ij} is the same as in the previous example. For $i = n$, $m_{ij} = 2$ unless $j = 1$ or $n - 1$, in which case $m_{ij} = 3$. Again, we can represent this as a graph (see Figure 2.4). Unlike A_{n-1}, the group $\tilde{A}_{n-1}$ is infinite.



(2.9)

Figure 2.2: $\tilde{A}_{n-1}$ Coxeter diagram

The complete classification of finite and affine Coxeter groups can be found in [BB05, Appendix A1]. Geometrically, there is realization of any Coxeter group as a reflection group. This allows us to make a connection between Coxeter groups and the representation theory of finite dimensional semi-simple Lie algebras.

A standard reference for this connection is [Bou08] explained how to derive a root system from the Cartan subalgebra of a finite dimensional semisimple Lie algebra using the roots of the Lie algebra. We construct a collection of hyperplanes perpendicular to the roots and further produce a group generated by the reflections associated to these hyperplanes. This group, called the Weyl group of the Lie algebra is a finite Coxeter group. There is also an inverse process by which we can recover a root system from the Coxeter group by constructing a special basis for $\mathbb{R}^{|S|}$ and a bilnear form whose values are derived from the Coxeter matrix. This connection between Coxeter groups and Weyl group explain our choice of W as the letter to represent a Coxeter group.

2.2.2 Simplicial Complexes

Before we move on to our discussion of buildings, we first review the definition of an abstract simplicial complex and some of its most relevant properties for future sections.

Definition 2.35. ([AB10, Definition A.1]) Given some vertex set V, an abstract simplicial complex Δ is a non-empty collection of finite subsets of V satisfying the following properties:

- If $A \in \Delta$ and $B \subset A$, then $B \in \Delta$.
- For every $v \in V$, $\{v\} \in \Delta$.

We call any element $A \in \Delta$ a simplex, say that it's rank is $r = |A|$ and that it's dimension is $r - 1$ (here the empty set has rank 0 and dimension -1) in Δ.

While this is a good definition, in some cases (specifically the one we will encounter), it is not the most practical to verify. Instead, we can note the following.

Remark 2.36. ([AB10, Section A.1.1]) If we view an abstract simplicial complex Δ as a poset, it satisfies the following two conditions:

1. For every A and B in Δ, the greatest lower bound $A \cap B$ exists.
2. For $A \in \Delta$, $\Delta_{\leq A}$, the subposet of Δ consisting of elements less than A, is isomorphic to the poset of subsets of a finite set.

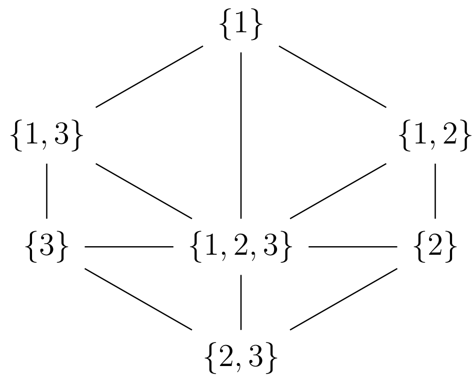
It turns out that we have the following proposition as well (see [AB10, Section A.1.1] for proof)

Proposition 2.37. *If a poset satisfies conditions 1 and 2 from the above remark, then it is isomorphic to an abstract simplicial complex.*

One special type of simplicial complex that we will often consider is a flag complex of some set.

Definition 2.38. ([AB10, Defintion A.4]) A flag in some poset is a chain (that is, a listing of elements $p_1, ..., p_k$ so that $p_1 \leq ... \leq p_k$). A flag complex of a set V has the elements of the power set of V as vertices. Simplices in this complex are flags in the poset of subsets of V.

Example 2.39. Suppose $V = \{1, 2, 3\}$. Then we can construct a flag complex as depicted below:



(2.10)

The simplicial complexes that we are interested in fall under a special class called chamber complexes.

Definition 2.40. ([AB10, Section A.1.3]) If Δ is a simplicial complex where all maximal simplices have the same dimension, a gallery in Δ is a sequence of adjacent (that is, sharing a codimension 1 simplex) maximal simplices.

Definition 2.41. ([AB10, Definition A.8]) A chamber complex is a simplicial complex where all maximal simplices have the same dimension and rank and where every pair of maximal simplices is connected by a gallery.

Given a set I, we can label the vertices of a simplicial complex Δ by I, which in turn induces a labeling of the simplices of Δ by subsets of I. When we have a chamber complex, we can require some extra structure on the way we choose this labeling.

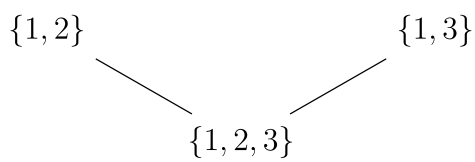
Definition 2.42. ([AB10, Page 665]) A labeling of a chamber complex Δ by a set I is a map $\ell : V(\Delta) \to I$, where $V(\Delta)$ is the set of vertices of Δ, such that the vertices of any chamber are in bijection with the elements of I. If a chamber complex admits a labeling, it is called labelable.

In [AB10], this is called a coloring and chamber complexes are called colorable. Finally, we would like some way to discuss the local structure of a simplicial complex. To do this, we can make the following definition.

Definition 2.43. ([Bro89, Section A.D]) The link of a simplex A in a simplical complex Δ (denoted $lk_{\Delta}A$) is the subcomplex of Δ of simplices B where $A \cap B = \emptyset$ and $A \cup B \in \Delta$.

Example 2.44. Consider the flag complex we constructed in the previous example. The union of the middle vertex with any other vertex is the edge between them, and the union of the middle vertex and line between two other vertices is the triangle with the middle vertex and the other two vertices. So the middle vertex is has empty intersection and union in Δ with every vertex and line not containing it. So it's link is the outside perimeter of the hexagon.

On the other hand the vertex $\{1\}$ has union in Δ with $\{1, 3\}$, $\{1, 2\}$, $\{1, 2, 3\}$ and any lines between those vertices. So its link is simply the following:



2.2.3 Coxeter Complexes and Buildings

We now work to define buildings of type W for some finite of affine Coxeter system (W, S). We must begin with the following definition.

Definition 2.45. ([AB10, Definition 2.12]) Let (W, S) be a Coxeter system. A subgroup of W is called special if it is $\langle J \rangle$ for some subset $J \subset S$. Any coset of a special subgroup is called a special coset.

It is easy to show that not all subgroups of a Coxeter subgroup are special (e.g. $\{(1\ 2\ 3), (1\ 3\ 2), 1\} \leq A_2$ is not special). We now define a simplicial complex using this notion of special subgroups and cosets of (W, S).

Definition 2.46. ([AB10, Definition 3.1]) The Coxeter complex of type (W, S) , denoted $\Sigma(W, S)$, is the poset of special cosets of W, ordered by the opposite of the inclusion relation.

Example 2.47. Take $n = 3$. The Coxeter complex of $A_{n-1} = S_n$ is given by the following (where each simplex is labeled by the coset it represents and we write the coset $\{w\} = w\langle 1 \rangle$ as w for $w \in W$):

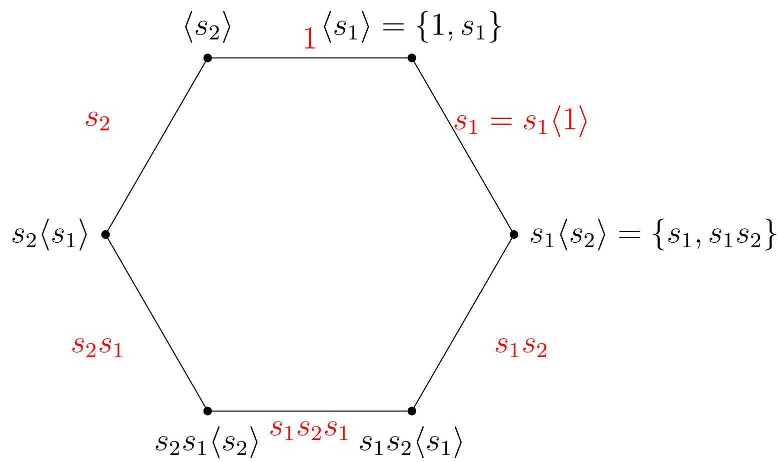


Figure 2.3: The A_2 Coxeter Complex

Example 2.48. The Coxeter complex of $\tilde{A}_{n-1}$ is partially given by Figure 2.4 below. Here the triangle are elements of the group, edges are cosets $w\langle S' \rangle$ with $|S|' = 1$ and vertices are cosets $w\langle S' \rangle$ with $|S|' = 2$.

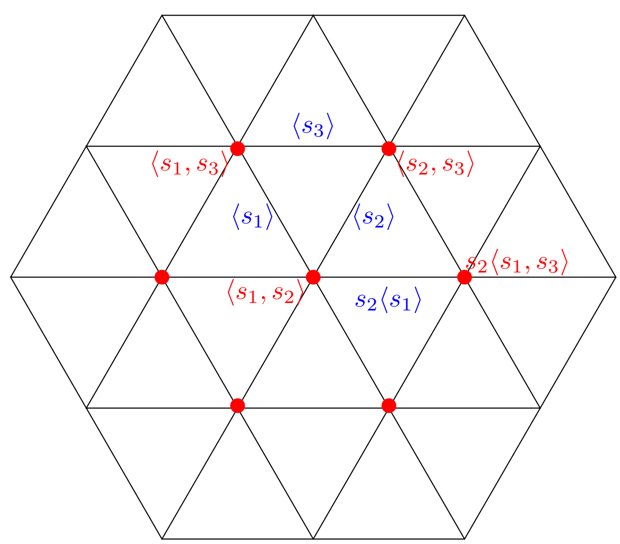


Figure 2.4: The $\tilde{A}_2$ Coxeter complex

Note that we have a invertible W action on $\Sigma(W, S)$ where $w \cdot w'\langle S' \rangle = ww'\langle S' \rangle$. We will make frequent use of this action when proving the following facts about $\Sigma(W, S)$.

Proposition 2.49. *([AB10, Theorem 3.5]) $\Sigma(W, S)$ is a chamber complex.*

Proof. ([AB10, Theorem 3.5]) To show this, we must show that $\Sigma(W, S)$ is simplicial. To do this, we use the strategy of checking the conditions of Remark 2.36.

First, to see that for any A and B that $A \cap B$, we recall that since we are working under the opposite of inclusion that we are working to find a least upper bound for the cosets $A = w'\langle S'\rangle$ and $B = w''\langle S''\rangle$. First, because the W action on $\Sigma(W, S)$ is invertible, we can shift A and B so that one is a special subgroup by acting on them by w'^{-1}, so that $A' = \langle S'\rangle$ and $B' = w\langle S''\rangle$ where $w = w'^{-1}w''$. Now, we note that by [Bro89, Corrolary 2.3C.3], there is a set S_w that is the smallest subset S' of S for which $w \in \langle S'\rangle$. The least upper bound of A' and B' is the smallest coset containing S', S'' and w, which we now see is $\langle S', S'', S_w\rangle$.

To show that $\Sigma(W, S)_{\leq A}$ is a Boolean poset for every A, we must only show this for a maximal simplex (i.e. a coset of $\{1\}$). We can again use the W action on $\Sigma(W, S)$. If we show this for $\{1\}$, it holds for every maximal coset. Now, the poset $\Sigma(W, S)_{\leq\{1\}}$ is simply the poset of special subgroups of W under the opposite of the inclusion relation.

Now, we show that the map from subsets of S to special subgroups of W given by $S' \mapsto \langle S'\rangle$ is a bijection. To show it is injective, see that if $S_1 \neq S_2$, and $s' \in S_1 \setminus S_2$, then $s' \in \langle S_1\rangle$ but $s' \notin \langle S_2\rangle$ since by the Coxeter relations, we cannot compose generators to get another generator. To show surjectivity, we must only note that every special subgroup is generated by a subset of S by definition. Since if $S_1 \subseteq S_2$ then $\langle S_1\rangle \leq \langle S_2\rangle$, this bijection gives us (special subgroups of $W)^{op} \simeq$ (subsets of S)^{op} as poset. But we also have a poset isomorphism (subsets of S)^{op} to the subsets of S by sending S' to $S - S'$. So we have shown (special subgroups of $W)^{op} \simeq$ (subsets of S) as desired.

We define the rank of a simplex $w\langle S'\rangle$ to be $|S| - |S'|$. So we can see that all maximal simplices have rank n, since they are $w\langle 1\rangle$ for some $w \in W$. To show that any two maximal chambers are connected by a gallery, suppose we have some chamber C represented by $w\langle 1\rangle$, where $w = w's_i$. See that $w\langle s_i\rangle = \{w, w'\}$ is a wall between C and C' represented by $w'\langle 1\rangle$. Now, write $w' = w''s_j$ and repeat this process to get a gallery from C to $\{1\}$. To get a gallery between any two chamber we must simply concatenate their repspective galleries to $\langle 1\rangle$. So $\Sigma(W, S)$ is a chamber complex, as desired. □

Now that we have shown that Coxeter complexes are indeed chamber complexes, we can show that they are labelable. From now on, we will say $\Sigma = \Sigma(W, S)$ for ease.

Proposition 2.50. *([AB10, Theorem 3.5]) Any Coxeter complex S is labelable by S. Moreover, the labeling ℓ is defined by $\ell(w\langle S'\rangle) = S - S'$.*

Proof. We need to show that the elements of a chamber are in bijection with the elements of S. Because of the W action on Σ we can once again prove this by making the argument for the chamber $\{1\}$. Now, see that the vertices of this chamber are simply the special subgroups generated by sets of size $n-1$. Under ℓ each of these subgroups has for a label a unique s_i. Also, by construction, there are exactly as many subgroups as elements of s_i so there is the desired bijection. Therefore Σ is labelable. □

In future chapters, we will refer mostly to the vertex labeling on Σ, as we will construct different labelings for the edges. We also want to make note of the following facts about the local structure of Coxeter complexes as they will come into play in later chapters.

Proposition 2.51. *([Bro89, Page 60]) For any $A \in \Sigma$, $lk_\Sigma A$ is isomorphic to a Coxter complex. More specifically, if $S'' = \ell(A)$, then $S' = S - S''$ and $W' = \langle S' \rangle$ then $lk_\Sigma A \simeq \Sigma(W', S')$.*

Corollary 2.52. *([AB10, Corollary 3.20]) If A is a codimension 2 simplex labled by a coset of $\langle S - \{s_i, i_j\} \rangle$, then the link of A in Σ is a $2m$-gon where $m = m(s_i, s_j)$.*

Now, we turn our attention from Coxeter complexes to buildings. We begin with the following definition.

Definition 2.53. ([AB10, Definition 4.1]) A building (often denoted Δ) is a simplicial complex that can be expressed as the union of a set $\mathcal{A} = \{\Sigma_i | i \in I\}$ of subcomplexes called a system of apartments satisfying the following axioms:

1. Every apartment Σ is a Coxeter complex.

2. For every pair of simplices A and B in Δ, there is some apartment in Δ containing both A and B.

3. If Σ_1 and Σ_2 are two apartments containing simplices A and B, there is an isomorphism $\Sigma_1 \to \Sigma_2$ fixing A and B pointwise.

By Proposition 2.49, a building is a chamber complex. If we pick maximal simplices A and A', by axiom 2, there is apartment Σ containing both. Since Σ is a Coxeter complex, and therefore a chamber complex, A and A' must have the same dimension and be connected by a gallery.

Remark 2.54. ([AB10, Remark 4.2]) What we have just defined as a building is in the literature sometimes called weak building, with the term building given to a chamber complex that satisfies our definition and is also thick (i.e., every codimension 2 simplex is contained in at last three chambers).

We note that the specific system of apartments is not a piece of the structure. A building may be equipped with several distinct systems of apartments. There is however, a canonical system of apartments for every building. It is true that the union of a collection of system of apartments is also a system of apartments (see [Bro89, Section 4.4]). So we have a maximal system of apartments (the union of all systems of apartments. We call this system the complete system of apartments.

Axiom 3 of a building can be replaced by two equivalent conditions. The first is:

Axiom 3.1: ([AB10, Remark 4.3]) Σ_1 and Σ_2 are apartments that both contain a chamber C and simplex A, then there is an isomorphism between Σ_1 and Σ_2 fixing A and C pointwise.

Proof. We show this new axiom implies Axiom 3. Choose any simplices A_1 and A_2, and apartments Σ_1 and Σ_2 containing A_1 and A_2. Now, choose some chamber $C_1 \in \Sigma_1$ so $A_1 \subseteq C_1$ and $C_2 \in \Sigma_2$ so $A_2 \subseteq C_2$. By axiom 2 of a building, there some apartment Σ' containing C_1 and C_2. Now, axiom 3.1 implies that there is an isomorphism $\phi_1 : \Sigma_1 \to \Sigma'$ fixing C_1 and A_2 and an isomorphism $\phi_2 : \Sigma' \to \Sigma_2$ fixing C_2 and A_1. So $\phi_2 \circ \phi_1$ is the isomorphism need to satisfy axiom 3. □

The second of the equivalent axioms is as follows (the proof of equivalence can be found in [AB10, Page 175]):

Axiom 3.2 ([AB10, Remark 4.4]): If Σ_1 and Σ_2 are apartments that both contain a chamber C, there is an isomorphism $\Sigma_1 \to \Sigma_2$ that fixes each simplex of $\Sigma_1 \cap \Sigma_2$.

Now, we wish to extend an important property of Coxeter complexes to buildings as well. From this point, take Δ to be an arbitrary building and $\mathcal{A}$ to be a system of apartments (not necessarily complete) for the building.

Proposition 2.55. *([AB10, Proposition 4.6]) A building Δ is labelable.*

Proof. ([AB10, Proposition 4.6]) To show this, first pick some chamber C in Δ and an arbitrary labeling ℓ on the vertices of C with some label set I. Now, see that for every apartment Σ containing C, we can use the W action on Σ to construct ℓ_Σ, the unique labeling of Σ consistent with ℓ. This labeling is also preserved by the isomorphism $\phi : \Sigma_1 \to \Sigma$ between two apartments containing C, since ϕ fixes C, $\ell_{\Sigma_1} \circ \phi$ and ℓ_Σ are both labelings of Σ_1 agreeing with ℓ and therefore the same labeling. So we can combine these labelings to get a labeling on the union of all apartments containing C. But, by axiom 2 of buildings, every simplex in Δ shares an apartment with C, so this union is Δ. Therefore, Δ is labelable. □

Given a labeling ℓ with index set I of Δ, we can fix an apartment Σ and intrepret the labeling set I as indexing the generating set S for the Coxeter complex. For any simplex A of type $I - \{i, j\}$, $m_{ij} = diam(lk_{\Sigma} A)$ (this is an 2m-gon due to similar reasoning to Corollary 2.52). We can then construct a Coxeter matrix $M = [m_{ij}]$. Since we have isomorphism between apartments that are labeling preserving from the previous proposition, this matrix M is the same for any choice of Σ. So every apartment in Δ is a Coxeter complex of the same type.

Definition 2.56. ([AB10, Proposition 4.7]) The type of a building Δ is the same as the type of any of its apartments.

Before we present some examples, we note that as in the case of Coxeter complexes, the link of any simplex in a building is also a building. We will talk more about this result in later chapters.

To present our first example, first suppose Q is a set with an incidence relation (that is, a reflexive and symmetric relation on Q). We can partition Q into set $Q_0, Q_1, ..., Q_{n-1}$. We call this partition an incidence geometry if two members of Q_i are incident only if they are equal. We can put many different sets of axioms on an incidence geometries. We will see both in the upcoming example and in later chapters that some types of incidence geometries correspond to different types of buildings. We call attention in particular to one type here.

Definition 2.57. ([AB10, Definition 4.17]) A projective plane is a incidence geometry $P = P_0 \cup P_1$, where we call P_0 points and P_1 lines subject to the following axioms.

1. Any two points are incident to a distinct line.

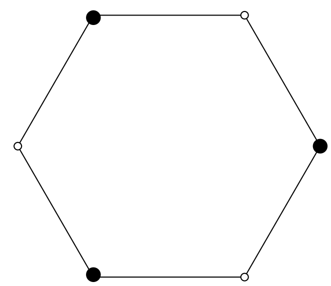
2. Any two lines intersect at a distinct point.

3. There exist 3 points that are non-colinear.

Example 2.58. ([AB10, Example 4.16]) Suppose that Δ is a building of rank 2. In this case, the chambers are lines, and codimension 1 simplices are points. Necessarily, this means that the apartments of Δ must be the Coxeter complex of a rank 2 Coxeter group (chambers are default labeled by $w\langle 1\rangle$ and so lines must be $w\langle s_i\rangle$, so $|S| - 1 = 1$). So our building has Coxeter type defined by the diagram

$$\bullet \overset{m}{\rule{3em}{0.4pt}} \bullet \tag{2.11}$$

We will consider the case where $m = 3$. This diagram corresponds to the system A_2, which as we have seen before has the Coxeter complex:



Here we partition the vertices into P_0 (white vertices) and P_1 (black vertices). We see that we can view this apartment as three lines (P_1) whose pairwise intersections are 3 distinct points (P_0). This suggests that the building Δ has the structure of a projective plane. This is indeed true and we can say this in general for any A_2 building. Indeed we can also say the converse, that the flag complex of a projective plane is always a type A_2 building ([AB10, Page 179]).

Concretely, let P be the projective plane over $\mathbb{F}_2$, which has 7 points (which are each on 3 lines) and 7 lines (with three points on each). This means that the corresponding building will have 14 vertices and 21 edges, as in Figure 2.5.

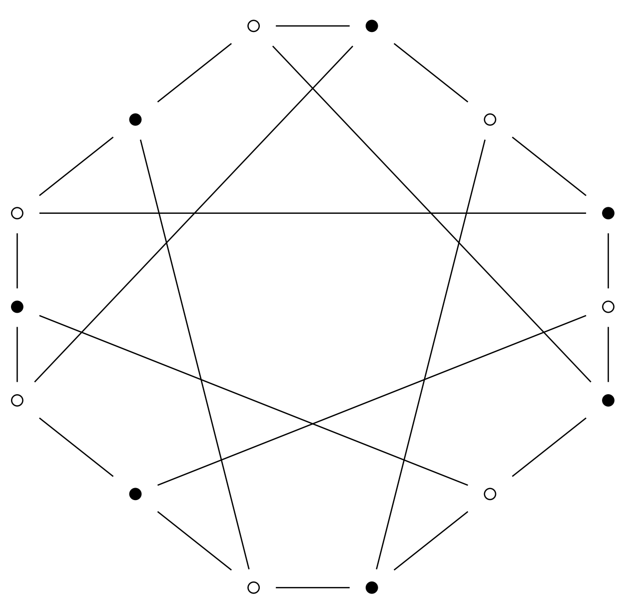


Figure 2.5: The building corresponding to the projective plane over $\mathbb{F}_2$

We will say more about this correspondence between type A buildings and projective geometries in a later chapter. If we were to change the value of m in our example, the resulting building would correspond to a different type of plane geometry (for example, the $m = 4$ case corresponds to polar geometry).

We would like to now present a concrete example of a building of affine type. To do this, we first need the following context.

Definition 2.59. ([AB10, Definition 6.106]) Suppose $\Bbbk$ is a field with unit group $\Bbbk^*$. A discrete valuation on $\Bbbk$ is a surjective map $v : \Bbbk^* \to \mathbb{Z}$ so that

$$v(k_1 * k_2) = v(k_1) + v(k_2)$$

and

$$v(k_1 + k_2) \geq \min(v(k_1), v(k_2))$$

for all $k_1, k_2 \in \Bbbk^*$ (we sometimes extend this definition to $\Bbbk$ by setting $v(0) = \infty$).

From this we have a subring A called the valuation ring of $\Bbbk$, defined as $A = \{k \in \Bbbk^* | v(k) \geq 0\}$. The units of this ring are $A^* = \{k \in \Bbbk | v(k) = 0\}$. Also, if we choose a element $\pi \in \Bbbk$ with $v(\pi) = 1$, then every $k \in \Bbbk^*$ can be written as $k = \pi^n u$ for $n \in \mathbb{Z}$ and $u \in A^*$. So we have a maximal principal ideal of A defined by πA and therefore $\overline{\Bbbk} = A/\pi A$ is a field, called the residue field of A.

Example 2.60. ([AB10, Example 6.108]) Let $\Bbbk = \mathbb{Q}$ and p be some prime. We can define what we will call the p-adic valuation on $\Bbbk$ by defining $v(k)$ to be the exponent of p, positive or negative, in the prime factorization of k. That is, if we write $k = p^m u$, where $m \in \mathbb{Z}$ and p does not divide u, then $v(k) = m$. See that the valuation ring of $\Bbbk$ is the ring $A = \{\frac{k_1}{k_2} | p \text{ does not divide } k_2\}$ (having a factor of p in the denominator is the only possible way to get a negative valuation). Here, the residue is $\mathbb{F}_p$, the finite field with p elements.

Proposition 2.61. *([AB10, Definition 6.110]) The valuation ring A is a principle ideal domain.*

Because of this, we can now use the theory of modules over principal ideal domains to talk more about A. In particular, we can take the vector space $V = \Bbbk^n$ and look at lattices in V (that is, we look at A-submodules $L \subset V$ where $L = Ae_1 \oplus ... \oplus Ae_n$ for some basis $\{e_1, ..., e_n\}$ of V.

If we consider two lattices L and L' and with we can choose a basis $e_1, ..., e_n$ for L such that

$L' = A\lambda_1 e_1 \oplus ... \oplus A\lambda_n e_n$. We can take the λ_i to be powers of π and these coefficients (called elementary divisors of L' with respect to L) are, up to order, unique.

Example 2.62. ([AB10, Section 6.9.2]) Given the setup provided, we can define an equivalence relation on lattices where L and L' are equivalent if $L' = \lambda L$ for some $\lambda \in \Bbbk^*$. We use $[L]$ to denote the equivalence class of L, or, if $L = Av_1 \oplus ... \oplus Av_n$, then we may label the equivalence class as $[[v_1, ..., v_n]]$. These equivalence classes will now serve as the vertex set of a building, but in order to fully define this building, we must have a notion of both type and incidence.

To determine type, we see that $GL_n(\Bbbk)$ acts on the lattice classes that we have defined. Also, see that under this action, the stabilizer of $[A^n]$ is $Z \cdot GL_n(A)$ where Z is the group of scalar multiples of the identity matrix. We note that $v(\det(G))$ is $0 \mod n$ for any matrix in this stabilizer. In light of this, we can look at an arbitrary class $\mathcal{L}$ and see that if $G \in GL_n(\Bbbk)$ such that $G[A^n] = \mathcal{L}$ will have $v(\det(G)) \mod n$ constant , so that we can set $v(\det(G))$ to be the type of $\mathcal{L}$. If $\mathcal{L} = [[v_1, ..., v_n]]$, then naturally $[v_1, .., v_n]$ will be one of those matrices and therefore $\text{type}(\mathcal{L}) = v(\det([v_1, ..., v_n])) \mod n$.

Now, we call two lattice classes $\mathcal{L}_1$ and $\mathcal{L}_2$ incident if we can find representatives L_1 and L_2 respectively such that $\pi L_1 \subset L_2 \subset L_1$. This is a symmetric incidence relations, and suggests that the elementary divisors of L_2 with respect to L_1 are some sequence $\{1, ..., 1, \pi, ..., \pi\}$. This means that L_1 and L_2 must belong to equivalence classes with different types, and so we have built an incidence geometry.

It can be show that the flag complex of this incidence geometry is a building of type $\tilde{A}_n$. The proof of this fact is complicated, and can be found in [AB10, Section 8.8C].

The crux of the omitted proof relies on an action of G on the building Δ. We often have groups acting on buildings in this way: as type-preserving automorphism that are simplical and send apartments to apartments. These actions are called strongly transitive if they are transitive on the pairs (Σ, C) of apartments and chambers. From these strongly transitive actions, we get the theory of BN-pairs . We can also use the theory in the opposite direction, defining an abstract BN-pair as follows.

Definition 2.63. ([AB10, Definition 6.55]) A BN-pair of a group G are subgroups B and N of G such that $G = \langle B, N \rangle$, $T = B \cap N$ is normal in N and $W = N/T$ is generated by a set S, satisfying the following:

- For all $s \in S$, $w \in W$, $(BsB)(BwB) \subseteq (BwB) \cup (BswB)$.

- For all $s \in S$, $sBs^{-1} \subseteq B$.

If we have a BN-pair for a group, then we can take $\Delta(G, B)$ to be the poset of parabolic subgroups of G (that is subgroups that contain a conjugate of B) under the opposite of the inclusion relation. This construction makes $\Delta(G, B)$ a building. Many of the classical examples of buildings come from the theory of BN-pairs.

2.2.4 Triangle Presentations

Finally, we will discuss another combinatorial object that arises from an affine building, introduced by Cartwright and his coauthors (see [CMSZ93], [Car95]). Recall that a group G can act on a building Δ by permuting the vertices of the building.

Definition 2.64. A group G's action on a set A is simply transitive if for all pairs $a_1, a_2 \in A$, there is a unique $g \in G$ where $g * a_1 = a_2$.

Example 2.62 uses the action of $PGL(n, \Bbbk)$ to construct a $\tilde{A}_{n-1}$ building Δ. Cartwright's goal when developing the theory of triangle presentations was to find subgroups of $PGL(n, \Bbbk)$ that act simply transitively on Δ. One could ask a similar question for any type of building, but the following proposition shows that it is only answerable in a special case.

Proposition 2.65. *([CMSZ93, Proposition 2.1]) Suppose Δ is an affine building whose Coxeter diagram is connected. Then there exists a group G that acts simply transitively on the vertices of Δ only if the building is of type $\tilde{A}_{n-1}$.*

Recall that we have a canonical labeling ℓ of the vertices of Δ with the integers $\{1, .., n\}$. A G-action is called type-rotating if for every $g \in G$ and vertex v in Δ, $\ell(g * v) = \ell(v) + c \mod n$ for a fixed c. Given a group G of type rotating automorphisms on Δ, [CMSZ93, Proposition 2.2] and [Car95, Theorem 2.6] show that there is a presentation of G in terms of generators and relations so that each generator g_x corresponds to a nearest neighbor vertex x of some fixed vertex v_0 in Δ, and all relations are of the form $g_x g_y g_z = 1$, for x, y, z nearest neighbor vertices of v_0. From this presentation, we also get an bijection σ between vertices of type k and type $n - k$ and list of 6 axioms for triples (x, y, z) where $g_z g_y g_z = 1$. This is the canonical example of a triangle presentation, and from this, we can obtain a general definition.

Definition 2.66. ([Car95, Page 46]) Let Π by any projective geometry of dimension $n \geq 3$. For $i \in [n-1]$, let $\Pi_i = \{u \in \Pi : \dim(u) = i\}$. Let $\sigma : \Pi \to \Pi$ be an involution such that $\sigma(\Pi_i) = \Pi_{n-i}$ for each i. A $\tilde{A}_{n-1}$-triangle presentation compatible with σ is a set $\mathcal{T}$ of triples (u, v, w), with $u, v, w \in \Pi$, satisfying the following properties.

1. Given $u, v \in \Pi$, then $(u, v, w) \in \mathcal{T}$ for some $w \in \Pi$ if and only if $\sigma(u)$ and v are distinct and incident.

2. If $(u, v, w) \in \mathcal{T}$, then $(v, u, w) \in \mathcal{T}$.

3. If $(u, v, w_1) \in \mathcal{T}$ and $(u, v, w_2) \in \mathcal{T}$, then $w_1 = w_2$.

4. If $(u, v, w) \in \mathcal{T}$ then $(\sigma(w), \sigma(v), \sigma(u)) \in \mathcal{T}$.

5. If $(u, v, w) \in \mathcal{T}$, then $dim(u) + dim(v) + dim(w) \equiv 0 \mod n$.

6. If $(x, y, u) \in \mathcal{T}$ and $(x', y', \sigma(u)) \in \mathcal{T}$, $dim(x) + dim(x') < n$ and $dim(y) + dim(y') < n$, then there exists a unique $w \in \Pi$, for which $(y', x, w) \in \mathcal{T}$ and $(y, x', \sigma(w)) \in \mathcal{T}$.

[Jon21] presents several alternate conditions for condition 6, which will prove useful to us.

Proposition 2.67. *([Jon21, Proposition 2.5]) Condition 6 in the above definition can be replaced by either of the following conditions:*

6'. If $(u, v, \sigma(r)), (r, w, \sigma(s)) \in \mathcal{T}$ and $dim(u) + dim(v) + dim(w) < n$, then there exists a unique $t \in \Pi$ such that $(u, t, \sigma(s))$, $(v, w, \sigma(t)) \in \mathcal{T}$.

6". If $(u, t, \sigma(s)), (v, w, \sigma(t)) \in \mathcal{T}$ and $dim(u) + dim(v) + dim(w) < n$, then there exists a unique $r \in \Pi$ such that $(u, t, \sigma(r))$, $(r, w, \sigma(s)) \in \mathcal{T}$.

Example 2.68. ([Jon21, Example 2.7]) Suppose $N = q^2 + q + 1$ for $q = p^n$, where p is prime. A cyclic planar difference set is a set $D \subset \mathbb{Z}/N\mathbb{Z}$ where every element of $\mathbb{Z}/N\mathbb{Z}$ can be written in a unique way as a difference of elements in D. Additionally, we call the cyclic planar difference set standard if D is invariant under multiplication by p.

From standard cyclic planar difference set, we can define an abstract projective plane where the points are the elements of $\mathbb{Z}/N\mathbb{Z}$, the lines are translates of D and the incidence relation is inclusion. We can also define a bijection σ from points to lines where $\sigma(z) = z + D$. From here, we can define the following sets:

$$T_1 = \{(z, z + d, z + (q + 1)d | z \in \mathbb{Z}/N\mathbb{Z}, d \in D\} \tag{2.12}$$

$$T_2 = \{(\sigma(z''), \sigma(z'), \sigma(z)) | (z, z', z'') \in T_1\} \tag{2.13}$$

$$T = T_1 \cup T_2 \tag{2.14}$$

We can show that T as defined is a triangle presentation. More details on this example and explicit examples of cyclic planar difference sets can be found in [Jon21].

CHAPTER

3

MODULE CATEGORIES FOR $\mathrm{Web}(\mathrm{SL}_n^-)$ FROM $\tilde{A}_{n-1}$ BUILDINGS

3.1 Introduction

In [Jon21], the author presents the construction of a fiber functor on a certain monoidal category $\mathrm{Web}(\mathrm{SL}_n^-)$ of type A webs in positive characteristic, which can be extended to a non-standard fiber functor on the $Rep(SL_{2k+1})$ as shown in [CEOP21, Example 7.3.4]. This functor is constructed through the use of a combinatorial structure called a triangle presentation of type $\tilde{A}_{n-1}$, as introduced in Section 2.2.4. The existence of a triangle presentation relies on the presence of a vertex-transitive action of a group on a building (see Proposition 2.65. This type of action requires the high level of symmetry found only in type A combinatorics. Thus we cannot apply the same ideas to obtain non-standard fiber functors in other types.

Instead of looking for fiber functors, it is natural to look more generally for *module categories* arising from the combinatorics of buildings, which may exist outside type A. There is a canonical module category for $\mathrm{Rep}(SL_n)$ whose underlying category is the category of vector bundles on the type A weight lattice Λ_W, which arises from restriction. In a certain sense, a locally finite building of type $\tilde{A}_{n-1}$ and order q (see note after Lemma 3.6) can be thought of as a q-analogue of Λ_W. This motivates the following result, which is the main theorem of this chapter.

Theorem A. *If* $\Bbbk$ *is a field of characteristic* $p \geq n-1$ *and* Δ *is an* $\tilde{A}_{n-1}$ *building of order* $q \equiv 1 \mod p$*, there is a monoidal functor* $\mathrm{Web}(SL_n^-) \to End(Vec(\Delta))$*, where both categories are defined over* $\Bbbk$*. When* n *is odd, this equips* $Vec(\Delta)$ *with the structure of a module category over* $Tilt(SL_n)$*.*

In the above theorem, $Vec(\Delta)$ denotes the category of vector bundles over the set of vertices of Δ. From this module category, we can recover the non-standard fiber functor introduced in [Jon21] by studying symmetries. Any action of a group G on Δ induces an action on the category $Vec(\Delta)$ and so we can consider its equivariantization $Vec(\Delta)^G$. Our second main theorem extends the module category of Theorem A to $Vec(\Delta)^G$.

Theorem B. *Let* $\Bbbk$ *be a field of characteristic* $p \geq n-1$*. For any type-rotating action of a group* G *on* Δ *there exists a monoidal functor* $\mathrm{Web}(SL_n^-) \to End(Vec(\Delta)^G)$*, equipping* $Vec(\Delta)^G$ *with the structure of a* $\mathrm{Web}(SL_n^-)$ *module category.*

When G acts simply transitively on an affine building in type A, then $Vec(\Delta)^G \simeq Vec$, and as we have $\mathrm{End}(Vec) \cong Vec$, we recover the fiber functor on the $\mathrm{Web}(\mathrm{SL}_n^-)$ category constructed in [Jon21].

In order to build the module category from Theorem A, we consider an intermediate tensor category $G(\Delta)$, called the *graph planar algebra* of Δ. Graph planar algebras were introduced in [Jon00], [Jon99] and [Mor10]. Our version varies slightly, but we show in Remark 3.12 how it fits into the pre-existing framework.

We expect similar results relating web categories and buildings should be true outside of type A. As outlined in Section 1.1, there have been many web categories defined in other types following the introduction of web categories in [Kup96]. In Chapter 4 of this dissertation, we consider this question in type C, taking the web category presentation of [BERT21] (or, more accurately, [Bod22]) that corresponds to the representation theory of $\mathfrak{sp}_{2n}$ and the Coxeter/Lie combinatorics in type C. We will show an elementary result and discuss challenges in duplicating our type A construction. Finally, we remark that as in [Jon21], the functors and natural transformations defining our module category make sense in characteristic 0, but do not satisfy the $\mathrm{Web}(\mathrm{SL}_n^-)$ relations. Instead they generate a new category related to the quantum automorphism group of the building, introduced and studied in [RV22], building on the previous work of [VV19]. It would be interesting to precisely clarify the relationship between $\mathrm{Web}(SL_n^-)$ and the quantum automorphism group of these graphs.

The structure of this chapter is as follows. In Section 3.2, we recall from Chapter 2 the main players in our work, namely $\tilde{A}_{n-1}$ buildings and the categories $\mathrm{Web}(\mathrm{SL}_n^-)$, and define $G(\Delta)$ for a locally finite building Δ of type $\tilde{A}_{n-1}$. Section 3.3 defines the functor $\mathrm{Web}(\mathrm{SL}_n^-) \to G(\Delta)$, while also proving some results about buildings of order 1 (which are Coxeter complexes). Section 3.4 then details the construction of $Vec(\Delta)^G$ as a $\mathrm{Web}(\mathrm{SL}_n^-)$ module category and provide examples of the category for several actions of G on Δ. The results of this chapter originally appear in [McG22] and much of the text is taken directly from this source.

3.2 Preliminaries

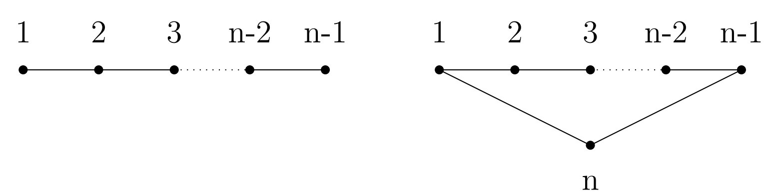
3.2.1 Buildings and Graph Planar Algebras

In this section, we will review the definitions of Coxeter complexes and buildings, with a focus on types $\tilde{A}_{n-1}$ and A_{n-1}. We will also state several facts about the structure of these objects and discuss their connections to each other and to the representation theory of the Lie algebra $\mathfrak{sl}_n$. We will then introduce graph planar algebras, and describe the construction of a graph planar algebra for any building of type $\tilde{A}_{n-1}$. We refer the reader to [AB10] and Section 2.2 for further background on Coxeter systems and simplical complexes.

Coxeter Complexes and Buildings

Recall from Definition 2.45 that the special subgroups of a Coxeter system (W, S) are the subgroups $\langle S' \rangle$ for some $S' \subseteq S$ and the special cosets are the cosets of theses subgroups. The Coxeter complex of (W, S), $\Sigma(W, S)$ or Σ if context is clear, is the poset of special cosets under the opposite of the inclusion relation. This poset is a labelable chamber complex (see Proposition 2.49).

Recall that we can represent a Coxeter system (W, S) as a graph called the Coxeter diagram of (W, S). In this chapter, we are interested in the groups with the following Coxeter diagrams.



(3.1)

These diagrams represent the Coxeter systems of type A_{n-1} and $\tilde{A}_{n-1}$ respectively. A group of type A_{n-1} is isomorphic to the symmetric group S_n and $\tilde{A}_{n-1}$ is the corresponding affine, irreducible system.

The link of a simplex X in a simplicial complex Σ (denoted $lk_\Sigma(X)$) is the subcomplex of all simplices that are both disjoint from and joinable to X (see Definition 2.43). If a is a vertex of Σ, then $lk_\Sigma(a)$ is the induced subcomplex with vertex set $V = \{b|b$ is connected by an edge to a in $\Sigma\}$. If $lk_\Sigma(a)$ has finitely many vertices for every vertex $a \in \Sigma$, we call Σ locally finite. We have the following results about the links of the A_{n-1} and $\tilde{A}_{n-1}$ Coxeter complexes.

Lemma 3.1. *The link (see Definition 2.43) of a vertex in a Coxeter complex of type $\tilde{A}_{n-1}$ is isomorphic to the Coxeter complex of type A_{n-1}*

Proof. To see this, notice that we identify each vertex v in our Coxeter complex with a coset of a special subgroup with generating size $n-1$. This special subgroup is isomorphic to A_{n-1}, and so we can get cosets of special subgroups of size $n-2$ contained in our initial vertex coset by first taking a coset of size $n-2$ in this isomorphic copy of A_{n-1} and then multiplying by the coset representative in $\tilde{A}_{n-1}$ on the left. We know that these cosets of size $n-2$ count two objects: edges adjacent to v in the Coxeter Complex $\tilde{A}_{n-1}$ and vertices in the Coxeter complex of type A_{n-1}. So we can identify a vertex adjacent to v by the coset related to it's edge. We must now show that this identification preserves adjacencies.

Suppose we have a triangle with vertices v, x and y in our Coxeter complex of type $\tilde{A}_{n-1}$. We must show that this occurs exactly when the cosets associated to the edges from v to x and v to y are connected in our isomorphic copy of A_{n-1}. We call these edges e_1 and e_2 respectively and refer to the coset related to an edge or vertex simply by the name we have given that edge or vertex. The existence of the triangle v, x, y implies that there is some coset with generating set of size $n-3$ that is contained in $v \cap x \cap y$. We call this coset t. Since e_1 and e_2 bound this triangle, we have $t \subset e_1$ and $t \subset e_2$. Now, these inclusions imply that t is a coset of the isomorphic copy of A_{n-1} that we had previously defined. So t as a coset defines the edge between e_1 and e_2 viewed as vertices in this type A_{n-1} complex, and so e_1 and e_2 are adjacent in this complex.

To show the opposite direction, if e_1 and e_2 are adjacent in our A_{n-1}. Then there exists some coset with generating set size $n-3$ that is contained in $e_1 \cap e_2$. But when we map into the type $\tilde{A}_{n-1}$ complex, we see that $e_1 \cap e_2 \subset v \cap x \cap y$, where x and y denote the opposite end of edges e_1 and e_2. So $t \subset v \cap x \cap y$ and this implies the existence of a triangle with vertices v, x, y which means that x and y are adjacent. So we have shown the desired isomorphism. □

Lemma 3.2. *The Coxeter complex of type A_n is isomorphic to the flag complex of proper non-empty subsets of $[n]$.*

Proof. First, we show that the vertices of the these structures are in bijection. The vertices of the Coxeter complex are cosets $w\langle S - \{s_i\}\rangle$, for some $w \in W$ and $i \in [n]$, where (W, S) is the Coxeter system of type A_n. We will identify $w\langle S - \{s_i\}\rangle$ with the subset $\{w(1), w(2), ..., w(i)\}$, and show that this map, which we call α, is a bijection.

First, we show that α is injective. Suppose that $\alpha(w\langle S - \{s_i\}\rangle) = \alpha(w'\langle S - \{s_j\}\rangle)$. First, see that this means $s_i = s_j$. So we have $\{w(1), ..., w(i)\} = \{w'(1)....w'(i)\}$. Applying w^{-1} to both subsets, we see that $\{1, ..., i\} = \{w^{-1}w'(1), ..., w^{-1}w'(i)\}$. This implies that $w^{-1}w' \in \langle S - \{s_i\}\rangle$. So we have $w\langle S - \{s_i\}\rangle = w'\langle S - \{s_i\}\rangle$ and so α is injective. To show that α is surjective, take some subset A of $[n]$. Suppose that $|A| = i$, and fix some ordering $a_1, ..., a_i$ of the elements of A. Now, take w_A to be some permutation so that $w(j) = a_j$ for all $j \in [i]$. See that $\alpha(w_A\langle S - s_i\rangle) = A$. So α is indeed a bijection between the vertex set of a type A_n Coxeter complex and that of a flag complex of proper non-empty subsets of $[n]$.

Now, we must show that two cosets are adjacent in the Coxeter complex if and only if the corresponding subsets under α are adjacent in the subset flag complex. To see this, first suppose we have adjacent cosets $w\langle S - \{s_i\}\rangle$ and $w'\langle S - \{s_j\}\rangle$. This implies that the cosets have non-zero intersection (more specifically that they both contain some special coset with generating set of size $n - 2$), and this means that if $i = j$, these are cosets are not adjacent but equal. So we assume without loss of generality that $i < j$ and proceed. See that this adjacency is equivalent to saying that $\langle S - \{s_i\}\rangle$ is adjacent to $w^{-1}w'\langle S - \{s_j\}\rangle$. So we will work with these cosets instead. See that $\langle S - \{s_i\}\rangle \cap w^{-1}w'\langle S - \{s_j\}\rangle \neq \emptyset$ if and only if $w^{-1}w' \in \langle S - \{s_i\}\rangle$. This inclusion means that $w^{-1}w'$ must stabilize the subsets $\{1, ..., i\}$ and $\{i + 1, ..., j\}$. So we have that $\{1, ..., i\} = \{w^{-1}w'(1), ..., w^{-1}w'(i)\}$. Now, since $i < j$, we have that $\{w^{-1}w'(1), ..., w^{-1}w'(i)\} \subset \{w^{-1}w'(1), ..., w^{-1}w'(j)\}$. So we have $\{1, ..., i\} \subset \{w^{-1}w'(1), ..., w^{-1}w'(j)\}$, which proves adjacency of the image of these cosets in the flag complex. This proves adjacency of the images of our original cosets as well.

Now, suppose we have subsets A, B with $A \subset B$ so that they are adjacent in our flag complex. We can find some permutation $w^* \in W$ so that $A = \{w^*(1), ..., w^*(i)\}$ and $B = \{w^*(1), .., w^*(j)\}$. So $\alpha^{-1}(A) = w^*\langle S - \{s_i\}\rangle$ and $\alpha^{-1}(B) = w^*\langle S - \{s_j\}\rangle$. See that $\alpha^{-1}(A) \cap \alpha^{-1}(B) = w^*\langle S - \{s_i, s_j\}\rangle$, and so $\alpha^{-1}(A)$ is adjacent to $\alpha^{-1}(B)$.

So we have adjacency in the Coxeter complex if and only if we have adjacency under the image of α in the 1-skeleton of the flag complex. This is enough to prove that the complexes are isomorphic. □

Note that the isomorphism in the above lemma is consistent with the canonical labelling of Σ (i.e. vertices with the same label are matched to subsets of the same size). We will use these facts throughout this chapter.

The $\tilde{A}_{n-1}$ Coxeter complex Σ appears in the representation theory of the Lie algebra $\mathfrak{sl}_n$. In particular, the $\mathfrak{sl}_n$ coroot lattice is embedded in Σ (see [AB10, Section 10.1.8]. If $n = 3$, it is easy to see that as graphs, Σ is isomorphic as a graph to the $\mathfrak{sl}_n$ weight lattice. We show in Section 3.3.2 the connection between the 1-skeleton of the $\tilde{A}_{n-1}$ Coxeter complex, the sl_n weight lattice, and the Cayley graph of a distinguished group that we will define in that section. A way to see this is by choosing a Weyl chamber as in [FH04, Lecture 14]. The collection of these Weyl chambers is isomorphic to a A_{n-1} Coxeter complex (or more specifically, the flag complex of proper subsets of $\{1, ..., n\}$ as previously defined).

A building of type $\tilde{A}_{n-1}$ is a simplicial complex which is built out of Coxeter complexes, originally introduced by Tits [Tit74, Definiton 3.1]. We recall Definition 2.53.

Definition 3.3. A building Δ is a simplicial complex with a distinguished set of subcomplexes called apartments that satisfy the following properties:

1. Each apartment is a Coxeter complex for some Coxeter system (W, S).
2. If A and B are simplices in Δ, then there is some apartment Σ_{AB} containing them both.
3. There is an isomorphism between any two apartments Σ and Σ' fixing the intersection $\Sigma \cap \Sigma'$.

Recall the collection $\mathcal{A}$ of apartments of Δ is called a system of apartments and that one building can be equipped with multiple possible systems of apartments. All apartments of a building must be Coxeter complexes of the same type (see Definition 2.56 and [AB10, Proposition 4.7]). So we will say Δ is type $\tilde{A}_{n-1}$ if its apartments are of type $\tilde{A}_{n-1}$. For examples of buildings, refer to Examples 2.58 and 2.62.

The labeling on each apartment Σ previously discussed can be extended to a consistent labeling of the building Δ (i.e. a labeling that restricts to the labeling of the Coxeter complex on any apartment Σ) - see 2.55. So a building is itself a labelable chamber complex. We can describe the links of vertices in building in a analogous way to those of Coxeter complexes.

Lemma 3.4. *The link of a vertex v_0 in a building Δ of type $\tilde{A}_{n-1}$ is a building of type A_{n-1}.*

Proof. To see this, first fix a vertex v_0 in a $\tilde{A}_{n-1}$ building Δ. Consider the subset $\mathcal{A}'$ of a system of apartments $\mathcal{A}$, where $\mathcal{A}'$ are precisely the apartments containing v_0. We know that for each $\Sigma \in \mathcal{A}'$, we have that the neighbors of $v+0$ in Σ form a A_{n-1} Coxeter complex Σ_*, and that every vertex adjacent to v_0 will be contained in some apartment $\Sigma \in \mathcal{A}'$. So we take Δ_* to be $\bigcup_{\Sigma\in\mathcal{A}'} \Sigma_*$. We will show that Δ_* is a building with system of apartments given by the collection of Σ_*'s.

First, suppose we have two simplices A_* and B_* in Δ_*. Since v_0 must is connected to every vertex of both A_* and B_*, we have simplices $A = A_* \cup \{v_0\}$ and $B = B_* \cup \{v_0\}$ in Δ. So we have some apartment $\Sigma \in \mathcal{A}'$ such that A and B are both in Σ (we know $\Sigma \in \mathcal{A}'$ since $a \in A$). Now, see that when we remove a, we have $A_* \subset \Sigma_*$ and $B_* \subset \Sigma_*$. So we have A_* and B_* in a common apartment in Δ_*, which proves axiom (B1).

Next, suppose that we have A_*, B_* contained in two apartments Σ_* and Σ'_* of Δ_*. We can again adjoin a to see that $A = A_* \cup \{v_0\}$ and $B = B_* \cup \{v_0\}$ are contained in both Σ and Σ'. Now, since Δ is a building, we have an isomorphism $\Sigma \to \Sigma'$ fixing A and B pointwise. Since this isomorphism fixes v_0 pointwise, it must take neighbors of a to neighbors of v_0. Since it also fixes A_* and B_*, we see that its restriction to Σ_* is an isomorphism $\Sigma_* \to \Sigma'_*$ which fixes A_* and B_* pointwise. So this proves axiom (B2). Therefore, we have Δ_* a building of type A_{n-1} as desired. □

In order to describe buildings of type A_n , we recall the following definition.

Definition 3.5. [Moo07, pg. 127] A finite projective geometry is a incidence geometry satisfying the following, where we call n the projective dimension of the geometry (A subspace S is a collection of points in the geometry such that for every two of its points, the line containing these points is also in S):

- There is a unique line through any two points.
- There exist three non-colinear points.
- Every line contains at least three points.
- A chain of (non-empty) subspaces (under the partial order of containment) has length at most $n+1$.
- Every line that is incident with two side of a triangle, and not with the vertices of the triangle, must be incident with the third side of the triangle.

Notice that this definition is a generalization of the definition of a projective plane (Definition 2.57). A locally finite building is a building where the link of every vertex is finite. Equipped with this terminology, we can state the following lemma due to Tits.

Lemma 3.6. *[Tit74, Theorem 6.3] A locally finite building of type A_{n-1} is isomorphic to a finite projective geometry of projective dimension $n-1$.*

The order of the building is defined as the order of the finite projective geometry arising as the link of (any) vertex. This gives us an interesting description of the relationship between Coxeter complexes and buildings. We recall that we can define the q-integer $[k] = \frac{q^k - q^{-k}}{q - q^{-1}}$ and $[k]!_q = [k]_q[k-1]_q...[2]_q[1]_q$ To see this, we first recall that in a finite projective geometry of algebraic dimension n and order q, the number of subspaces of dimension k is $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!_q}{[k]!_q[n-k]!_q}$ (see [Moo07, Page 121]). If we set $q = 1$, this is simply equal to $\binom{n}{k}$. This is of course the number of subsets of size k of an n element set. So in the philosophy of [Tit57], a flag complex of proper non-empty subsets of $[n]$ can be thought of as a finite projective geometry of dimension n over Tits' degenerate "field of order 1". Reversing the logic, this suggests that we can think of general finite type A_{n-1} buildings as q-analogues of the A_{n-1} Coxeter complex.

Now we consider this in the affine case. Lemma 3.3 and the facts stated we can say that $\tilde{A}_{n-1}$ Coxeter complexes are equivalent to $\tilde{A}_{n-1}$ buildings of "order 1". Since the 1-skeleton of the $\tilde{A}_{n-1}$ Coxeter complex is isomorphic as a graph to the Cayley graph of the weight lattice of $\mathfrak{sl}_n$, we can say that a locally finite $\tilde{A}_{n-1}$ building is a q-analogue of this weight lattice. Another way to say this, is that the Coxeter complex is the degenerate building over the field of order 1 as imagined in [Tit57].

As we finish our discussion of how finite projective geometry ties into the theory of buildings, we state several facts that we will use in our proof of Lemma 3.23 and Theorem 3.24. Recall that in algebraic dimension ≥ 4, these questions are problems in linear algebra, as we must only consider classical projective geometries. In algebraic dimension 3, we must turn to the theory of projective planes. Both of the facts below follow from the basic theory of projective planes (see [Moo07, Chapter 6] and in particular Theorem 6.3 for justification). Note that the projective dimension of the geometries in these Lemmas is $n-1$.

Lemma 3.7. *In a finite projective geometry of algebraic dimension n and order q, the number of subspaces of algebraic dimension k is $\begin{bmatrix} n \\ k \end{bmatrix}_q$*

Lemma 3.8. *In a finite projective geometry of algebraic dimension n and order q, the number of subspaces of algebraic dimension k containing some fixed m-dimensional subspace is $\begin{bmatrix} n-m \\ k-m \end{bmatrix}_q$*

Finally, we note that for subspaces V and W of a vector space U, we have $dim(V+W) = dim(V) + dim(W) - dim(V \cap W)$. This fact also holds for projective planes, and is easy to see by inspection of the definition.

3.2.2 The Graph Planar Algebra $G(\Delta)$

We can view the 1 skeleton of a type $\tilde{A}_{n-1}$ building Δ as a graph which we call Γ_Δ, where we define the vertex and edge sets of Γ_Δ as the 0 and 1 simplices of Δ respectively. This graph is undirected, but we can easily view it as as directed graph by replacing each edge with two directed edges with opposite sources and targets. Recall that every building has a vertex labeling that is consistent with the labeling of its apartments. So we can label $V(\Gamma_\Delta)$ by the elements of the set $[n]$ in a way that is consistent with the labeling on Δ (where a vertex v has label $\ell(v)$) . We use this vertex labeling to define a labeling on $E(\Gamma_\Delta)$ by labeling an edge from x to y with $\ell(x) - \ell(y) \mod n$. We see then that if the edge $x \to y$ has label k, then edge $y \to x$ has label $n - k$ mod n. We will abuse notation slightly by using Γ_Δ to refer to this directed graph as well. In fact, we will almost exclusively use the directed version of Γ_Δ.

Remark 3.9. Recall that for any vertex v_0 in Δ a $\tilde{A}_{n-1}$ building of order q, that $lk_\Delta(v_0)$ is isomorphic to a A_{n-1} building. This is in turn isomorphic to a finite projective geometry of order q. So we can choose to identify every vertex in $lk_\Delta(v_0)$ (or equivalently all the edges in Γ_Δ originating at v_0) with a proper subspace of an n dimensional vector space in a way that the label of an edge in Γ_Δ will be equal to with the dimension of the subspace to which it is identified (see the proof of [Tit74, Theorem 6.3]). Incidence in the projective geometry corresponds to inclusion of subspaces. So a cycle $v_0 \to x \to y \to v_0$ exists in Γ_Δ if and only if one of the subspaces associated to x and y is contained in the other.

Now, equipped with an encoding of any $\tilde{A}_{n-1}$ building as a directed, edge-labeled graph, we make the following definitions.

Definition 3.10. If p is a path in Γ_Δ with edges $e_1, ..., e_k$, then the type of p (notation $type(p)$), is the tuple of length k where $(type(p))_i = \ell(e_i)$.

Definition 3.11. For a graph Γ whose edges are labelled by natural numbers, we define the category $G(\Gamma)$ over a field $\Bbbk$ as the category whose:

- Objects are finite sequences of natural numbers (selected from the edge labels of Γ).
- A morphism between objects σ and τ is a linear functional

$$f : \Bbbk(\{(p_1, p_2) |\ type(p_1) = \sigma,\ type(p_2) = \tau\}) \to \Bbbk$$

where $f((p_1, p_2)) = 0$ unless the starting and ending points of p_1 and p_2 coincide.

- A composition of morphisms $f : \sigma \to \tau$ and $g : \tau \to \omega$ is defined on a matched pair (p_1, p_2) of type (σ, ω) as follows:

$$(g \circ f)((p_1, p_2)) = \sum_{p' \in P'} f((p_1, p')) * g((p', p_2)) \tag{3.2}$$

  where P' is the set of paths of type τ whose starting and ending vertices coincide with p_1 and p_2.

This category is a strict monoidal category. The monoidal product acts in the following way

- For sequences σ and τ, $\sigma \otimes \tau = (\sigma_1, ..., \sigma_k, \tau_1, ..., \tau_m)$.
- For morphisms $f : \sigma \to \tau$ and $g : \omega \to \mu$,

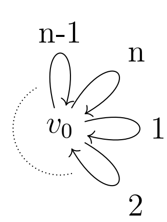
$$(f \otimes g)((p_1 \otimes q_1, p_2 \otimes q_2)) = f((p_1, p_2)) * g((q_1, q_2))$$

  .

The monoidal unit is the empty sequence.

The main focus of our discussion will be $G(\Gamma_\Delta)$ for Δ a locally finite building of type $\tilde{A}_{n-1}$. We will use the short hand $G(\Delta)$ for this category. The categories we have described above are instances of *graph planar algebras* originally introduced by V.F.R. Jones in the context of subfactors (see [Jon00], [Jon99] and [Mor10]) Our version is slightly different but very similar in spirit.

Remark 3.12. Looking to the work in [Mor10] we see that $G(\Delta)$ follows the authors' definition of a graph planar algebra in the following way. Let G be the following graph



Let $\pi : \Gamma_\Delta \to G$ be the homomorphism sending every element in $V(\Gamma_\Delta)$ to the single vertex v_0 in G and every edge with label i in $E(\Gamma_\Delta)$ to the edge in G labeled i. Now, we can use the structure of $\tilde{A}_{n-1}$ buildings to see that choosing $\delta_k = \begin{bmatrix} n \\ k \end{bmatrix}_q$ and $d(v) = 1$ for all $v \in V(\Delta_\Gamma)$ gives us Perron-Frobenius data for this homomorphism.

Notice that Γ_Δ is not finite, and thus does not exactly satisfy [Mor10]'s definition of a bidirected graph, but that this construction only requires Γ_Δ to be locally finite, which it is. We then see that $G(\pi)$ as defined in ([Mor10], definition 2.5) is equivalent to our previously defined $G(\Delta)$. Because of this connection, we will often refer to $G(\Delta)$ as the graph planar algebra of Δ.

Finally, we introduce the following definition which will help us discuss this category locally.

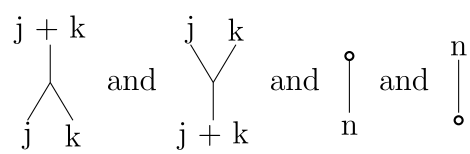
Definition 3.13. For a list of composable morphisms $f_1, f_2, \ldots, f_k$ in $G(\Delta)$ where $f_i : \sigma_i \to \sigma_{i+1}$, a labeling is a specific choice of paths of types $\sigma_1, \ldots, \sigma_n$ with common initial and final vertices. For example each choice of p' in 3.2 gives us a labeling (p_1, p', p_2) of f, g.

Notice that we can use this idea to reword the definition of composition in Definition 3.11 by saying that we can compose a list of morphisms $f_1, f_2, \ldots, f_k$ by summing over all labelings of this list and taking the product of values in each labeling. For a specific evaluation of this map (i.e. $f_1 \circ f_2 \circ f_k(p_1, p_2)$), we will consider only labelings starting with p_1 and ending with p_2. Also, if we have a morphism that is the sum of multiple web pictures, we can evaluate it by independently summing over labelings of each summand and then adding the values together.

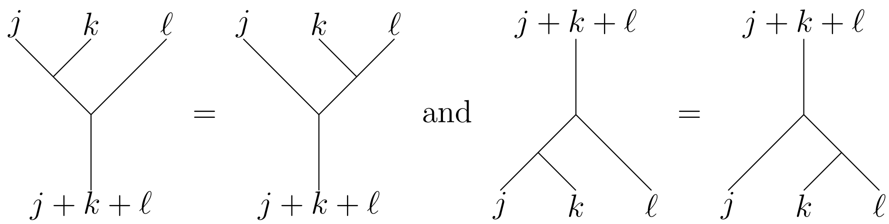
3.2.3 The Category $\mathrm{Web}(\mathbf{SL}_n^-)$

In this section, we introduce the category $\mathrm{Web}(\mathrm{SL}_n^-)$ as an an example of a class of categories often called "web categories." (first found in [Kup96]). These categories, described by diagrammatic generators and relations, derive their name from their morphisms, as complicated compositions in these category have pictorial descriptions resembling spiderwebs. We will examine the representation theoretic context of $\mathrm{Web}(\mathrm{SL}_n^-)$ in Subsection 3.2.3.

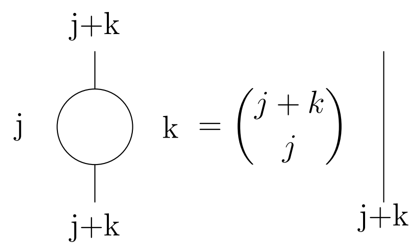
$\mathrm{Web}(\mathrm{SL}_n^-)$ was introduced in [Jon21] as an extension of the categories described in [BEAEO20]. These are closely related to the sl_n web categories originally introduced in [CKM14], though our categories do not have a chosen pivotal structure. Formally, the category $\mathrm{Web}(\mathrm{SL}_n^-)$ is the category whose objects are finite sequences in $[n]$ and whose morphisms are generated by the following diagrams (called webs):



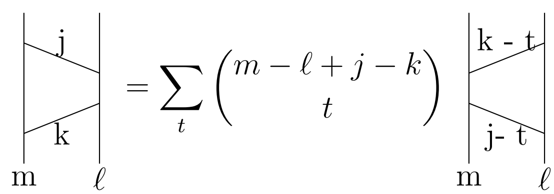
Here, j and k can be any natural number labels, such that $j + k \leq n$. If $j + k > n$, then the morphism does not exist. The collection of morphisms in this category is just the collection of formal $\Bbbk$ linear combinations of horizontal and vertical compositions of these generating morphisms. The morphisms satisfy the following relations.



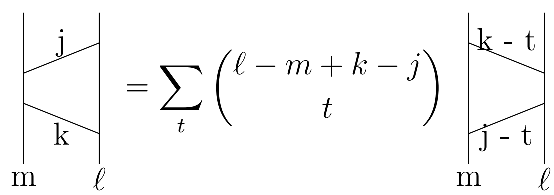
(3.3)



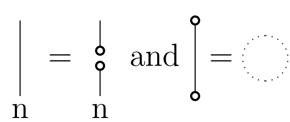
(3.4)



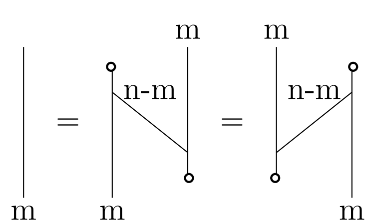
(3.5)



(3.6)



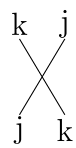
(3.7)



(3.8)

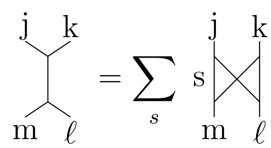
We will call relation 3.3 the associativity and coassociativity relation, relation 4.5 the bigon bursting relation and relations 3.5 and 3.6 the square switch relations. We define a tensor product on objects of $\mathrm{Web}(\mathrm{SL}_n^-)$ as concatenation of lists and on morphisms as horizontal composition of pictures. This construction makes $\mathrm{Web}(\mathrm{SL}_n^-)$ a strict monoidal category (see [EGNO15] and Section 2.1.1 for background information on monoidal categories). Furthermore, we see that the last relation for morphism shows that we have duals in Web(SL_n^-). So Web(SL_n^-) is also a rigid monoidal category.

Remark 3.14. The category $PolyWeb(GL_n)$ is essentially Web(SL_n^-) without lollipops and any relations involving them. In [BEAEO20], the category $PolyWeb(GL_n)$ is originally defined with three types of generators, the third being



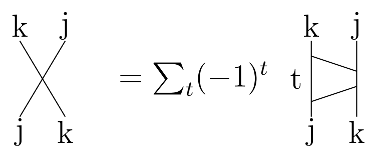
,

and relations 3.3, 3.4 and the following:



(3.9)

It turns out that including a crossing generator is redundant, as we can use the relations to show that



(3.10)

Using this rewriting of the crossing generator, relations 3.5 and 3.6 can replace relation 3.9.

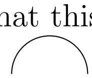
Web($\mathbf{SL}_n^-$) in Context

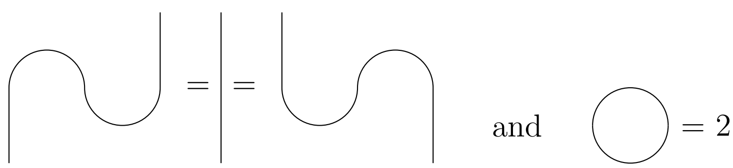
A quotient of a monoidal category C is a dominant monoidal functor $C \to D$ for a monoidal category D. One way to derive Web(SL_n^-) is as a quotient of a larger web category $PolyWeb(GL_n)$. In fact, this is how [Jon21] presents this category.

The theory of web categories gives us a pictorial way to describe categories of representations. For example, the larger category $PolyWeb(GL_n)$ is equivalent to the category of polynomial representations of GL_n over an algebraically closed field. When $n = 2k + 1$, we have that Web(SL_n^-) is equivalent to a web category Web(SL_n^+), with almost identical structure (the major distinction between these categories are the coefficents attached to relation 6). If we take these categories over an algebraically closed field $\Bbbk$, we have that the Karoubi envelope of Web(SL_n^-) is equivalent to $Tilt(SL_n)$, the category of titling modules for

SL_n (see [Jon21, Remark 3.4]). As a result (and because [GMP+23, Theorem 3.21] show that the Karoubi completion is categorical), functors from $Tilt(SL_{2k+1})$ are completely described by functors out of $\mathrm{Web}(\mathrm{SL}^-_{2k+1})$, where we must simply define images for all generating morphisms that satisfy the $\mathrm{Web}(\mathrm{SL}^-_n)$ relations.

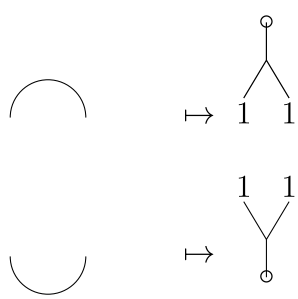
In characteristic 0, $Tilt(SL_n)$ is equivalent to the category $Rep(SL_n)$, but in positive characteristic it is not. Instead, it is simply a full subcategory of $Rep(SL_n)$. In this setting, $Tilt(SL_n)$ can be characterized as the subcategory of $Rep(SL_n)$ whose behavior mimics that of $Rep(SL_n)$ in characteristic 0. Additionally, $Rep(SL_n)$ is the Abelian envelope of $Tilt(SL_n)$ [CEOP21].

When n is even, we believe that $\mathrm{Web}(\mathrm{SL}^-_n)$ corresponds to tilting modules for a $q = -1$ deformation of SL_n. For example, when $n = 2$, we have that $\mathrm{Web}(\mathrm{SL}^-_2) \simeq TLJ(2)$, the Temperley-Lieb-Jones category (see [KL94], [Tur94], [Che14]) with loop parameter $\delta = 2$. Remember that this category is the $\Bbbk$-linear strict monoidal category whose morphisms are generated by  and  with relations:



(3.11)

The monoidal equivalence mentioned above comes from extending the following identification to all of $TLJ(2)$.



(3.12)

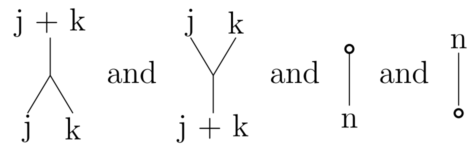
(3.13)

3.3 Embedding $\mathrm{Web}(\mathbf{SL}^-_n)$ in $G(\Delta)$

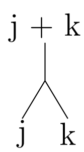
Our goal for this section is to define a functor from $\mathrm{Web}(\mathrm{SL}^-_n)$ to $G(\Delta)$ for any locally finite type $\tilde{A}_{n-1}$ building Δ. We will do this by specifying the images of the generating maps in $\mathrm{Web}(\mathrm{SL}^-_n)$. We will then explore what it would mean to have these images satisfy the $\mathrm{Web}(\mathrm{SL}^-_n)$ relations, and introduce the language we will use to ultimately prove the existence of this functor.

3.3.1 Defining the Maps

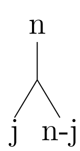
Recall, the generating maps of the category Web(SL_n^-) are:



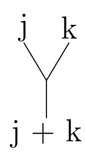
Now, notice that in both Web(SL_n^-) and $G(\Delta)$ the objects are sequences of integers in $\{1, ..., n\}$. We map a sequence of integers in Web(SL_n^-) to the corresponding sequence $\mod n$ in $G(\Delta)$. This allows us to identify $n \in$ Web(SL_n^-) with 0 in $G(\Delta)$. We now define the images of the generating maps in $G(\Delta)$.

For $j + k < n$, 

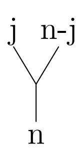
$\mapsto \mathbb{1}_{j,k} : \Bbbk(\{(p_1, p_2)|\ \mathit{type}(p_1) = (j, k) \text{ and } \mathit{type}(p_2) = (j + k)\}) \to \Bbbk$



$\mapsto \mathbb{1}_{j,n-j} : \Bbbk(\{(p_1, p_2)|\ \mathit{type}(p_1) = (j, n - j) \text{ and } \mathit{type}(p_2) = \emptyset\})$

For $j + k < n$, 

$\mapsto \mathbb{1}_{j+k} : \Bbbk(\{(p_1, p_2)|\mathit{type}(p_1) = (j + k) \text{ and } \mathit{type}(p_2) = (j, k)\}) \to \Bbbk$



$\mapsto \mathbb{1}_{\emptyset} : \Bbbk(\{(p_1, p_2)|\ \mathit{type}(p_1) = \emptyset \text{ and } \mathit{type}(p_2) = (j, n - j)\})$

$\mapsto \mathbb{1}_{\emptyset,\emptyset} : \Bbbk(\{(p_1, p_2|\ \mathit{type}(p_1) = (\emptyset) \text{ and } \mathit{type}(p_2) = \emptyset\}) = \Bbbk \to \Bbbk$

$\mapsto \mathbb{1}_{\emptyset,\emptyset} : \Bbbk(\{(p_1, p_2|\ \mathit{type}(p_1) = \emptyset \text{ and } \mathit{type}(p_2) = (\emptyset)\}) = \Bbbk \to \Bbbk$

Here, $\emptyset$ represents the path with no edges. We must use this when we have a label n, since we do not have an edge label $n \equiv 0$ in Γ_Δ (this comes from the fact that the two distinct vertices of the same type cannot be connected in the $\tilde{A}_{n-1}$ Coxeter complex). The map $\mathbb{1}_{j,k}$ evaluates to 1 on a matched pair (p, q), if pq, the concatenation of p and q is a cycle of length 2 or 3, and 0 otherwise (for $\mathbb{1}_{\emptyset,\emptyset}$, this simply evaluates to 1 on every vertex). Similarly, the other maps defined also evaluate to 1 on every 2 and 3 cycle and 0 elsewhere. To show that this construction indeed gives us a functor $\text{Web}(\text{SL}_n^-) \to G(\Delta)$, we must show that the maps defined above satisfy the $\text{Web}(\text{SL}_n^-)$ relations.

Since the $\text{Web}(\text{SL}_n^-)$ relations are pictorial, in order to show that the maps we have chosen in $G(\Delta)$ satisfy these relations we wish to have a pictorial understanding of them as well. To do this, we notice that the generator of $\text{Web}(\text{SL}_n^-)$ correspond to triangles and length 2 loops in Γ_Δ. Specifically, each string in one of these generator represents an edge whose label matches that of the string. As a result, each region bounded by two or more strings can be labeled with a vertex of Γ_Δ. In order for a composition to have a valid labeling as in Definition 3.13 we must have a collection of compatible triangles in Γ_Δ (by compatible, we mean triangles that share an edge where two $\text{Web}(\text{SL}_n^-)$ generators are composed). Also, note that the maps $\mathbb{1}_{j,k}, \mathbb{1}_{j+k}, \mathbb{1}_{j,n-j}, \mathbb{1}$ take the values 0 or 1. Any composition of these maps is a function that simply counts the number of triangle arrangements that satisfy this composition. Of course, as before, evaluating a map at a specific pair of paths fixes the initial and final paths in a labeling. See that this fixes some vertices in our triangle arrangement as well (If we draw this arrangement of triangles on top of our pictorial composition, these fixed vertices will appear in the boundary regions of our picture).

3.3.2 The Degenerate Case

To aid in our proof of the existence of the desired functor for all $\tilde{A}_{n-1}$ buildings, we explore the special case where $q = 1$. In this case, note that as previously discussed we can treat a collection of finite subsets of a set of size n as a finite projective geometry of order 1. We will now show in detail, the connection between the $\mathfrak{sl}_n$ weight lattice and the $\tilde{A}_{n-1}$ Coxeter complex. We make the following definition to provide an alternative presentation of the weight lattice. This definition will help us to prove the existence of the functor we defined in the previous section.

Definition 3.15. [Jon21, Example 2.11] For some natural number n, let $x = \{1, ..., n\}$, S be the collection of proper nonempty subsets of X and define a function $\sigma : S \to S$ where $\sigma(x) = X \setminus x$ for all $x \subset X$. We can partition X into subsets Π_i for $1 \leq i \leq n-1$, where Π_m

is the set of subsets of size m. Then if we set

$$T_1 = \{(A, B, C) \in S \times S \times S | A, B, C \text{ are pairwise disjoint, and } A \cup B \cup C = X\}$$

$$T_2 = \{(A, B, C) \in S \times S \times S | (\sigma(C), \sigma(B), \sigma(A)) \in T_1\}$$

then we can define $\mathcal{T} = T_1 \cup T_2$. We can also define a group

$$\Gamma_{\mathcal{T}} = \langle g_a | a \in S, g_a g_b g_c = 1 \text{ if and only if } (a, b, c) \in \mathcal{T}, g_a g_{\sigma(a)} = 1 \rangle.$$

Note that this means $A, B, C \in T_1$ if and only if $A \cap B = \emptyset$ and $\sigma(C) = A \cup B$. We will occasionally call $\mathcal{T}$ the degenerate triangle presentation, from the work of [CMSZ93], as the q-analogue of the structure we have defined is Cartwright's $\tilde{A}_{n-1}$ triangle presentations. In particular $\mathcal{T}$ satisfies the following properties.

- [Jon21, Proposition 2.5] If $dim(A)+dim(B)+dim(C) < n$, then $(A, B, \sigma(D)), (D, C, \sigma(E)) \in \mathcal{T}$ if and only if $(A, F, \sigma(E)), (B, C, \sigma(F)) \in \mathcal{T}$, where D, E, F are unique.

Lemma 3.16. *If $(A, B, C) \in \mathcal{T}$, then $|A| + |B| < n$ if and only if $(A, B, C) \in T_1$.*

Proof. First, we show that if $(A, B, C) \in T_1$, then $|A| + |B| < n$. To see this, note $(A, B, C) \in T_1$ means that A, B and C are disjoint sets whose union is X, then $|A| + |B| + |C| = n$, so $|A| + |B| = n - |C| < n$.

Now, to show the converse, we will show that if $(A', B', C') \in T_2$, then $|A'| + |B'| > n$. To do this, see that $(A', B', C') \in T_2$ implies $(\sigma(C'), \sigma(B'), \sigma(A')) \in T_1$. By the rotational symmetry of triangle presentations and out definition of T_1, we have $(\sigma(B'), \sigma(A'), \sigma(C')) \in T_1$ as well. See that now from the first paragraph, $|\sigma(B')| + |\sigma(A')| < n$. But now $|A'| + |B'| = n - |\sigma(A')| + n - |\sigma(B')| = 2n - (|\sigma(B')| + |\sigma(A')|) > n$. So $|A'| + |B'| > n$ for every element of T_2, which completes the proof. $\square$

One of our goals of this section is to show the connection between the Coxeter complex of type $\tilde{A}_{n-1}$ and the $\mathfrak{sl}_n$ weight lattice. To do this, we first define a set of functionals that generate the weight lattice.

Definition 3.17. ([FH04, pg. 177]) For a diagonal matrix in $\mathbb{C}^{n \times n}$, the linear functional L_i, is :

$$L_i \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} = a_i$$

Proposition 3.18. *$\Gamma_{\mathcal{T}}$ is isomorphic to the sl_n weight lattice Λ_W.*

Proof. To construct an isomorphism $\phi : \Lambda_W \to \Gamma_{\mathcal{T}}$, we first come up with an alternate notation for an element g of $\Gamma_{\mathcal{T}}$. Since $g = g_{A_1}...g_{A_k}$ for subsets $A_1, ..., A_k \subseteq X$, we can represent g as the multiset that is the union of $A_1, ..., A_k$. For example $g_{\{1\}}g_{\{1,2\}}$ is identified with the multiset $\{1, 1, 2\}$ or $\{1^2, 2^1\}$. This description is well defined since all generators in $\Gamma_{\mathcal{T}}$ commute. Now we can define ϕ as the map that takes the weight $a_1L_1 + ... + a_nL_n$ to the element of $\mathcal{T}$ represented by $\{1^{a_1}, ..., n^{a_n}\}$. It is easy to see that ϕ is a homomorphism.

We must now show that if two weights are connected in the weight lattice, that the corresponding vertices are connected in $\mathcal{T}$. We consider the weights $\alpha = a_1L_1 + ... + a_nL_n$ and $\beta = b_1L_1+...+b_nL_n$. This means that $(a_1L_1+...+a_nL_n)-(b_1L_1+...+b_nL_n) = c_1L_1+...+c_nL_n$ where $c_i \in \{0, 1\}$ (this is since all weights directly adjacent to 0, and therefore appear as the difference between adjacent weights, are $\Sigma_{i=1}^n a_iL_i$, with $a_i \in \{0, 1\}$. So $\phi(\alpha)$ and $\phi(\beta)$ will differ by a multi-set where each element has multiplicity 0 or 1, which is just a subset of $\{1, ..., n\}$. In $\Gamma_{\mathcal{T}}$, the multi-sets of $\phi(\alpha)$ and $\phi(\beta)$ differing by a subsets of $\{1, .., n\}$ means that the group elements differ by a generator, and are connected in $\Gamma_{\mathcal{T}}$.

To show that ϕ is a isomorphism, first note that surjectivity follows from $a_1L_1 + ... + a_nL_n$ being in Λ_W for all sequences $a_1, ..., a_n$. The only question for injectivity is that weights can be described by multiple sequences of integers. This is due to the fact that $L_1 + ... + L_n = 0$ in the weight lattice. Suppose we have a weight $L_{i_1} + ... + L_{i_n} = -L_{j_1} + ... + -L_{j_n}$. See that $\phi(L_{i_1} + ... + L_{i_n}) = g_{\{i_1,...,i_n\}}$ and $\phi(-L_{j_1} + ... + -L_{j_n}) = g^{-1}_{\{j_1,...,j_n\}}$. The relation on the weight lattice means that $\{i_1, ..., i_n\} \cup \{j_1, ..., j_n\} = \{1, ..., n\}$ where the union is disjoint, and therefore $.g_{\{i_1,...,i_n\}} = g^{-1}_{\{j_1,...,j_n\}}$ by definition of $\Gamma_{\mathcal{T}}$. This argument extends to any two sequences of integers describing the same weights because ϕ is a homomorphism, and so ϕ is injective. □

Recall that the facts that we stated in Section 3.2.1 give us a connection between edges in a Coxeter complex Σ and subsets of a set of order n. If we take the graph planar algebra of Σ, we will see that the order of the subsets corresponds directly to the label of the edge. This definition motivates the following theorem.

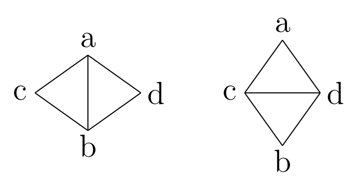
Theorem 3.19. *If $\mathcal{T}$ is the degenerate triangle presentation of type $\tilde{A}_{n-1}$, then the Cayley Graph of the group $\Gamma_{\mathcal{T}}$ has flag complex isomorphic to Σ, the Coxeter complex of type $\tilde{A}_{n-1}$ (or equivalently, the Cayley graph of $\Gamma_{\mathcal{T}}$ (with generators corresponding to subsets of X) is isomorphic to the 1-skeleton of Σ).*

This theorem is a re-imagining of [Car95, Theorem 2.5] for the $\mathbb{F}_1$ case, and the proof follows similarly to the proof of that theorem. It holds since the Coxeter complex is the building of order 1, and $\mathcal{T}$ as defined is the triangle presentation of order 1.

An important consequence of Theorem 3.19 is that all triangles in the $\tilde{A}_{n-1}$ Coxeter complex must have edge labels summing to n (this follows from property E on page 46 on [Car95]). This must hold for any $\tilde{A}_{n-1}$ building as well, since the edge labeling is derived from the vertex labeling, which is consistent between a building and any of its apartments.

The connection between the Cayley graph of $\Gamma_{\mathcal{T}}$ and the 1-skeleton of Σ give us a correspondence between cycles $a \to b \to c \to a$ in the graph planar algebra of Σ and elements $(u, v, w) \in \mathcal{T}$ (namely, that (u, v, w) are the subsets labeling the edges in the cycle). So we can say that the collection of cycles in $G(\Sigma)$ also satisfies the following property.

Property 3.20. If there exists vertices $a, b, c, d \in \Sigma$ so that labels of edges $a \to d$, $d \to b$ and $b \to c$ sum to less than n, $a \to b \to c \to a$ and $a \to d \to b \to a$ are cycles in $G(\Sigma)$ if and only if $a \to c \to d \to a$ and $b \to d \to c \to b$ are cycles in $G(\Sigma)$. That is, we have the left picture as a subgraph of $G(\Sigma)$ if any only if we have the right picture as a subgraph of $G(\Sigma)$



(3.14)

[Jon21] gives a definition of a functor from $\mathrm{Web}(\mathrm{SL}_n^-)$ to $Vec(\Gamma_{\mathcal{T}})$ where $\mathcal{T}$ is any triangle presentation, for defining field of characteristic $p \geq n-1$ and $q \equiv 1 \mod p$. We will show that when $\mathcal{T}$ is the degenerate triangle presentation as defined in Definition 3.15 (and therefore is the $\mathfrak{sl}_n$ weight lattice), the restriction on the characteristic of the field is unnecessary. We show this in the following theorem.

Theorem 3.21. *If Γ is the $\mathfrak{sl}_n$ weight lattice, then there exists a functor $\mathrm{Web}(SL_n^-) \to G(\Gamma)$ as defined in Section 3.3.1 and [Jon21, Section 4], where the relevant categories may be defined over any field $\Bbbk$.*

Proof. Recall that we must show that the images of the $\mathrm{Web}(\mathrm{SL}_n^-)$ maps in $Vec(\Gamma)$ (as defined in section 3.3.1) satisfy the $\mathrm{Web}(\mathrm{SL}_n^-)$ relations. We will show this by using the presentation of the weight lattice stated in Definition 3.15. This allows us to assign to each edge of the weight lattice (and therefore, strings in the $\mathrm{Web}(\mathrm{SL}_n^-)$ maps and relations) a proper subset of $[n]$. Our first goal is to prove the square switch relations in full generality ([Jon21, Lemma

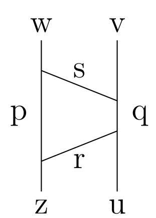
4.1] gives a proof where $\mathrm{char}(\Bbbk) = p \geq n-1$). We will show the proof of equation 3.5. First, we will denote the functions in $\mathrm{Hom}(m \otimes \ell, m-(k-j) \otimes \ell + (k-j))$ given by the left and right hand side of the equation as L and R respectively. So for some $(z,u) \in \Pi_m \times \Pi_\ell$, we have that

$$L(z \otimes u) = \sum_{(w,v) \in \Pi_{m-(k-j)} \times \Pi_{\ell+(k-j)}} L_{z,u,w,v} w \otimes v$$

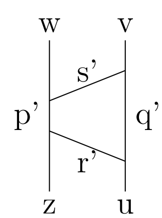
$$R(z \otimes u) = \sum_{(w,v) \in \Pi_{m-(k-j)} \times \Pi_{\ell+(k-j)}} R_{z,u,w,v} w \otimes v.$$

We will show that $L_{z,u,w,v} = R_{z,u,w,v}$ for all z,u,w,v. To see this, first note that by definition of $\mathrm{Web}(\mathrm{SL}_n^-)$, the elements of the triangle presentation represented by the generating morphisms will always have $|A|+|B| < n$ and so we can say $(A,B,\sigma(C)) \in T_1$ by Lemma 3.16, which implies that $A \cup B = C$. We will make heavy use of this fact in our argument. We will use the shorthand $\delta = k-j$ throughout.

First, fix $z \in \Pi_m, w \in \Pi_\ell, w \in \Pi_{m-\delta}, v \in \Pi_{\ell+\delta}$, with $|z \cap w| = m-i$ for some k. If $j \leq k$, we have $i \in [\delta, d]$ and if $c > d$ we have $i \in [0,d]$. Note that $L_{z,u,w,v}$ is the number of tuples (p,r,q,s) that satisfy the left hand side of the picture below (i.e. choices of subset so that every trivalent vertex is labeled by an element of $\mathcal{T}$). Similarly, $R_{z,u,w,v}$ is sum of all allowable t of the number of tuples (p',r',q',s') that satisfy the right hand side of the picture below.



$$= \sum_t \binom{m-\ell+j-k}{t}$$



First, suppose that we have a valid tuple (p,q,r,s), where $(p,r,\sigma(z)), (r,u,\sigma(q)), (s,v,\sigma(q))$, $(p,s,\sigma(w)) \in \mathcal{T}$. See that if $|z \cap w| = m-i$, then $|z \setminus w| = i$ and $|w \setminus z| = i - \delta$. Since $p \cup r = z$ and $p \cup s = w$, then we have $p \subseteq z \cap w$. We must then have $z \setminus w \subseteq r$ and $w \setminus z \subseteq s$. Also, we must have $z \setminus w \nsubseteq s$ and $w \setminus z \nsubseteq r$, $r \cap u = \emptyset$ and $s \cap v = \emptyset$. So we must have $z \setminus w \subseteq v \setminus u$ and $w \setminus z \subseteq u \setminus v$. If these two conditions are not met, $L_{z,u,w,v} = 0$. So we assume that they are true and proceed with our proof.

Now, we can say that $p = (z \cap w) \setminus j$, where $|j| = k-i$. See that this forces $r = (z \setminus w) \cup j$, $s = (w \setminus z) \cup j$ and $q = u \cup j \cup (z \setminus w)$. These values will make the (p,r,q,s) satisfy this morphism. We are free to pick any subset $j \subset (z \cap w) \setminus u$. So we have $L_{z,u,w,v} = \binom{|(z \cap w) \setminus u|}{k-i}$.

Now, consider the right hand side diagram and suppose we have a valid tuple (p', r', q', s'). See that we have $z \cup r' = p'$ and $w \cup s' = p'$ and so we must have that $z \cup w \subseteq p'$. Also, see that we must then have $w \setminus z \subseteq r'$ and $z \setminus w \subseteq s'$. Also, see we have $z \setminus w \not\subseteq r'$, $w \setminus z \not\subseteq s'$ and so we have $z \setminus w \subseteq v \setminus u$ and $w \setminus z \subseteq u \setminus v$. If these two conditions are not met, $R_{z,u,w,v} = 0$. So we have $L_{z,u,w,v}$ and $R_{z,u,w,v}$ are 0 under the same conditions.

Now, see that $p' = (z \cup w) \cup j'$ where $|j'| = k - t - i$. Also, see that this forces $s' = (z \setminus w) \cup j'$, $r' = (w \setminus z) \cup j'$ and $q' = u \setminus r'$. We are free to pick any subset $j \subseteq u \setminus (z \cup w)$. Also, we must have $k - i - t \geq 0$, so we will have $t \in [0, k - i]$. So $R_{z,u,w,v} = \sum_{t=0}^{k-i} \binom{m-\ell-\delta}{t} \binom{|u \setminus (z \cup w)|}{k-i-t}$.

We must show that

$$\binom{|(z \cap w) \setminus u|}{k - i} = \sum_{t=0}^{k-i} \binom{m - \ell - \delta}{t} \binom{|u \setminus (z \cup w)|}{k - i - t}$$

We will do this by showing $|(z \cap w) \setminus u| = m - \ell - \delta + |u \setminus (z \cup w)|$ and then applying the Chu-Vandermonde identity.

First, see that $|z \cap w \setminus u| = |z \cap w| - |u \cap z \cap w|$ and similarly $|u \setminus (z \cup w)| = |u| - |u \cap (z \cup w)|$. We can further decompose $|u \cap (z \cup w)|$ as $|u \cap (z \setminus w)| + |u \cap (w \setminus z)| + |u \cap z \cap w|$. Substituting these into desired equality, we now have

$$|z \cap w| - |u \cap z \cap w| = m - \ell - \delta + |u| - |u \cap (z \setminus w)| - |u \cap (w \setminus z)| - |u \cap z \cap w|$$

Now, we know that $z \setminus w \subset v \setminus u$, so the third to last term is simply 0. Also, $|u \cap (w \setminus z)| = |w \setminus z|$ since $w \setminus z \subset u \setminus v$. So we can further simplify to

$$|z \cap w| = m - \ell - \delta + |u| - |w \setminus z|$$

Remembering $|z \cap w| = m - i$, $|u| = \ell$ and $|w \setminus z| = i - \delta$ will give us the desired equality. Thus (5) holds. The proof of (6) follows similarly.

The proof of the other $\mathrm{Web}(\mathrm{SL}_n^-)$ relations follows exactly like the proof of theorem 4.2 in [Jon21]. $\square$

3.3.3 The Functor for Buildings

Now, we move towards a functor from $\mathrm{Web}(\mathrm{SL}_n^-)$ to the graph planar algebra of any building Δ of type $\tilde{A}_{n-1}$. We will lean heavily on the correspondence between edges connected to a distinguished vertex in Δ and elements of a finite projective geometry of order n that we established in Section 3.3. When we examine the graph planar algebra of Δ, the labels of edges in our graph will coincide with the dimensions of the corresponding subspaces. In the degenerate case studied above, we were able to extend our labeling of edges at one vertex to a consistent labeling of every edge in the diagram. However, in general, we do not have the symmetry needed for this labeling. Instead, we must center any argument on a specific vertex and only use subspaces to describe the link of this vertex.

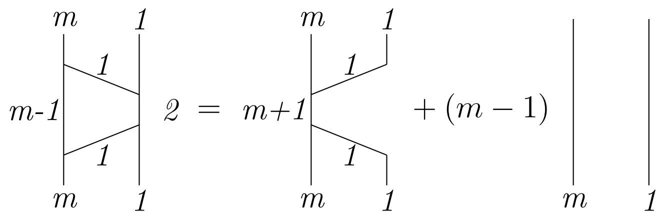
We must consider one more property of an $\tilde{A}_{n-1}$ building. This property will prove essential to showing that the square switch relations are satisfied in the graph planar algebra of Δ.

Lemma 3.22 (The Tetrahedron Property)**.** *Suppose Δ is a locally finite building of type $\tilde{A}_{n-1}$ and Γ_Δ is as defined in Section 3.2.2. If cycles $a \to b \to c \to a$ and $a \to d \to b \to a$ exist in Γ_Δ, where the sum of labels on edges $a \to d$, $d \to b$ and $b \to c$ is less than n, then c and d are connected and therefore the cycles $a \to c \to d \to a$ and $b \to d \to c \to b$ exist.*

Proof. Consider the 2-simplices C and D with vertex sets a, b, c and a, b, d respectively. By the axioms of a building, these simplices must lie in a common apartment. Hence there exists some Coxeter complex $\Sigma_{CD} \in \mathcal{A}$ so that a, b, c, d are vertices in Σ_{CD}. So in the graph $\Gamma_{\Sigma_{CD}}$, we have cycles $a \to b \to c \to a$ and $a \to d \to b \to a$. Now, use Property 3.20 to show that $a \to c \to d \to a$ and $b \to d \to c \to b$ exists in $\Gamma_{\Sigma_{CD}}$. But since $\Gamma_{\Sigma_{CD}}$ is embedded in Γ_Σ, we have that these cycles exist in Γ_Σ as well. $\square$

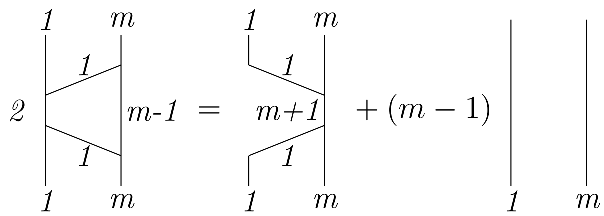
This property derives its names from it's implication that if we have two faces of a tetrahedron in our building Δ we must have the entire tetrahedron in Δ as well. Equipped with this useful property and the language of finite projective geometry to describe our buildings locally, we now prove a special case of the square switch relations which [Jon21] shows is sufficient to prove them in general.

Lemma 3.23. *For a locally finite $\tilde{A}_{n-1}$ building Δ of order q, if $\Bbbk$ is a field of characteristic p, where $q \equiv 1 \mod p$, then the following special cases of the square switch relation are* *satisfied by our embedding of* Web(SL_n^-) *into the graph planar algebra of* Δ *over* $\Bbbk$*.*



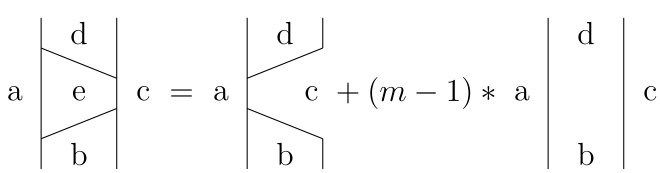
(3.15)

and



(3.16)

Proof. We will give the proof of the first relation, with the second being analogous. We must show that the left and right hand sides of 3.15 have the same evaluation for every pair of paths (p_1, p_2) where p_1, p_2 have type $(m, 1)$. The choice of (p_1, p_2) will fix the vertices labeling the boundary regions of our diagram. As mentioned in Section 3.2.2, evaluating each side of the equation amounts to summing over all labelings of each side whose first element is p_1 and final element is p_2. Note that since the image of the generating morphisms in Web(SL_n^-) evaluate to either 0 or 1 for any input paths, we need not worry about multiplying values in our sum, instead just adding 1 to our total for every valid labeling. Recall also from Section 3.2.2 that on the right hand side we will evaluate each morphism independently and add the result. To begin, we set $p_1 = a \to b \to c$ and $p_2 = a \to d \to c$, and note that this totally determines the right hand side, while on the left side we will count over all e such that $p' = a \to e \to c$ gives us a labelling (p_1, p', p_2). This will correspond to the following diagram.



First, note that we must have $m < n$, since we cannot have edges with label n in Γ_Δ. If $m = 1$, this relation is trivial. So we assume $1 < m < n$. We define $L_{a,b,c,d}$ as the number of labelings of the left hand side consistent with our choice of boundary vertices, and $R_{a,b,c,d}$ as the number of labelings of the right hand side consistent with our chosen boundary. We must show that $L_{a,b,c,d} = R_{a,b,c,d}$ when evaluated in the field $\Bbbk$.

First, let's assume that $b \neq d$. We will show that $L_{a,b,c,d} \leq 1$ and $R_{a,b,c,d} \leq 1$ and then that $L_{a,b,c,d} = 1$ precisely when $R_{a,b,c,d} = 1$. Suppose $L_{a,b,c,d} \neq 0$. We then have an e such that we have edges $a \to e$ of type $m-1$, $e \to b$ and $e \to d$ of type 1, and $e \to c$ of type 2 in Γ_Δ. Then we have cycles $a \to e \to b \to a$ and $a \to e \to d \to a$ in Γ_Δ. Now, since e and b are connected to both a and each other in Δ they are also connected in $lk_\Delta(a)$. We can apply Remark 3.9 to $lk_\Delta(a)$ and assign a subspace $V(a \to x)$ for each edge from a to $x \in lk_\Delta(a)$. Recall that the dimension of $V(a \to x)$ corresponds to the label of $a \to x$ (i.e. $V(a \to b)$ and $V(a \to d)$ are m-dimensional, while $V(a \to e)$ is $m-1$ dimensional). So by Remark 3.9, we have $V(a \to e) \subset V(a \to b)$ and similarly $V(a \to e) \subset V(a \to d)$ (which implies $V(a \to e) \subset V(a \to b) \cap V(a \to d)$). Now, $b \neq d$ implies $V(a \to b) \neq V(a \to d)$. Since $\dim(V(a \to b)) = \dim(V(a \to d)) = m$, then $V(a \to b) \cap V(a \to d)$ has dimension at most $m-1$. $V(a \to e)$ is $m-1$ dimensional and so we must have that $V(a \to e) = V(a \to b) \cap V(a \to d)$. So if $\dim(V(a \to b) \cap V(a \to d)) < m-1$, $L_{a,b,c,d} = 0$. If $\dim(V(a \to b) \cap V(a \to d)) = m-1$, then $V(a \to b) \cap V(a \to d)$ labels a unique vertex, and this vertex is the only possible e. So $L_{a,b,c,d}$ is at most 1.

Now we show that $R_{a,b,c,d}$ is at most 1. First, notice that $b \neq d$ implies that the second term of this sum is 0. Now, notice that all regions of the first term are predetermined by our choice of a, b, c, d. So there is at most one way to fill in the diagram. We now give a condition for the existence of this filling. If we associate edges $a \to x$ with vector spaces as before, we see that $V(a \to b) \subset V(a \to c)$ and $V(a \to d) \subset V(a \to c)$. So we have that $V(a \to b) + V(a \to d) \subset V(a \to c)$. Edge $a \to c$ is type $m+1$ and therefore $V(a \to c)$ has dimension $m+1$. However, $b \neq d$, implies $\dim(V(a \to b) + V(a \to d)) \geq m+1$ (since as before $b \neq d$ implies $V(a \to b) \neq V(a \to d)$). So in order to fill in this diagram, we must have $\dim(V(a \to b) + V(a \to d)) = m+1$. In this case $V(a \to b) + V(a \to d) = V(a \to c)$ and $R_{a,b,c,d} = 1$. Otherwise, $R_{a,b,c,d} = 0$.

Now, suppose that $R_{a,b,c,d} = 1$. This means that $V(a \to b) + V(a \to d)$ has dimension $m+1$. Since $\dim(V(a \to b) + V(a \to d)) + \dim(V(a \to b) \cap V(a \to d)) = \dim(V(a \to d)) + \dim(V(a \to b))$, we have that $\dim(V(a \to b) \cap V(a \to d)) = m-1$. We choose e to be the unique vertex with edge $a \to e$ such that $V(a \to e) = V(a \to b) \cap V(a \to d)$. Thus we have cycles $a \to e \to b \to a$ and $a \to e \to d \to a$ in our graph. Now, notice that $R_{a,b,c,d} = 1$ also implies that we have cycles $a \to b \to c \to a$ and $a \to d \to c \to a$ in our graph. If we suppose $m < n-1$, we can apply the tetrahedron property to the pair of cycles

$$a \to b \to c \to a, a \to e \to b \to a \text{ and } a \to d \to c \to a, a \to e \to d \to a$$

and use uniqueness of edges in our graph to see that we have an edge $e \to c$. So $e \to c \to b \to e$, $e \to c \to d \to e$ and $e \to a \to c \to e$ are cycles in our graph. The first two give us a consistent filling of the left hand side of the equation and so $L_{a,b,c,d} = 1$.

If $m = n - 1$, then $a = c$ and we can directly see that having $a \to b \to c \to a$ and $a \to e \to b \to a$ implies $e \to c \to b \to e$, and similarly for d, without having to use the tetrahedron property.

Now, suppose that $L_{a,b,c,d} = 1$. This means that $\dim(V(a \to b) \cap V(a \to d)) = m - 1$. So $\dim(V(a \to b) + V(a \to d)) = m + m - (m - 1) = m + 1$. Thus, we have that $V(a \to b) + V(a \to d)$ represents some edge $a \to g$ with label $m + 1$, where we have cycles $a \to b \to g \to a$ and $a \to g \to d \to a$ in our graph. If $m < n - 1$, we can apply the tetrahedron property to the pairs

$$a \to b \to g \to a, \ a \to e \to b \to a \text{ and } a \to d \to g \to a, \ a \to e \to d \to a$$

to see that we have cycles $e \to b \to g \to e$, $a \to e \to g \to a$ and $e \to d \to g \to e$ in our graph as well. This shows us that we have a vertex g in $G(\Delta)$ that is connected to a, b, d and e. We also know that labels of the edges $b \to g$ and $d \to g$ are the same as the labels of $b \to c$ and $d \to c$. If the edge $a \to c$ exists, then it will also have the same label as $a \to g$. Also, recall that the labels of edges in any cycle must sum to $0 \mod n$. We know that $a \to e$ has label $m - 1$ and $g \to a$ has label $n - (m + 1)$, so $e \to g$ must be labeled with 2.

Now we need that $g = c$. To see this, we forget the subspaces with which we labelled edges starting at a. Instead, we apply Remark 3.9 to $lk_\Delta(e)$ and label each edge with target vertex x and label i_x with a subspace $W(e \to x)$ of dimension i_x. So based on the the labeling of $lk_\Delta(e)$ of e, we have 1-dimensional subspaces $W(e \to b)$ and $W(e \to d)$. Since we have shown that g is connected to both b and d, the 2 dimensional subspace $W(e \to g)$ contains $W(e \to b) + W(e \to d)$. But since $b \neq d$, we have $W(e \to b) + W(e \to d)$ is 2-dimensional, so $W(e \to g) = W(e \to b) + W(e \to d)$. Note c is also connected to both b and d. We can repeat this reasoning with $W(e \to c)$ to see that $W(e \to c) = W(e \to b) + W(e \to d)$. So we must have $W(e \to c) = W(e \to g)$. Since each edge is labeled by a unique subspaces, that gives us $g = c$. This means that we have an edge $a \to c$ with label $m + 1$ and therefore $R_{a,b,c,d} = 1$.

If $m = n - 1$, then $m + 1 = n$, so $a = c$. Thus the cycles on the right hand side are simply loops $a \to b \to a$ and $a \to d \to a$.

Now, suppose $b = d$. We have that

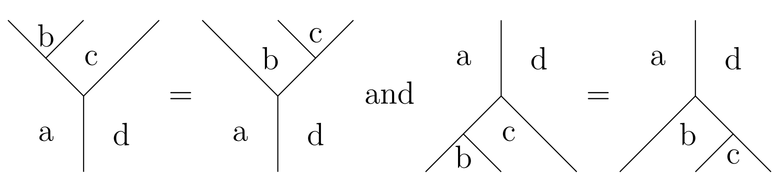
$$R_{a,b,c,b} = \begin{cases} m & a \to b \to c \to a \text{ in } \Delta \\ m-1 & \text{otherwise} \end{cases}$$

Now, we consider $L_{a,b,c,b}$, and see that we are counting vertices e so that $a \to e \to b \to a$ and $e \to b \to c \to e$ are cycles in Δ. To count these, we apply Remark 3.9 to $lk_\Delta(b)$. That is, we define an $n-m$ dimensional subspace $S(b \to a)$, a $n-1$ dimensional subspace $S(b \to e)$ and a 1 dimensional subspace $S(b \to c)$. Now, our previous statement about existence of cycles is equivalent to requiring $S(b \to c) \subset S(b \to e)$ and $S(b \to a) \subset S(b \to e)$. So we must choose for $S(b \to e)$ any $n-1$ dimensional subspace containing the subspace $S(b \to a) + S(b \to c)$. If $a \to b \to c \to a$ is a cycle in Δ, we have $S(b \to c) \subset S(b \to a)$. So we are choosing an $n-1$ dimensional subspace containing a fixed $n-m$ dimensional subspace. By Lemma 3.8, there are $\begin{bmatrix} m \\ m-1 \end{bmatrix}_q$ ways to to do this, which in our case ($q \equiv 1 \mod p$) reduces to m. If $a \to b \to c \to a$ is not a cycle, we are instead counting the number of $n-1$ dimensional subspace containing a fixed $n-m+1$ dimensional subspace. Again by Lemma 3.8, there are $\begin{bmatrix} m-1 \\ m-2 \end{bmatrix}_q$ ways to do this, which in our case reduces to $m-1$. In either case, we have $L_{a,b,c,b} = R_{a,b,c,b}$. □

Since the square switch relations are the most complicated of the $\text{Web}(\text{SL}_n^-)$ relations, this lemma almost completely gives us the following theorem.

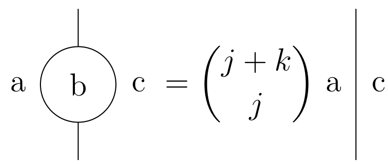
Theorem 3.24. *Suppose Δ is an $\tilde{A}_{n-1}$ building of order q and $G(\Delta)$ is the graph planar algebra of Δ over a field $\Bbbk$ of characteristic $p \geq n-1$ where $q \equiv 1 \mod p$. Then the maps we defined as the images of the $\text{Web}(SL_n^-)$ maps in the graph planar algebra satisfies the $\text{Web}(SL_n^-)$ relations. Therefore these maps define a functor from $\text{Web}(SL_n^-)$ to the graph planar algebra of Δ over $\Bbbk$.*

Proof. **Associativity and Co-associativity:** We must show that our maps satisfy relation 3.3. We do this by labeling the regions in the following ways



See that the left hand side of these equations are 1 if $a \to b \to c \to a$ and $a \to c \to d \to a$ are cycles in our graph and that the right hand side of these equations are 1 if $a \to b \to d \to a$ and $b \to c \to d \to b$ are cycles in our graph. Now, note that these cycles are directly related by the tetrahedron property. So by Lemma 3.22, both sides of this equation are 1 at exactly the same time (The edge sum condition on the tetrahedron property ensures that both sides of the equality are valid morphisms in $G(\Delta)$).

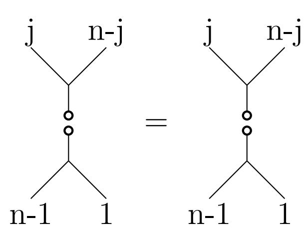
Bigon-Bursting: We must show that our maps satisfy relation 4.5. We do this by labeling the regions in the following ways



We consider $lk_\Delta(a)$ and assign subspaces to each label. We see that $V(a \to b)$ is a j dimensional subspace and that $V(a \to c)$ is a $j+k$ dimensional subspace. On the left hand side of this equation, we are counting the number of b's so that $a \to b \to c \to a$ is a cycle in our graph. Notice that this is equivalent to counting the number of j dimensional subspaces that are incident to the $j+k$ dimensional subspace $V(a \to c)$. We can do this by counting $n-j$ dimensional subspaces containing a fixed $n-(j+k)$ dimensional subspace. By Lemma 3.8, there are $\begin{bmatrix} n-(n-(j+k)) \\ n-j-(n-(j+k)) \end{bmatrix}_q = \begin{bmatrix} j+k \\ k \end{bmatrix}_q$. In our setting this equals $\binom{j+k}{k} = \binom{j+k}{j}$, as we have chosen $q \equiv 1 \mod p$. But this is exactly the value of the right hand side, since we have labelled this diagram in a unique way by definition.

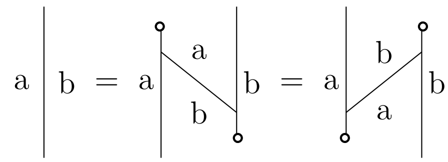
Square Switch Relations: Proposition 3.3 in [Jon21] says that Lemma 3.23 gives us the general square switch relations whenever $1 \leq j, k \leq n-2$. So we must consider the cases where we have $j = n-1$ or $k = n-1$. We will show the case that $k = n-1$ for the first square switch relation and proofs of the other cases will follow similarly.

In our setup, if no edges have labels n, $k = n-1$ then $m = n-1$ and $\ell = 1$. So if we look at the possible values of t for the right hand side of the equation, we see that the only possible value of t giving a non-zero value is $j-1$. So our relation reduces to the following diagram,



which is trivial. The other cases similarly give trivial equalities of diagrams. So we see that the square switch relations hold in general.

SL_n^- **Relations:** We must show that our maps satisfy relation 3.8. We do this by labeling the regions in the following ways



The left hand side of this equation is 1 whenever there is an edge $a \to b$ of type m. Both the center and right hand side of this equality are 1 whenever there are loops $a \to b \to a$ and $b \to a \to b$, where $a \to b$ has type m and $b \to a$ has type $n-m$. These conditions are exactly equivalent given the structure of our graph. □

3.4 Constructing Module Categories

Recall from Section 2.1.3 that module categories for a tensor category $\mathcal{C}$ are a categorification of the concept of R-modules for a ring R. For a tensor category $\mathcal{C}$, a $\mathcal{C}$-module category $\mathcal{M}$ can be characterized by a category $\mathcal{M}$ along with a monoidal functor $\mathcal{C} \to End(\mathcal{M})$. More details concerning module categories can be found in [EGNO15, Section 7.1] and Section 2.1.3. Definitions 3.32, 3.34, and 3.33 are taken from [EGNO15, Section 2.7]. Our goal is to use the functor we established in the previous section as an intermediate step to a module category. We will then show that symmetries of the underlying building yield new module categories under the equivariantization construction. As a special case, we recover the main results of Jones ([Jon21]) when the symmetries are simply transitive. When $n = 2k+1$ and $\Bbbk$ is algebraically closed, these will also be $Tilt(SL_{2k+1})$ module categories (see Section 3.2.3).

3.4.1 Building a module category

We introduce a new category whose structure relies on an $\tilde{A}_{n-1}$ building Δ. We will then establish a connection between this category and $G(\Delta)$ as in Definition 3.11.

Definition 3.25. For a locally finite building Δ of type $\tilde{A}_{n-1}$ with set of 0-simplices (i.e. vertices) $V(\Delta)$ the category $Vec(\Delta)$ is defined as the category where:

- Objects are tuples $V = (V_i)_{i \in V(\Delta)}$ of vector spaces.
- Morphisms from V to W are collections of linear transformations $(f_i : V_i \to W_i)_{i \in V(\Delta)}$.
- Composition of morphisms is a componentwise operation.

While $Vec(\Delta)$ ties a vector space to each vertex of our building (and thus of Γ_Δ), it does not give us any information about which vertices are connected. We would like to use the structure of the 1-simplices of Δ (i.e. the edges of Γ_Δ) to construct a endofunctor on $Vec(\Delta)$ in a way that encodes the data of Γ_Δ. Consider the following collection of functors.

Definition 3.26. For $m \in \{0, ..., n-1\}$, define the functor $F_m : Vec(\Delta) \to Vec(\Delta)$ on objects as $(F_m(V))_i = \oplus_{k\in E(m,i)} V_k$, where $E(m,i) = \{k \in V(\Delta) \mid \text{there exists an edge } i \to k \text{ with label } m \text{ in } \Gamma_\Delta\}$ and on morphisms as $(F_m(f))_i = \oplus_{k\in E(m,i)} f_k$.

Consider the composition of functors of this form. For example

$$(F_{m_1}F_{m_2}(V))_i = \oplus_{j\in E(m_1,i)}(F_{m_1}(V))_j = \oplus_{j\in E(m_1,i)}(\oplus_{k\in E(m_2,j)}V_k).$$

This double sum on the right hand of side of this equality can equivalently be indexed by all paths starting at i whose edges have labels m_1 and m_2 respectively. So we see that the composition of these functors is encoding paths whose edge labels have a certain type. Recall that the type of a path $a_1 \to a_2 \to ... \to a_k$ is $\sigma = (\sigma_1, ..., \sigma_{k-1})$ where the edge with source a_i has label σ_i. We will write F_σ for the composition of functors $F_{\sigma_1}...F_{\sigma_{k-1}}$.

Now, we will consider a special class of objects of $Vec(\Delta)$ which we will later show serves in some sense as a generating set for the entire category.

Definition 3.27. For each vertex $a \in V(\Delta)$ define an object $\Bbbk_a \in Vec(\Delta)$ where $(\Bbbk_a)_b = \Bbbk$ if $b = a$ and $(\Bbbk_a)_b = 0$ otherwise.

Now, consider $F_\sigma(V)$ for some path type σ. Notice that for arbitrary $a \in V(\Delta)$, $(F_\sigma(V))_a$ is a direct sum $\bigoplus V_b$ for all vertices b that are a σ-type path away from a. This means that $(F_\sigma(V))_a \simeq \bigoplus_b(\bigoplus_{i=0}^{dim(V_b)} \Bbbk)$ for all compatible b. We show F_σ commutes with direct products. So we can simply move them inside the direct product and direct sums, and study $(F_\sigma(\Bbbk_a))_b$ instead.

Lemma 3.28. *For any path type σ, $F_\sigma(\bigoplus_{i\in I} V^i) \cong \bigoplus_{i\in I} F_\sigma(V^i)$ for all $V^i \in Ob(Vec(\Delta))$.*

Proof. We will show this isomorphism component-wise. For a sum $\bigoplus_{i\in I} V^i$ of objects in $V(\Delta)$ and a functor F_m for $m \in \{0, ..., n-1\}$,

$$(F_m(\bigoplus_{i\in I} V^i))_j = \bigoplus_{k\in E(m,j)} (\bigoplus_{i\in I} V^i)_k = \bigoplus_{k\in E(m,j)} (\bigoplus_{i\in I} V^i_k)$$

We can then switch the order of sums to see that

$$\bigoplus_{k\in E(m,j)}(\bigoplus_{i\in I}V_k^i)=\bigoplus_{i\in I}(\bigoplus_{k\in E(m,j)}V_k^i))=\bigoplus_{i\in I}(F_m(V^i))_j$$

So we have the equality for each component of F_m and therefore in general. □

We will see that these F_σ form a set of distinguished objects in $End(Vec(\Delta))$ that allow us to embed $G(\Delta)$ into $End(Vec(\Delta))$. To this end, we have the following theorem, which gives us a correspondence between morphisms in $G(\Delta)$ and $End(Vec(\Delta))$.

Theorem 3.29. *For functors F_σ as defined in Definition 3.26,*

$$Nat(F_\sigma,F_\tau)\cong Hom_{G(\Delta)}(\sigma,\tau)$$

as $\mathbb{k}$-vector spaces.

Proof. Consider fixed path types σ and τ and choose an arbitrary natural transformation $\eta: F_\sigma\to F_\tau$. Notice that η is composed of maps η_V for each $V\in Vec(\Delta)$. Each η_V is further broken down into component maps $(\eta_V)_a$ for each vertex $a\in V(\Delta)$. By the previous paragraph, we need only consider the maps $(\eta_{\mathbb{k}_a})_b$ for vertices a and b. We can then recover the overall structure of the transformation from these maps.

First, we see that if $\sigma=(\sigma_1)$, then $(F_\sigma(\mathbb{k}_a))_b=\bigoplus_{c\in E(b,\sigma_1)}(\mathbb{k}_a)_c$. which is $\mathbb{k}$ if there exists an edge $b\to a$ of type σ_1 and 0 otherwise. So for a path type σ of arbitrary length, we see that $(F_\sigma(\mathbb{k}_a))_b=\bigoplus_{p\in P(\sigma,b,a)}\mathbb{k}$, where we define $P(\sigma,b,a)$ as the set of paths of type σ that start at b and end at a. Notice that $\bigoplus_{p\in P(\sigma,b,a)}\mathbb{k}\simeq\mathbb{k}[P(\sigma,b,a)]$, the $\mathbb{k}$ vector space with σ paths between b and a as a basis. So $(F_\sigma(\mathbb{k}_a))_b\simeq\mathbb{k}[P(\sigma,b,a)]$. Similarly, $(F_\tau(\mathbb{k}_a))_b\simeq\mathbb{k}[P(\tau,b,a)]$. So we can think of $(\eta_{\mathbb{k}_a}))_b$ as a map from the $\mathbb{k}$-linear span of $P(\sigma,b,a)$ to the $\mathbb{k}$-linear span of $P(\tau,b,a)$. Since Δ is locally finite, these are both finite dimensional vector spaces. This yields a finite matrix $M=(m_{q,p})_{p\in P(\sigma,b,a),q\in P(\tau,b,a)}$ for $(\eta_{\mathbb{k}_a})_b$. For every pair (p,q) we have a chosen element of $\mathbb{k}$, namely the matrix entry $m_{q,p}$. We can rearrange data we are given to form a linear functional from the $\mathbb{k}$-vector space spanned by the elements of $P(\sigma,b,a)\times P(\tau,b,a)$ to $\mathbb{k}$.

Now, we collect all of the data from all choices of b in $(\eta_{\mathbb{k}_a})_b$. So we can determine a functional from the space spanned by all matched pairs of paths with types (σ,τ) with final vertex a. Furthermore, the collection of maps $\eta_{\mathbb{k}_a}$ for all vertices $a\in V(\Delta)$ then encodes the data of a linear functional $f_{\sigma,\tau}$ from the space spanned by all matched paths of type (σ,τ) (call the matrix we get from this aggregation $M(\eta)$). This functional is uniquely determined by our choices of matrices M for each pair of vertices (a, b). On the other hand, for an arbitrary functional $f : \Bbbk[(p_1, p_2) : type(p_1) = \sigma, type(p_2) = \tau] \to \Bbbk$, we can determine entries for the matrix M for a given a and b by saying $M_{q,p} = f(qp)$. Now, f is by definition in $Hom(\sigma, \tau) \in G(\Delta)$. So, this is an explicit construction of the isomorphism between $Hom_{G(\Delta)}(\sigma, \tau)$ and $Nat(F_\sigma, F_\tau)$ as desired. □

Now, if $\eta \in Nat(F_{\sigma_1}, F_{\tau_1})$ and $\mu \in Nat(F_{\sigma_2}, F_{\tau_2})$, we consider the natural transformation $\eta \otimes \mu \in Nat(F_{\sigma_1}F_{\sigma_2}, F_{\tau_1}F_{\tau_2})$. We can describe a component of this transformation as the following composition

$$(F_{\sigma_1}F_{\sigma_2}(\Bbbk_m))_n \xrightarrow{(\eta_{F_{\sigma_2}(\Bbbk_m)})_n} (F_{\tau_1}F_{\sigma_2}(\Bbbk_m))_n \xrightarrow{F_{\tau_1}(\mu_{\Bbbk_m})_n} (F_{\tau_1}F_{\tau_2}(\Bbbk_m))_n.$$

We can identify $(F_{\sigma_1}F_{\sigma_2}(\Bbbk_m))_n$ with the $\Bbbk$-vector space of paths of type $\sigma_1\sigma_2$ (here listing path types consecutively means concatenation of lists) from n to m, and similarly with $(F_{\tau_1}F_{\sigma_2}(\Bbbk_m))_n$ and $(F_{\tau_1}F_{\tau_2}(\Bbbk_m))_n$. So we can represent these maps by matrices as before, and their composition by matrix multiplication. Thus, we have matrices $M = M(\eta_{F_{\sigma_2}(\Bbbk_m)})_n)$ and $M' = M(F_{\tau_1}(\mu_{\Bbbk_m})_n)$.

Lemma 3.30. *For the matrices M and M' as defined above, we have the following*

$$M_{q'p',qp} = \begin{cases} 0 & p \neq p' \\ M(\eta)_{q',q} & p = p' \end{cases} \quad M'_{q''p'',q'p'} = \begin{cases} 0 & q' \neq q'' \\ M(\mu)_{p'',p'} & q'' = q' \end{cases}$$

Proof. We will show this for a general object X in $Vec(\Delta)$ and as a result will have the desired statement. We begin with showing the first claim. Define a map $e^{p_0,p_0'} : F_{\sigma_2}(X) \to F_{\sigma_2}(X)$ as the map from the vector space of σ_2 paths to itself sending the path p_0 to p_0' and every other path to 0. Now, $F_{\sigma_1}(e_{p_0,p_0'})$ will act on $\sigma_1\sigma_2$ paths by fixing the σ_1 path and applying $e_{p_0,p_0'}$ to the σ_2 path. Let $E^{p_0\to p_0'}$ be the matrix of $F_{\sigma_1}(e_{p_0,p_0'})$. So we have that $E^{p_0\to p_0'}_{q_1p_1,q_1p_2} = \delta_{q_1=q_2}\delta_{p_1=p_0'}\delta_{p_2=p_0}$. Since we will consider a fixed p_0 and p_0' in the argument, take $E = E^{p_0\to p_0'}$. Now, we can repeat this process with $F_{\tau_1}(e_{p_0,p_0})$ to get a matrix $\overline{E} = \overline{E}^{p_0\to p_0'}$ for this map where $\overline{E}_{q_1'p_1,q_2'p_2} = \delta_{q_1'=q_2'}\delta_{p_1'=p_0'}\delta_{p_2'=p_0}$.

Now, consider the following naturality diagram.

$$\begin{array}{ccc} F_{\sigma_1}(F_{\sigma_2}(X)) & \xrightarrow{F_{\sigma_1}(e_{p_0,p_0'})} & F_{\sigma_1}(F_{\sigma_2}(X)) \\ \downarrow \eta_{F_{\sigma_2}(X)} & & \downarrow \eta_{F_{\sigma_2}(X)} \\ F_{\tau_1}(F_{\sigma_2}(X)) & \xrightarrow{F_{\tau_1}(e_{p_0,p_0'})} & F_{\tau_1}(F_{\sigma_2}(X)) \end{array}$$

Commutativity of this diagram is simply saying that $\overline{E}M = ME$. This equality will put some restrictions on the entries of M. Now, we will expand the matrix multiplication at an entry and see that

$$\overline{E}M_{q_1'p_1,q_2p_2} = \sum_{q'p\in P(\tau_1\sigma_2)} \overline{E}_{q_1'p_1,q'p}M_{q'p,q_2p_2} = \sum_{q'p\in P(\tau_1\sigma_2)} \delta_{q_1'=q'}\delta_{p_1=p_0'}\delta_{p=p_0}M_{q'p,q_2p_2} = M_{q_1'p_0,q_2p_2}$$

where we also carry the information that $p_1 = p_0'$. In all other cases, this matrix entry will be 0. Now, see also that

$$ME_{q_1'p_1,q_2p_2} = \sum_{qp\in P(\sigma_1,\sigma_2)} M_{q_1'p_1,qp}E_{qp,q_2,p_2} = \sum_{qp\in P(\sigma_1,\sigma_2)} M_{q_1'p_1,qp}\delta_{q=q_2}\delta_{p=p_0'}\delta p_2 = p_0 = M_{q_1'p_1,q_2p_0'}$$

where we carry the extra condition that $p_2 = p_0$. Now, we combine these expressions to see that $M_{q_1'p_0,q_2p_2} = M_{q_1'p_1,q_2p_0'}$ where we must have $p_1 = p_0'$ and $p_2 = p_0$ for this entry to be non-zero. This last condition shows that our matrix entry can be rewritten as $M_{q'p_0,q_2p_0}$. So any non-zero entry of M will have fixed σ_2 component.

Now, we show this fact for M'. To see this, see that $F_{\tau_1}(\mu_{\mathbb{k}_m})_n : F_{\tau_1}(F_{\sigma_2}(\mathbb{k}_m))_n \to F_{\tau_1}(F_{\tau_2}(\mathbb{k}_m))_n$. Now, expanding this statement based on the action of F_{τ_1}, we get this alternate form of the map.

$$\bigoplus_{k\to n\in P(\tau_1)} (\mu_{\mathbb{k}_m})_k : \bigoplus_{k\to n\in P(\tau_1)} F_{\sigma_2}(\mathbb{k}_m)_k \to \bigoplus_{k\to n\in P(\tau_1)} F_{\tau_2}(\mathbb{k}_m)_k$$

Since this map is a component wise map, there is no way to map from one competent (that is a fixed τ_1 path) to another. So we must have that M' has non-zero entries only when $q'' = q'$. □

Now, consider the matrix $M'M$. See that

$$M'M_{q''p'',qp} = \sum_{q'p'\in P(\tau_1,\sigma_2)} M'_{q''p''q'p'}M_{q'p',qp} =$$

$$\sum_{q'p' \in P(\tau_1,\sigma_2)'} M(\mu)_{p'',p'} M(\eta)_{q'q} \delta_{p'=p} \delta_{q'=q''} = M(\mu)_{p'',p'} M(\eta)_{q'q}$$

since all other terms have either $p' \neq p$ or $q' \neq q''$. As before, these maps on components generalize to way to describe $\eta \otimes \mu$ and its image $f_{\eta\otimes\nu}$ as a matrix. Now, see that under the image of the isomorphism described earlier,

$$f_\eta \otimes f_\mu(q_1p_1, q_2p_2) = f_\eta(q_1, q_2) f_\mu(p_1p_2)$$

by definition of the tensor product in $G(\Delta)$. So we will have $f_\eta \otimes f_\mu$ acts the same way as $f_{\eta\otimes\mu}$.

In $End(Vec(\Delta))$, $F_\sigma \otimes F_\tau = F_\sigma F_\tau$ is a sum over paths of type $\sigma\tau$. Also, see that in $G(\Delta)$, $\sigma \otimes \tau$ is simply the concatenated sequence $(\sigma_1, ..., \sigma_n, \tau_1, ..., \tau_m)$ so that $F_\sigma \otimes F_\tau = F_{\sigma\otimes\tau}$. From this, we can extend Theorem 3.29 to the following.

Theorem 3.31. *There exists a monoidal embedding $G(\Delta) \to End(Vec(\Delta))$.*

This theorem means that when our categories are defined over a field $\Bbbk$ whose characteristic $p \geq n-1$ is such that $q \equiv 1 \mod p$ we can extend our functor $\mathrm{Web}(\mathrm{SL}_n^-) \to G(\Delta)$ as stated in section 5 to one from $\mathrm{Web}(\mathrm{SL}_n)^-$ to $End(Vec(\Delta))$. Theorems 3.24 and 3.31 together give us the proof of theorem A.

3.4.2 Equivariantization

Now, suppose that G is a group acting on Δ by permuting the vertices and preserving adjancency. We can use the action of G to create other interesting $\mathrm{Web}(\mathrm{SL}_n^-)$-module categories. Define $Cat(G)$ as the monoidal category where objects are group elements, the only morphisms are identities and the tensor product is group multiplication. Also, recall $Aut(\mathcal{C})$ is the category of auto-equivalences on $\mathcal{C}$ where morphisms are natural isomorphisms of functors (See [EGNO15, Section 2.7] for these definitions).

Definition 3.32. Given a group G and a category $\mathcal{C}$, an action of G on $\mathcal{C}$ is a monoidal functor $A : Cat(G) \to Aut(C)$. We denote this $g \mapsto A_g \in Aut(\mathcal{C})$, and the "tensorator" isomorphisms $\gamma_{g,h} : A_g \circ A_h \simeq A_{gh}$ for all $g, h \in G$.

Definition 3.33. An G-equivariant object (X, u) in $\mathcal{C}$ is $X \in Ob(\mathcal{C})$ together with a choice of a family of isomorphisms $u = (u_g : A_g(X) \to X)_{g\in G}$ such that the following diagram commutes.

$$\begin{array}{ccc} A_g(A_h(X)) & \xrightarrow{A_g(u_h)} & A_g(X) \\ \Big\downarrow{\scriptstyle \gamma_{g,h}} & & \Big\downarrow{\scriptstyle u_g} \\ A_{gh}(X) & \xrightarrow{\quad u_{gh} \quad} & X \end{array}$$

Definition 3.34. The collection of G-equivariant objects of $\mathcal{C}$ forms a category $\mathcal{C}^G$, called the G-equivariantization of $\mathcal{C}$. Morphisms in $Hom_{\mathcal{C}^G}((X,u),(Y,v))$ are simply morphisms f in $Hom_{\mathcal{C}}(X,Y)$ such that the following diagram commutes.

$$\begin{array}{ccc} A_g(X) & \xrightarrow{A_g(f)} & A_g(Y) \\ \Big\downarrow{\scriptstyle u_g} & & \Big\downarrow{\scriptstyle v_g} \\ X & \xrightarrow{\quad f \quad} & Y \end{array}$$

Our goal is to extend the functor constructed in Theorem 3.31 to one from $\mathrm{Web}(\mathrm{SL}_n^-)$ to $End(Vec(\Delta)^G)$ and then explore the effects of different G actions on Δ. As we begin, note that an element $g \in G$ acts on Δ by permuting the vertices. Thus, $(A_g(V))_i = V_{g^{-1}(i)}$ for every vertex $i \in V(\Delta)$. As $A_g(A_h(V)) = A_{gh}(V)$, we can choose $\gamma_{g,h}$ to be the identity maps. Also, recall that a type rotating action of G on Δ is one where $type(g(i)) = type(i) + c$ mod n for some fixed c and every $i \in V(\Delta)$, $g \in G$. Our next task is to find an analogue of our functors F_a from Definition 3.26 on $Vec(\Delta)^G$. We will make use of the following lemma in our discussion.

Lemma 3.35. *If G acts on Δ by a type rotating action, then the functors A_g that come from the corresponding action on $Vec(\Delta)$ commute with the functors F_m as defined in 3.26 for all $g \in G$, $m \in \{0, ..., n-1\}$. That is, $F_m A_g = A_g F_m$ for all $m \in \{0, ..., n-1\}$ and $g \in G$.*

Proof. We will show that $F_m(A_g(V))_i = A_g(F_m(V))_i$ for all $V \in Vec(\Delta)$ and $i \in V(\Delta)$. To do this, see that

$$F_m(A_g(V))_i = \bigoplus_{k \in E(m,i)} (A_g(V))_k = \bigoplus_{k \in E(m,i)} V_{g^{-1}(k)}$$

and that

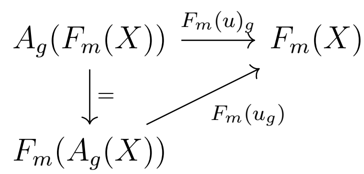
$$A_g(F_m(V))_i = F_m(V)_{g^{-1}(i)} = \bigoplus_{k \in E(m,g^{-1}(i))} V_k$$

we have equality between these two expressions whenever $k \in E(m,i)$ implies that $g^{-1}(k) \in E(m, g^{-1}(i))$. But of course, this occurs for any type-rotating action of G. So we have that these two classes of functors commute. $\square$

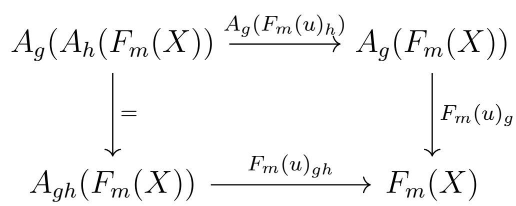
Now, we turn our attention to the category $Vec(\Delta)^G$ and make the following definition, which will allow us to begin to extend our result in Theorem 3.31 to this category.

Definition 3.36. For a given $m \in \{0, ..., n-1\}$, define a functor $F_m^G \in End(Vec(\Delta)^G)$ as acting on a object (X, u) in the following way, $F_m^G(X, u) = (F_m(X), F_m(u))$, where $F_m(X)$ is as defined in 3.26 and $F_m(u)_g = F_m(u_g)$.

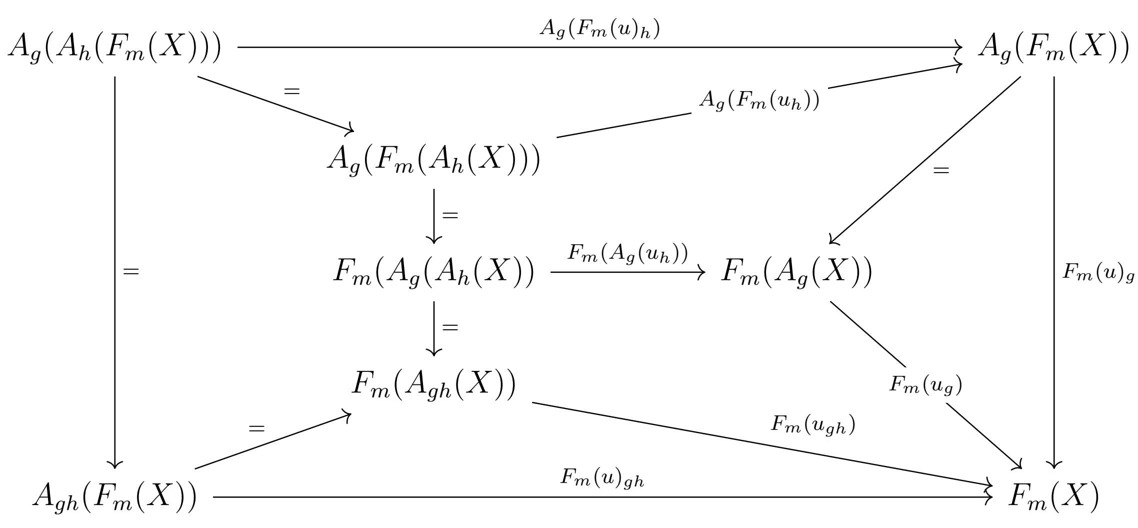
We arrive at this definition for $F_m^G(u)$ in the following manner. By Lemma 3.35, we have that $A_g(F_m(X)) = F_m(A_g(X))$. So we have the following diagram



This diagram shows us that the definition we made is the obvious choice for $F_m(u)$. Now we must verify that $F_m^G(X, u)$ is indeed an equivariant object. Recall that this means that we must show that the following diagram commutes.



To do this, we expand the diagram above as follows



(3.17)

We see that the pentagon on the left commutes because of equality. All three triangles and the top quadrilateral commute by our definition of $F_m(u)$. The bottom center quadrilateral commutes since X is an equivariant object and this square is simply F_m applied to the square in the Definition 3.34 and the top center quadrilateral commutes by Lemma 3.35. So $F_m^G(X, u)$

is indeed an equivariant object as desired and F_m^G for $m \in \{0, ..., n-1\}$ gives us a collection of endofunctors on $Vec(\Delta)^G$.

Now, equipped with our functors, we turn to the study of natural transformations between these functors. We want to show that a subcategory of $G(\Delta)$ can be mapped to these functors and thus embedded into $End(Vect(\Delta)^G)$. Then we will show that the image of our Web(SL_n^-) maps are contained in this subcategory, and so this correspondence will allow us to build a functor from Web(SL_n^-) to $End(Vec(\Delta)^G)$.

Recall that the motivation for Theorem 3.31 was the isomorphism between $Nat(F_\sigma, F_\tau)$ and $Hom_{G(\Delta)}(\sigma, \tau)$. So we wish to find some condition on natural transformations in $Vec(\Delta)^G$ that allows us to construct a similar isomorphism between a subclass of morphisms in $G(\Delta)$ and these distinguished natural transformations. To do this, we must define what the G-action does to a natural transformation $\eta \in Nat(F_\sigma, F_\tau)$.

Definition 3.37. For a natural transformation $\eta \in Nat(F_\sigma, F_\tau)$ and a group G acting on $Vec(\Delta)$, let $g(\eta) \in Nat(F_\sigma^G, F_\tau^G)$ be the natural transformation with components $g(\eta)_{(X,u)} = F_\tau(u_g)A_g(\eta_X)F_\sigma(u_g)^{-1}$ for $g \in G$.

Lemma 3.38. *If $\eta \in Nat(F_\sigma, F_\tau)$ is invariant under the action of G on $Vec(\Delta)$ (that is, the component maps η_X and $g(\eta)_{(X,u)}$ are equal for any choice of u), then η induces a natural transformation $\eta^G \in Nat(F_\sigma^G, F_\tau^G)$.*

Proof. Suppose we have $\eta \in Nat(F_\sigma, F_\tau)$ which is invariant under the action of G. This means that for every $g \in G$ and $X \in Vec(\Delta)$, we have that $\eta_X = F_\tau(u_g)A_g(\eta_X)F_\sigma(u_g)^{-1}$. This equality gives us that the following diagram commutes.

$$\begin{array}{ccc} A_g(F_\sigma(X)) & \xrightarrow{A_g(\eta_X)} & A_g(F_\tau(X)) \\ \downarrow{\scriptstyle F_\sigma(u_g)} & & \downarrow{\scriptstyle F_\tau(u_g)} \\ F_\sigma(X) & \xrightarrow{\quad \eta_X \quad} & F_\tau(X) \end{array}$$

This diagram is exactly the diagram we need to show that $(\eta_X)_{X \in Vec(\Delta)}$ is a collection of equivariant morphsims. Naturally of these maps follows from naturality of η, so we have for each (X, u) a map $\eta^G_{(X,u)}$, where the collection $\eta^G \in Nat(F_\sigma^G, F_\tau^G)$. ☐

Now, if η is invariant under the action of G, we will have that $M(\eta)_{p,q} = M(\eta)_{g(p),g(q)}$ for all paths (p, q) of appropriate type. This means in our Theorem 3.31 correspondence, the map $f \in Hom_{G(\Delta)}(\sigma, \tau)$ is invariant under the action of G on Δ. In fact, we can also reverse this statement. If $f \in Hom_{G(\Delta)}(\sigma, \tau)$ is invariant under the action of G (that is $f(pq) = f(g(p)g(q))$ for any $g \in G$ and p, q suitable paths), then the natural transformation it is mapped to also has this property and so carries an induced $Vec(\Delta)^G$ natural transformation.

Lemma 3.39. *The image of any web in* $\mathrm{Web}(SL_n^-)$ *under the functor defined in Theorem 3.24 is invariant under any type-rotating action of* G *on* Δ.

Proof. Recall that the images of the $\mathrm{Web}(\mathrm{SL}_n^-)$ webs in $G(\Delta)$ are morphisms whose co-domain is simply $\{0, 1\}$. So we must only show that the existence of the left triangle implies that of the right triangle.

$$\begin{array}{ccc} a & \xrightarrow{j} & b \\ & \searrow^{j+k} & \downarrow{\scriptstyle k} \\ & & c \end{array} \quad , \qquad \begin{array}{ccc} g(a) & \xrightarrow{j} & g(b) \\ & \searrow^{j+k} & \downarrow{\scriptstyle k} \\ & & g(c) \end{array}$$

Since the action of G is a graph automorphism, $g(a)$, $g(b)$ and $g(c)$ will be connected. Now, since the action of G is type-rotating, we have that $type(a) - type(b) = type(g(a)) - type(g(b))$ and so all edges will have the same labels as well. So $\mathbb{1}_{j,k}((a \to b \to c, a \to c)) = \mathbb{1}_{j,k}((g(a) \to g(b) \to g(c), g(a) \to g(c)))$. The argument that $\mathbb{1}_{j+k}$ and the other generating webs are invariant follows similarly. □

The previous two lemmas allow us to extend Theorem 3.31 to the category $\mathrm{Web}(\mathrm{SL}_n^-)$. This then gives us the following theorem.

Theorem 3.40. *For a fixed* n, *if* $\Bbbk$ *is a field of characteristic* $p \geq n - 1$ *and* Δ *is a locally finite* $\tilde{A}_{n-1}$ *building of order* $q \equiv 1 \mod p$, $Vec(\Delta)^G$ *has the structure of a* $\mathrm{Web}(SL_n^-)$*-module category, where the action is by the equivalence in Theorem 3.31 pre-composed with the functor defined in theorem 3.24.*

This theorem, along with Theorem 3.24 is enough to prove theorem B. Now, we have a general action of $\mathrm{Web}(\mathrm{SL}_n^-)$ on $Vec(\Delta)^G$, where the specific structure of $Vec(\Delta)^G$ is dependent on the action of G. We can consider certain classes of actions and what the functor $\mathrm{Web}(\mathrm{SL}_n^-) \to Vec(\Delta)^G$ looks like under these classes.

Example 3.41. Recall that a simply transitive action of G on Δ is one for which there exists a unique $g \in G$ where $g(x) = y$ for every $x, y \in V(\Delta)$. [CMSZ93] and [Car95] show that when G acts simply transitively on Δ, we can reduce the combinatorial information in Δ to a simpler object called a triangle presentation of type $\tilde{A}_{n-1}$. [Jon21] further showed that there exists a fiber functor from $\mathrm{Web}(\mathrm{SL}_n^-)$ that in certain setting extends to a fiber functor on $Tilt(SL_{2k+1})$ whose structure comes from these triangle presentations. Now, when

G admits a simply transitive action on Δ, that $Vec(\Delta)^G \simeq Vec$. So we recover the existence of the previously discovered fiber functor as a special case of Theorem 3.24.

Example 3.42. In general, we have for any transitive action of G on Δ, that $Vec(\Delta)^G \simeq Rep(Stab(*))$, where $*$ is any vertex in Δ. So, if we consider the stabilizers of various actions, we can realize some representation categories as $\mathrm{Web}(\mathrm{SL}_n^-)$ module categories.

Example 3.43. A natural place to start is to consider the action of $PGL_n(K)$ on the building described in Example 2.62. See that under this action, the stabilizer of the standard lattice ($Ae_1 \oplus ... \oplus Ae_n$ where $e_1, ..., e_n$ is the standard basis for K^n) is $SL_n(A)$. So, through this action, we have that $Rep(SL_n(A))$ is a $\mathrm{Web}(\mathrm{SL}_n^-)$ module category.

CHAPTER

4

TOWARDS A TYPE C EMBEDDING

4.1 Introduction

As mentioned in Chapter 3, an advantage of the module category construction of Theorem 3.40 is that we can construct a class of module categories for every $\tilde{A}_{n-1}$ building, not just those that admit a simply transitive group action. Because type A is the only type that allows this type of action (see Proposition 2.65), our results give us hope that buildings of other types can be used to build interesting module categories for the corresponding web categories.

In this chapter, we turn our attention to type C. In particular, we are interested in buildings (and therefore Coxeter complexes) of type $\tilde{C}_n$. Tits ([Tit74]) states that finite buildings of Type C are isomorphic to polar spaces. Detailed work on Type $\tilde{C}_n$ buildings has been done by Shemanske in [She07] and his student Setyadi in [Set05] and [Set07].

On the web category side of the picture, the most complete presentation of type C, or $\mathfrak{sp}_{2n}$, spiders comes in [BERT21]. This paper provides an analogous category to [CKM14]'s $\mathfrak{sl}_n$ spiders. Before this, conjecture and partial results were found in [RT22] and [Bod22]. The latter gives a constructive proof of the correctness of a category of $\mathfrak{sp}_4$ which serves as a basis for the results of this chapter.

Our initial foray into establishing a connection between webs and affine buildings takes $n = 2$. We will use the Coxeter complex of type $\tilde{C}_2$, which is isomorphic as a graph to the sp_4 weight lattice, to construct a graph, which will vary slightly from the analogous graph we defined in type C. We then define the graph planar algebra on this graph exactly as in Definition 3.11. Finally, we define the category $\mathrm{Web}(\mathrm{SP}_4^-)$ as a modification of the $\mathfrak{sp}_4$ spiders defined in [Bod22]. We then construct a functor from $\mathrm{Web}(\mathrm{SP}_4^-)$ to our graph planar algebra by translating the maps Bodish defines on the fundamental representations of $\mathfrak{sp}_4$ to maps in a graph planar algebra. By taking $q = -1$, throughout we can show these satisfy the appropriate web relations, which leads to the following theorem.

Theorem 4.12. *The functor η as defined in Section 4.3 is a monoidal functor and thus provide an embedding of* $\mathrm{Web}(SP_4^-)$ *into* $G(\bar{\Gamma}_{\tilde{C}_2})$.

The structure for this chapter is as follows. Section 4.2 will discuss the relevant background information, including buildings and webs in type C and an introduction to the representation theory of $\mathfrak{sp}_4$. In Section 4.3 we will present our construction of an embedding of $Web(\mathfrak{sp}_4)$, by constructing a graph planar algebra, defining embedding maps, and showing they satisfy the appropriate web relations. Finally, in Section 4.4, we discuss the shortcomings of this embedding, the differences between type C and type A and potential future steps.

4.2 Background

4.2.1 Buildings in type C

Recall from Section 2.2 that a building is associated to a specific Coxeter system (W, S). In this chapter, we will be interest in the group $\tilde{C}_n$ and its finite counterpart C_n. The Coxeter diagrams of these systems are show in Figures 4.1 and 4.2, where each diagram has n vertices. We are particularly interested in the Coxeter complex of type $\tilde{C}_n$. A portion of the 1-skeleton of this simplical complex is pictured in Figure 4.3. From inspection, we can see that as a graph, this 1-skeleton is isomorphic to the Cayley graph of the weight lattice of the Lie algebra $\mathfrak{sp}_4$. We believe that this is true for $n > 2$ as well, giving us a similar isomorphism as

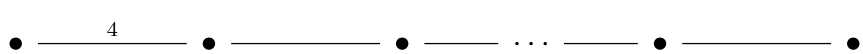


Figure 4.1: The C_n Coxeter diagram

Figure 4.2: The $\tilde{C}_n$ Coxeter diagram

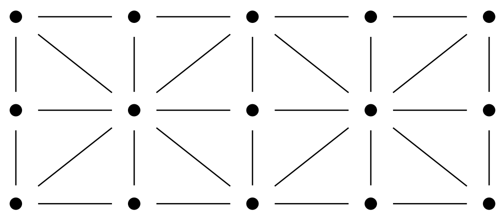


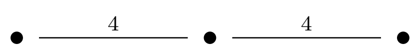
Figure 4.3: A portion of the $\tilde{C}_2$ Coxeter complex

we have in type A (see Section 3.3.2).

To highlight a major difference between type C and type A, we first make the following definition.

Definition 4.1. [Ron09, Page 35] A vertex v of the $\tilde{C}_n$ (or $\tilde{A}_{n-1}$) is a special vertex (in the Coxeter complex Σ), if $lk_\Sigma(v)$ is the Coxeter complex of type C_n (or A_n). A vertex in a building is special if it is special in every apartment in which it is contained.

In type C, as in type A, the type of a vertex is defined by the generator s which is removed from the special subgroup underlying the coset it represents. Recall that 2.51 states that the link of this vertex is the Coxeter complex of the system that is $(W, S - \{s\})$. So we can determine the special vertices of a Coxeter complex by looking at the corresponding Coxeter diagram. In type A, every vertex is special. If we look at the Coxeter diagram of $\tilde{C}_2$,



(4.1)

see that the vertices corresponding to the end generators are special, but the middle generator is not. We can directly observe this in the Coxeter complex as well, where we have vertices of both degree 4 and degree 8. When we increase n, we will get more non-special vertices. In addition, not all of these vertices will have the same degree (the link of the vertices will correspond to the system $C_k \times C_{n-k}$ for some k. The $\tilde{C}_2$ example is the Coxeter complex of $C_1 \times C_1$).

In Chapter 3, we made heavy use of Tits' discovery that buildings of type A_{n-1} have the structure of finite projective geometries. We now discuss the analogous characterization in type C.

Definition 4.2. [Tit74, Definition 7.1] A polar space is a set S together with a set of distinguished subsets L called subspaces that satisfy the following axioms.

- A subspace, together with the subspaces it contains, forms a projective space.
- The intersection of two subspaces is a subspace.
- Given a subspace X and a point (1-dimensional subspace) p not in X, there is a unique subspace Y where $X \cap Y$ is $n-2$ dimensional and Y contains p. Furthermore, Y contains all points of X colinear with P.
- There exist two disjoint $n-1$ dimensional subspaces.

Theorem 4.3. *([Tit74, Theorem 7.4]) A (weak) building of type C_n is a polar space of rank n.*

In order to fully take advantage of the above theorem, two things must happen. First, as we saw above, only some of the vertices of a $\tilde{C}_n$ building can be described using C_n building theory. We hope to modify these results in some way to work for non-special vertices as well. Second, there are very few direct ways to count subspaces in a polar space. So understanding more about the combinatorics of polar spaces is a future goal.

4.2.2 Representation Theory of $\mathfrak{sp}_{2n}$

In this section, we introduce the underlying structure of the Lie algebra $\mathfrak{sp}_4(\mathbb{C})$ and more specifically the $n=2$ case of $\mathfrak{sp}_4(\mathbb{C})$. We discuss the basic structure of this Lie algebra and its roots, weights and dominant weights. We then look at the representations of $\mathfrak{sp}_4(\mathbb{C})$, and identify two fundamental representations that will prove the foundation to our connection between $\mathfrak{sp}_4(\mathbb{C})$ and C_2 webs. This discussion will follow closely the discussion in Lecture 16 of [FH04].

To begin, suppose we have a non-degenerate, skew-symmetric bilinear from $Q : V \times V \to \mathbb{C}$ where V is a $2n$-dimensional vector space. We then have the following definition.

Definition 4.4. The Lie algebra $\mathfrak{sp}_{2n}(\mathbb{C})$ is the vector space of endomorphisms $A : V \to V$ for which $Q(Av, w) + Q(v, Aw) = 0$ for all $v, w \in V$. The bracket is the commutator.

It turns out that this algebra does not depend on the specific form Q chosen. So we choose a generic Q and use it to describe $\mathfrak{sp}_{2n}(\mathbb{C})$ in a way that hides its dependence on the form Q. Suppose

$$Q(x, y) = x^T My, \text{ where } M = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

Now, consider the condition that for any $X \in \mathfrak{sp}_{2n}(\mathbb{C})$, $Q(Xv, w) + Q(v, Xw) = 0$ for all $v, w \in V$. This means that

$$(Xv)^T Mw + v^T M(Xw) = v^T(X^T M)wv^T(MX)w = 0$$

for all $v, w \in V$. Note that since choosing $v = e_i$ and $w = e_j$ gives us $v^T(X^T M)wv^T(MX)w = (X^T M)_{ij} + (MX)_{ij}$, we must have $X^T M + (MX) = 0$. Suppose we have a X satisfying the above. Then, if

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \text{ then } X^T M = \begin{pmatrix} -C^T & A^T \\ -D^T & B^T \end{pmatrix} \text{ and } MX = \begin{pmatrix} C & D \\ -A & B \end{pmatrix}$$

So, $X \in \mathfrak{sp}_{2n}(\mathbb{C})$ if B and C are symmetric matrices and $A = -D^t$. From now on, we will use this characterization of the lie algebra, as it allows us to describe $\mathfrak{sp}_{2n}(\mathbb{C})$ simply by properties of its elements and not by conditions on a form Q.

Our ultimate goal is to describe the representations of $\mathfrak{sp}_{2n}(\mathbb{C})$. In order to do this, we first define the Cartan subalgebra which in $\mathfrak{sl}_n(\mathbb{C})$ the Cartan subalgebra is simply the subalgebra of diagonal matrices. The natural choice for the Cartan subalgebra of $\mathfrak{sp}_{2n}(\mathbb{C})$ is the subalgebra of diagonal matrices that satisfy the conditions stated above. A basis for this subalgebra $\mathfrak{h}$ is the set $\{E_{i,i} - E_{n+i,n+i} | 1 \leq i \leq n-1\}$ (we will denote $E_{i,i} - E_{n+1,n+i}$ as H_i). Note that under the standard representation, $E_{i,i} - E_{n-1,n-1}$ fixes e_1, negates e_{n+i} and sends every other standard basis vector to 0. Also, we have a dual basis $L_i \in \mathfrak{h}^*$ (as in Defintion 3.17), where as usual $\langle L_i, H_j \rangle = \delta_{ij}$.

Now, in order to find the roots and weights of $\mathfrak{g} = \mathfrak{sp}_{2n}(\mathbb{C})$, we consider the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ Consider a matrix unit $E_{i,j} \in \mathfrak{gl}_n(\mathbb{C})$. See that under action of $\mathfrak{h}$, $E_{i,j}$ is sent to itself under H_i, its negation under H_j and 0 otherwise. The same is true for $E_{n+j,n+i}$ and so we define a matrix $X_{i,j} = E_{i,j} - E_{n+j,n+i}$ that will be an eigenvector for the action of $\mathfrak{h}$ with eigenvalue $L_i - L_j$.

Unlike in the $\mathfrak{sl}_n(\mathbb{C})$ case, this is not the only class of eigenvectors and eigenvalues for this action. Table **??** shows the complete list of eigenvectors and their corresponding eigenvalues. We note that the computation to discover these is similar to that in the paragraph above. So from this, we can deduce that the roots of the $\mathfrak{sp}_{2n}(\mathbb{C})$ are $\pm L_i \pm L_j$ for all $1 \leq i, j \leq n$. Our next step in our discussion is to find the subalgebras of $\mathfrak{sp}_{2n}(\mathbb{C})$ that are isomorphic to $\mathfrak{sl}(2\mathbb{C})$. This will help us determine the weights of $\mathfrak{sp}_{2n}(\mathbb{C})$ (this is analogous to the discussion leading

Table 4.1: Eiegenvalues/vectors of $\mathfrak{sp}_{2n}(\mathbb{C})$

Eigenvector	Eigenvalue
$Y_{ij} = E_{i,n+j} + E_{j,n+i}$	$L_i + L_j$
$Z_{ij} = E_{n+i,j} + E_{n+j,i}$	$-L_i - L_j$
$U_i = E_{i,n+i}$	$2L_i$
$V_i = E_{n+i,i}$	$-2L_i$

to [FH04, Proposition 12. 15] for $\mathfrak{sl}_n$. For each root α, we must find a vector H_α so that H_α take s the place of h in some copy of $\mathfrak{sl}(2,\mathbb{C})$. For example, see that $[X_{ij}, X_{ji}] = H_i - H_j$, so that $H_{L_i - L_j}$ must be the multiple of $H_j - H_j$ such that its adjoint action on X_{ij} is $2X_{ij}$. See that

$$ad(H_i - H_j)(X_{ij}) = ((L_i - L_j)(H_i - H_j)) * X_{ij} = 2X_{ij} \quad (4.2)$$

. So $H_{L_i-L_j} = H_i - H_j$. We can use a similar method to determine H_α for opposite roots $L_i + L_j$ and $L_i - L_j$. See we have that $[Y_{ij}, Z_{ij}] = H_i + H_j$ and $ad(H_i + H_j)(Y_i j) = 2Y_{ij}$. So we have $H_{L_i+L_j} = H_i + H_j$ and similarly $H_{L_i-L_j} = H_i - H_j$.

A similar calculation using $\pm 2L_i$ and $[U_i, V_i]$ give us $H_{\pm 2L_i} = \pm H_i$. So our set H_α that we desired is the set $\{H_i, \pm H_i \pm H_j | i \neq j\}$. The weight lattice Λ_W is the set of linear forms on $\mathfrak{h}$ that are integral on all of the H_α. In this case, this is all integral combinations of the L_i.

Our final goal for general n is to find a description of a Weyl chamber for $\mathfrak{sp}_{2n}(\mathbb{C})$ and thus the fundamental weights. To do this, we can choose a linear functional ℓ on the weight lattice where $\ell(\sum_{i=1}^{n-1} a_i L_i) = c_1 a_1 + ... + c_n a_n$ for some $c_1 \geq c_2 \geq ... \geq c_n > 0$. This gives $R^+ = \{L_i + L_j\}_{i \leq j} \cup \{L_i - L_j\}_{i<j}$ as the set of positive roots. So we can define a Weyl chamber $\mathcal{W} = \{a_1 L_1 + ... + a_n L_n | a_1 \geq ... \geq a_n \geq 0\}$. The walls of this chamber will occur when there is exactly one equals sign in the sequence of a_i.
For the rest of the paper, we will consider $\mathfrak{sp}_4(\mathbb{C})$, the $n = 2$ case. We begin by stating the following theorem from [FH04, Section 16.2].

Theorem 4.5. *There is a unique irreducible representation Γ_α of $\mathfrak{sp}_4(\mathbb{C})$ with highest weight α for any α in the intersection of the closed Weyl chamber $\mathcal{W}$ with the weight lattice.*

In $\mathfrak{sp}_4(\mathbb{C})$, we have as weights integral combinations of L_1 and L_2. So the α in question are non-negative linear combinations of L_1 and $L_1 + L_2$ which as we saw before, are the walls of the Weyl chamber. Define $\Gamma_{a,b}$ as the representation whose highest weight is $aL_1 + b(L_1 + L_2)$.

First, we consider the standard representation of $\mathfrak{sp}_4(\mathbb{C})$ as endomorphisms of V, a 4-dimensional vector space with basis $\{e_1, e_2, e_3, e_4\}$. The eigenvalues of the basis vectors are $L_1, L_2, -L_1, -L_2$. These are the weights of the representation and so the highest weight is L_1. So, as a representation of $\mathfrak{sp}_4(\mathbb{C})$, $V = \Gamma_{1,0}$.

Our next wish is to find the the representation $\Gamma_{0,1}$. To do this, we must study the exterior square $\Gamma^2 V$. In this representation, the eigenvalues are pairwise sums of distinct weights of V (i.e. $L_1+L_2, L_1-L_2, L_2-L_1, -L_1-L_2$ and 0 with multiplicity 2). However, this representation is not irreducible, as it is isomorphic to $W \oplus \mathbb{C}$. As a representation $\mathbb{C}$ has weight 0. So W is the irreducible representation with weights $L_1 + L_2, L_1 - L_2, L_2 - L_1, -L_1 - L_2$ and 0. The highest weight here is $L_1 + L_2$ and so W is the representation $\Gamma_{0,1}$.

The highest weights L_1 and $L_1 + L_2$ of V and W are not only fundamental weights of $\mathfrak{sp}_4(\mathbb{C})$ but they are also generators for the dominant weights. Adopting the notation that $\overline{\omega_1} = L_1$ and $\overline{\omega_2} = L_2$, we can define the category $Fund(\mathfrak{sp}_4(\mathbb{C}))$ as the subcategory of $Rep(\mathfrak{sp}_4(\mathbb{C}))$ generated by the fundamental weights ω_1 and ω_2. Finally, note that $V \otimes V \simeq 2V \oplus W \oplus \mathbb{C}$ and so there is a one dimensional space of maps $V \times V \to W$ (ie. $V(\omega_1) \otimes V(\omega_1) \to V(\omega_2)$). This subspace of maps will form the foundation for our diagrammatic description of the category $Fund(\mathfrak{sp}_4(\mathbb{C}))$.

4.2.3 Webs in Type C

Recall that webs are a class of diagrammatic categories designed with the purpose of being isomorphic to the representation theory of finite lie algebras and their quantum deformations. The general theory of webs and spiders was begun by Greg Kuperberg in [Kup96] which defined spiders for rank 2 lie algebras A_2, B_2 and G_2. This opened the problem of finding the correct categories of spiders (which may also be called webs) for all other finite type lie algebras.

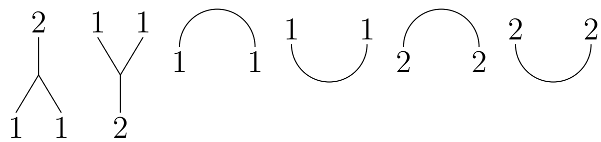
In this paper, we will discussing some of the work done in solving the type C case. In particular, we will be looking at Elijah Bodish's presentation of these webs in [Bod22]. We choose to present this work and not Kuperberg's original construction because it has recently been extended to a presentation of webs for $\mathfrak{sp}_{2n}(\mathbb{C})$ (see [BERT21]).

We now define $Web(\mathfrak{sp}_4)$. Recall that we define the quantum integers are defined as

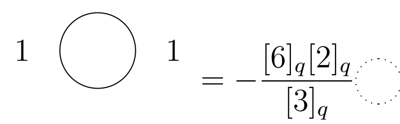
$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$$

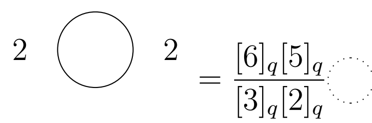
Also, let $\mathcal{A} = \mathbb{Z}[q, q^{-1}, \frac{1}{[2]_q}]$. We can now state Bodish's definition of C_2 webs.

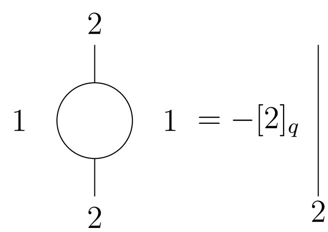
Definition 4.6. The category $Web(\mathfrak{sp}_4)$ has objects generated by 1 and 2 (i.e. sequences with 1 and 2 as the entries). The generating morphisms are

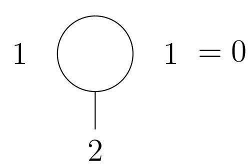


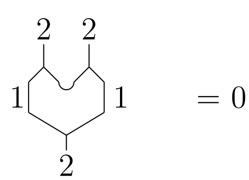
where we note that these morphisms and their compositions form a coherent graphical calculus (that is, we can bend a leg of a morphism in any direction we desire and still have equality). Additionally, we impose the following relations on our morphisms.

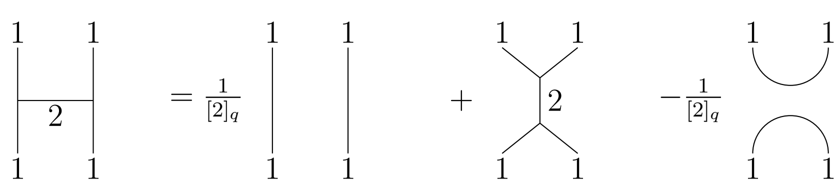
 (4.3)

 (4.4)

 (4.5)

 (4.6)

 (4.7)

 (4.8)

Note that the coefficent in equations 1 and 2 are quantum integers by the following equalities:

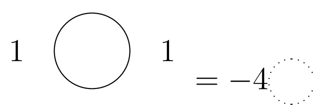
$$[5]_q - [1]_q = \frac{[6]_q[2]_q}{[3]_q} \tag{4.9}$$

$$[7]_q - [5]_q + [3]_q = \frac{[6]_q[5]_q}{[3]_q[2]_q}. \tag{4.10}$$

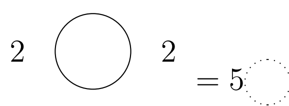
The category $Web(\mathfrak{sp}_4)$ is a modification of Kuperberg's original presentation of the B_2 spider. Bodish's definitions of the relations are scaled so that they will better conform to $\mathfrak{sp}_4$ representation theory. In addition, the all relations exception the last have integer coefficients, which make the category more useful combinatorially.

We will use a slightly modified version of $Web(\mathfrak{sp}_4)$ for our embedding. By taking $q = -1$ in every quantum integer above, we get a web category whose relations have rational coefficients.

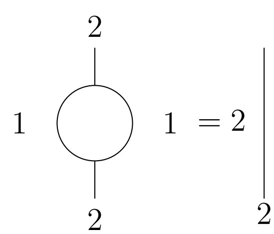
Definition 4.7. The category $\mathrm{Web}(\mathrm{SP}_4^-)$ has the same objects and generating morphisms as $Web(\mathfrak{sp}_4)$, but the relations are as follows:



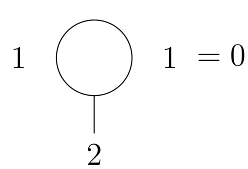
(4.11)



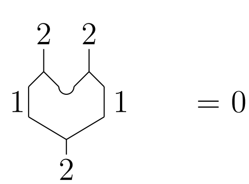
(4.12)



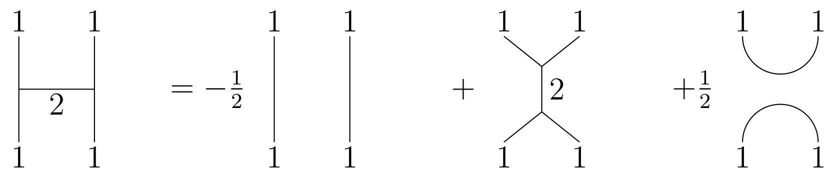
(4.13)



(4.14)



(4.15)



(4.16)

Recall that in the representation theory of $\mathfrak{sp}_4$, we have a 1 dimensional space of maps from $V \otimes V$ to W. The first generating morphism of $Web(\mathfrak{sp}_4)$ will corresponds to this space of maps.

4.3 Constructing an Embedding

4.3.1 Choosing A Graph

In order to construct an embedding of the Web(SP$_4$) category, we must define a functor from Web(SP$_4$) to some graph planar algebra. But first, we must define the graph that our graph planar algebra is defined over and also pick a labeling of this graph. Recall that in type A, this graph is quite simple to come up with, and we can use the following definition found in [McG22].

Definition 4.8. The graph $\Gamma_{\tilde{A}_n}$ is the 1-skeleton of the $\mathfrak{sl}_n$ weight lattice (or $\tilde{A}_{n-1}$ Coxeter complex, with each 1 - simplex replaced by two opposite edges. The label of each edge $s \to t$ is $\ell(t) - \ell(s) \mod n$, where the labelling ℓ of vertices is the canonical labeling of the Coxeter complex.

Since we also have that the weight lattice and Coxeter complex are isomorphic in type C, we could make a similar definition for our type C graph. However, we choose to instead make the following definition, which is more closely related to the representation theoretic origins of this simplical complex.

Definition 4.9. The graph $\Gamma_{\tilde{C}_2}$ is the 1-skeleton of the $\mathfrak{sp}_4$ weight lattice, with each 1-simplex replaced by two opposite edges. These edges are both labeled by 1 if they represent a weight in the first fundamental representation of $\mathfrak{sp}_4$ and 2 is they represent a weight in the second fundamental representation of $\mathfrak{sp}_4$.

Pictorially, the above definition means that every horizontal or vertical edge has label 2 and every diagonal edge has label 1. We can also talk about the following graph, which we call the partial completion of $\Gamma_{\tilde{C}_2}$.

Definition 4.10. The partial completion of $\Gamma_{\tilde{C}_2}$, denoted $\bar{\Gamma}_{\tilde{C}_2}$, is $\Gamma_{\tilde{C}_2}$ with additional label 2 edges connecting every unconnected diagonally adjacent pair of vertices, and loops with label 2 on every vertex in the graph.

We can now define the graph planar algebra as in Definition 3.11 of $\bar{\Gamma}_{\tilde{C}_2}$, which we will call $G(\bar{\Gamma}_{\tilde{C}_2})$. We now have a category to use at the target category in our embedding of $\text{Web}(\text{SP}_4^-)$.

4.3.2 Defining an Embedding

In order to define an embedding of $\text{Web}(\text{SP}_4)$ into $G(\bar{\Gamma}_{\tilde{C}_2})$, we must first fix an orientation for the graph $\bar{\Gamma}_{\tilde{C}_2}$. This allows to talk about the direction of each edge not just in the sense of the label, but in the sense of the exact weight that the edge corresponds to (Note that traveling the opposite direction on the edge corresponds to the opposite weight). The directions of an edges around a fixed vertex are described in Figure 4.4 where a tuple (a, b) corresponds to the weight $aL_1 + b(L_1 + L_2)$.

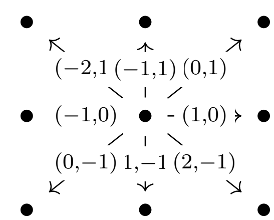


Figure 4.4: The directions in the Coxeter complex of $\tilde{C}_2$

Now, we can define a functor $\eta : \text{Web}(\text{SP}_4) \to G(\bar{\Gamma}_{\tilde{C}_2})$ by where it sends the morphisms of $\text{Web}(\text{SP}_4)$. Define maps $cap_1, cup_1, cap_2, cup_2, f_1, f_2$ in $G(\bar{\Gamma}_{\tilde{C}_2})$ whose values are given by the Tables 4.2-4.7 below. Then, Equations 4.17 and 4.18 define η.

Table 4.2: Values of $cup_1 : \Bbbk(\{(\emptyset, p_1) | \ type(p_1) = (1,1)\})$

Directions of p_1	Value of cup_1
(1,0), (-1,0) (-1,1), (1,-1)	-1
(1, -1), (-1, 1) (-1, 0), (1,0)	1

Table 4.3: Values of $cap_1 : \mathbb{k}(\{(p_1, \emptyset)|\ type(p_1) = (1,1)\}) \to \mathbb{k}$

Directions of p_1	Value of cap_1
(1,0), (-1,0) (-1,1), (1,-1)	1
(1, -1), (-1, 1) (-1, 0), (1,0)	-1

Table 4.4: Values of $cup_2 : \mathbb{k}(\{(\emptyset, p_1)|\ type(p_1) = (2,2)\}) \to \mathbb{k}$

Directions of p_1	Value of cup_2
(0,1), (0,-1) (0,-1), (0, 1)	1
(2, -1), (-2, 1) (-2, 1), (2,-1)	-1
(0,0), (0,0)	$-\frac{1}{2}$

Table 4.5: Values of $cap_2 : \mathbb{k}(\{(p_1, \emptyset)|\ type(p_1) = (2,2)\}) \to \mathbb{k}$

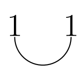
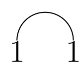
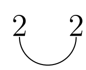
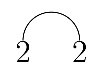
Directions of p_1	Value of cap_2
(0,1), (0,-1) (0,-1), (0, 1)	1
(2, -1), (-2, 1) (-2, 1), (2,-1)	-1
(0,0), (0,0)	-2

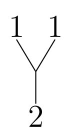
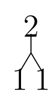
Table 4.6: Values of $f_1 : \mathbb{k}(\{(p_1, p_2)|\ type(p_1) = (2)$ and $type(p_2) = (1,1)\} \to \mathbb{k}$

Directions of p_1	Directions of p_2	Value of f_1
(0, 1) (2, -1) (-2, 1) (0, 1) (0, -1) (2, -1) (-2, 1) (0, -1) (0,0) (0,0)	(1,0),(-1,1) (1,0),(1,-1) (-1,1),(-1,0) (-1,1),(1,0) (1,-1),(-1,0) (1,-1),(1,0) (-1,0),(-1,1) (-1,0),(1,-1) (1,0), (-1,0) (1,-1), (-1, 1)	-1
(0,0) (0,0)	(-1,0), (1,0) (-1,1), (1, -1)	1

Table 4.7: Values of $f_2 : \Bbbk(\{(p_1, p_2)|\ type(p_1) = (1, 1) \text{ and } type(p_2) = (2)\} \to \Bbbk$

Directions of p_1	Directions of p_2	Value of f_1
(1,0),(-1,1)	(0, 1)	-1
(1,0),(1,-1)	(2, -1)	
(-1,1),(-1,0)	(-2, 1)	
(-1,1),(1,0)	(0, 1)	
(1,-1),(-1,0)	(0, -1)	
(1,-1),(1,0)	(2, -1)	
(-1,0),(-1,1)	(-2, 1)	
(-1,0),(1,-1)	(0, -1)	
(1,0), (-1,0)	(0,0)	-1/2
(1,-1), (-1, 1)	(0,0)	
(-1,0), (1,0)	(0,0)	1/2
(-1,1), (1, -1)	(0,0)	

 $\mapsto cup_1$  $\mapsto cap_1$  $\mapsto cup_2$  $\mapsto cap_2$ (4.17)

 $\mapsto f_1$  $\mapsto f_2$ (4.18)

4.3.3 The Proof of the Embedding

The proof of the existence of this functor is a verification that our maps $cup_1, cap_1, cup_2, cap_2, f_1$ and f_2 satisfy the relations in Web(SP$_4$). Just as in [McG22] (and Chapter 3), we can do this by identifying edges in $\bar{\Gamma}_{\tilde{C}_2}$ with strings in Web(SP$_4$), which in turn leads us to identify regions in the web pictures with vertices of $\bar{\Gamma}_{\tilde{C}_2}$. We must show that the relations in Web(SP$_4$) hold for any choices of vertices $a - b - c - d$ labeling the outside regions. Note that in $\bar{\Gamma}_{\tilde{C}_2}$, all vertices are essentially identical (i.e. they have the same edges in and out in the same directions), so we can pick a generic a. However, once we fix this a, the choice of b, c, d will determine the values of the morphisms in $G(\bar{\Gamma}_{\tilde{C}_2})$ as these are dependent on the direction of the edges between the vertices. So we must consider every possible choice of directions. Consider Figure 4.5, which we will use to enumerate the cases we must check.

Lemma 4.11. *The morphisms in the image of* $\eta : \mathrm{Web}(SP_4^-) \to G(\bar{\Gamma}_{\tilde{C}_2})$ *satisfy the relation 4.16.*

Proof. In order to begin verifying this relation, we must first enumerate all possible directions

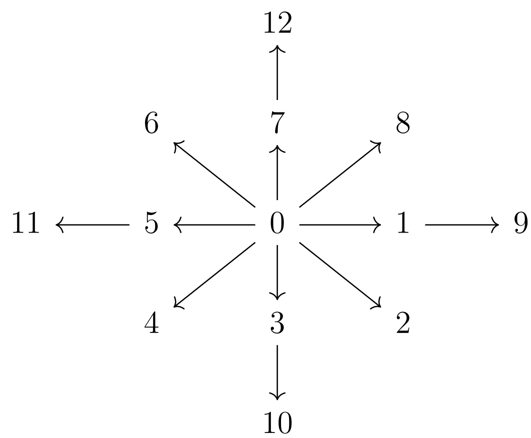
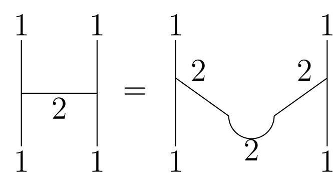


Figure 4.5: The near neighbors of a vertex in the graph $G(\bar{\Gamma}_{\tilde{C}_2})$

for b, c and d. See Table 4.8, where the number for b,c,d refer to Figure 4.5. We will show the verification of this relation for case 1, and note that verification of the other cases follows almost identically.

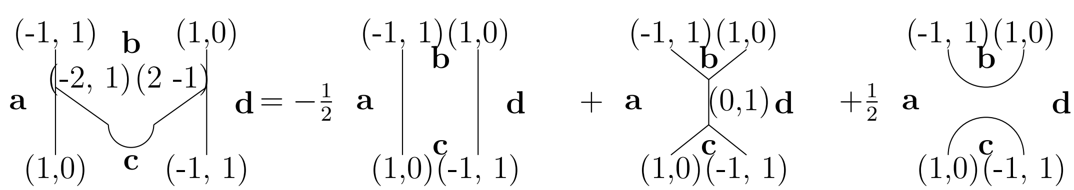
To start, note that we have the following equality which allows us to evaluate the morphism on the left hand side of the relation.



(4.19)

Refer to [Bod22, Notation 2.3] to interpret the two "forks" morphisms in this equality. Note that we can then use this interpretation to say that the evaluation of one of these morphisms in $G(\bar{\Gamma}_{\tilde{C}_n})$ is the product of the corresponding coefficent of f_1 and cap_1.

From here, we can label the edges in our diagram with the appropriate directions from case 1.



(4.20)

Table 4.8: Possible Values for b, c and d

Case #	Choice of b	Choice of c	Choice of d
1	7	1	8
2	1	7	8
3	7	1	0
4	1	7	0
5	3	1	2
6	1	3	2
7	3	1	0
8	1	3	0
9	5	3	4
10	3	5	4
11	5	3	0
12	3	5	0
13	7	5	6
14	5	7	6
15	7	5	0
16	5	7	0
17	7	7	0
18	5	5	0
19	1	1	0
20	3	3	0
21	1	1	9
22	3	3	10
23	5	5	11
24	7	7	12
25	7	7	8
26	7	7	6
27	1	1	8
28	1	1	2
29	3	3	2
30	3	3	4
31	5	5	4
32	5	5	6

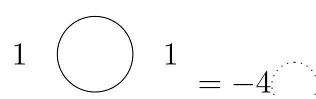
First, note that in the first morphism on the right hand side, b and c label the same region and so should correspond to the same vertex. However, in case 1, this is not true, so the morphism evaluates to 0. We can also see this by looking at the directions on the edges, as we have labeled one string with two separate directions. Similarly, the third morphism on the right hand side is also 0, since a and d label the same region but correspond to different vertices in our graph in case 1. We can see that evaluating the string also gives 0, since cups and caps only have nonzero evaluation, when the strings label opposite directions.

So all we must do is show that the evaluation of the left hand side is equal to that of the second morphism on the right hand side. See that on the right hand side, we are looking at $f_2 \circ f_1$, and so we must multiply the values of these morphisms. By definition, this is $-1 * -1 = 1$. Now, on the left hand side, we compute this constant by multiplying the corresponding value of cup_2 (-1), with the evaluations of the left and right forks (computed as described above), which are -1 and 1 respectively. So, the value of this morphism is 1 as well, and we have shown equality of this relation in case 1. The other cases proceed similarly, in each case assigning the proper directions and evaluating the maps from the tables in section 3.1. □

Now that we have tackled the most complicated Web(SP_4) relation, we are ready to prove that η is indeed a monoidal functor.

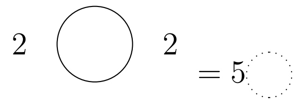
Theorem 4.12. *When* Web(SP_4) *is defined over a field of characteristic p such that $q \equiv -1 \mod p$, the functor η as defined is a monoidal functor and thus provide an embedding of* Web(SP_4) *into* $G(\bar{\Gamma}_{\tilde{C}_n})$

Proof. Recall that proving the existence of this functor is verifying the following relations for $cap_1, cup_1, cap_2, cup_2, f_1$ and f_2 as defined in Definition 4.6



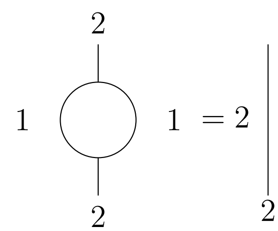
(4.21)

The left hand side of this relation has two regions, inside and outside of the circle. See that for any given choice of vertex a for the outside region, there are four choices of 1-labelled edges to traverse first in one direction and then the other to end up at a. Also, note that the values of cup_1 and cap_1 for a direction pair $(x, y), (-x, -y)$ are always either -1 and 1, or 1 and -1. So each choice of a internal vertex b gives us a coefficent of -1 which we add together to get -4 as the total value of the right hand side, which is what we needed.



(4.22)

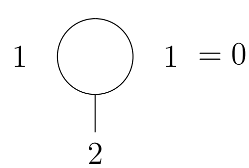
Notice that again, when evaluating the left hand side, we must fix an outer vertex a and consider all possible inner vertices b. This picture denotes traversing both directions of an edge with label 2, so there are 5 choices for b. Notice that $cap_2 = cup_2$ for any $(x, y), (-x, -y)$ pair and so each b contributes a term of 1 to the final evaluation. So the value of the LHS morphism is 5, which what is desired.



(4.23)

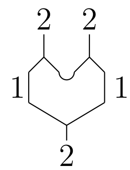
For this relation, we call the outer regions a and c respectively, and the inside region b. We first tackle the case where $a \neq c$. To start, first see that for any choice of a and c, there are two possible choices for b (a and c are opposite corners of a square in the graph $\bar{\Gamma}_{\tilde{C}_n}$, and these choices of b correspond to traveling around the sides of the square from a to c). Note that for any non $(0, 0)$ direction of a 2-labelled edge, that the coefficents of f_1 and f_2 are -1. So as long as $a \neq c$, this morphism evaluates to 1 for every choice of b. Since there are two choice of b, the final value of the morphism is 2 which is equal to the right hand side

When $a = c$, the direction of the two edge is $(0, 0)$ and we now have four options for the vertex b. Each choice corresponds to a different direction, but the value of f_1 times the value of f_2 in all cases is $\frac{1}{2}$. So the sum over all possible values of b is again 2 as desired. So this relation is true for any choice of a and c.



(4.24)

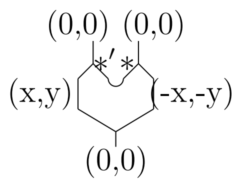
To verify this relation, note that as in the first two relations, we have only one outside region. In fact, the only way that this differs from 4.3 is that the first morphism in the composition is f_2, instead of cup_1. So again, for a given a, we have four choices of b, the vertex labeling the inner region of the diagram. Also, note that since we have only one outer region, the direction of the 2 edge must be $(0,0)$. To evaluate this morphism, we must add it's value for each choice of b (which corresponds to a choice of direction for the edges). This value is the product of the values of f_2 and cap_1, which is 1 when the direction of the left most 1 edge is $(-1,1)$ or $(1,-1)$ and -1 when this edges is direction is $(1,0)$ or $(-1,0)$. So the sum of all possible b is 0, as desired.



$= 0$

(4.25)

For this equation, first see that if all of the edges labeled 2 have directions that are not $(0,0)$, this configuration never appears in the graph planar algebra. So we consider two cases, when all outside edges have direction (0,0) and when only one does (Note that two edges having direction (0,0) forces the third to have this direction as well). First, if all edges have direction $(0,0)$ we are looking at the web diagram labeled with directions as follows:



(4.26)

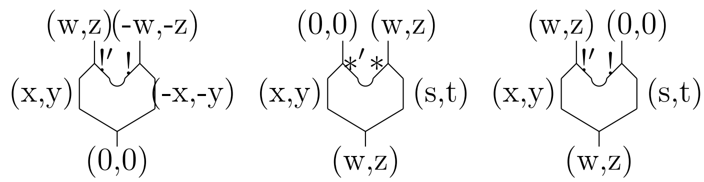
where $* = (x,y)$ and $*' = (-x,-y)$. So we see that the value of this morphism for a given direction (i.e. a choice of internal vertex b) will be the product of the value of f_1 and the square of f_2 for the directions $(x,y),(-x,-y)$ and $(0,0)$ and the value of cup_1 for $(-x,-y),(x,y)$. The values for each direction are shown in the tables below. We can easily see that the sum

Table 4.9: Evaluation of Equation 4.26

Initial Direction	f_1	f_2^2	cup_1	Total Evaluation
(1,0)	-1	1/4	1	-1/4
(-1,0)	1	1/4	-1	-1/4
(1,-1)	-1	1/4	-1	1/4
(-1,1)	-1	1/4	1	1/4

of these is 0, as desired.

Now, if only one edge has direction (0,0), we must fall into one of the following three cases for directions on diagram.



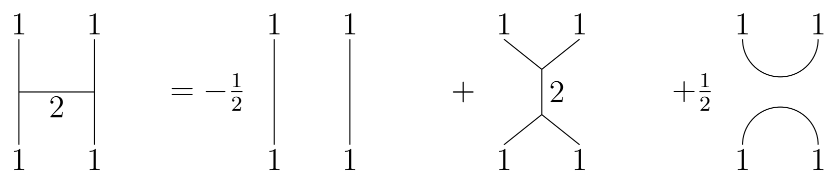
(4.27)

where here $*, *'$ are as in the previous case, and $!' = (s,t)$, $! = (-s,-t)$. Now, the choice of two of (x,y), (s,t) and (w,z) will completely determine the third (or else the morphism will evaluate to 0 and we need not consider it). Each of these three cases must sum to 0 over all choices of vertices in the inner region. Note also that if we label the outer regions a, b and c going clockwise from the left side, the three cases we have break down into $a = c$, $a = b$, and $b = c$ respectively. When enumerating these cases, we note that we must only choose $(s,t) \neq \pm(x,y)$, since $(s,t) = (x,y)$ will give f_2 the value of 0, and $(s,t) = (-x,-y)$ puts exactly in the previously solved case. Again, we must multiply the values of f_1, f_2 and cup_1 for the relevant direction pairs for each case. We enumerate the cases and do the computations for each case in the Tables 4.10 - 4.12 below.

Table 4.10: Case 1: $a = c$

(x,y)	(s,t)	f_1	Left f_2	Right f_2	cup_1	Value of Composition
(1,0)	(-1, 1)	-1	-1	-1	-1	1
(1,0)	(1, -1)	-1	-1	-1	1	-1
(-1,0)	(-1, 1)	1	-1	-1	-1	-1
(-1,0)	(1, -1)	1	-1	-1	1	1
(-1,1)	(-1, 0)	1	-1	-1	1	1
(-1,1)	(1, 0)	1	-1	-1	-1	-1
(1,-1)	(-1, 0)	-1	-1	-1	1	-1
(1,-1)	(1, 0)	-1	-1	-1	-1	1

In all three tables, the sum of the far column is 0, thus fully verifying the relation.



(4.28)

Table 4.11: Case 2: $a = b$

(x,y)	(s,t)	f_1	Left f_2	Right f_2	cup_1	Value of Composition
(1,0)	(-1, 1)	-1	-1/2	-1	1	-1/2
(1,0)	(1, -1)	-1	-1/2	-1	1	-1/2
(-1,0)	(-1, 1)	-1	1/2	-1	-1	-1/2
(-1,0)	(1, -1)	-1	1/2	-1	-1	-1/2
(-1,1)	(-1, 0)	-1	1/2	-1	1	1/2
(-1,1)	(1, 0)	-1	1/2	-1	1	1/2
(1,-1)	(-1, 0)	-1	-1/2	-1	-1	1/2
(1,-1)	(1, 0)	-1	-1/2	-1	-1	1/2

Table 4.12: Case 3: $b = c$

(x,y)	(s,t)	f_1	Left f_2	Right f_2	cup_1	Value of Composition
(1,0)	(-1, 1)	-1	-1	-1	-1	1
(1,0)	(1, -1)	-1	-1	1	1	1
(-1,0)	(-1, 1)	-1	-1	-1	-1	1
(-1,0)	(1, -1)	-1	-1	1	1	1
(-1,1)	(-1, 0)	-1	-1	-1	1	-1
(-1,1)	(1, 0)	-1	-1	1	-1	-1
(1,-1)	(-1, 0)	-1	-1	-1	1	-1
(1,-1)	(1, 0)	-1	-1	1	-1	-1

The proof of Lemma 3.1 is the verification of this relation. So we have shown all relations hold, and so η is a valid monoidal functor. □

4.4 Discussion of Results

Now that we have a weight lattice motivated graph planar algebra embedding for Web(SP_4), we look at why we needed to make several modifications to the graph we used for this embedding. We wish to contrast this to type A, in which we have an embedding into the graph planar algebra of the graph embedding in the most natural way.

In type A, the labelling on the Coxeter complex corresponds directly to the labelling of the weight lattice by fundamental weights (that is, a vertex of type k will correspond to a subspace of size k in, which in turn corresponds to the k_{th} fundamental weight for a certain labeling). In particular, this means there is a bijection between edges in the weight lattice and edges in the coxeter complex that preserves labels. In type C, this bijection does not exist. There are several ways we can see this. First, if we consider "label 1" vertex in the C_2 Coxeter complex/weight lattice, we see that it has degree 4, and that all of the edges correspond to weights in the first fundamental representation of sp_4. So these edges are all labeled 1 in the weight lattice. However, in the Coxeter complex, the label is simply the label of the target vertex minus that of the source. So we have edges labeled 1, with target vertex labeled 2, and vice versa, in the Coxeter complex. Now consider the case of a vertex labeled 2, that has edges labeled 1 and 2 both in the Coxeter complex and weight lattice. In the Coxeter complex, the 1-edges go from vertices labeled 2 to vertices labeled 0. But these edges have label 2 in weight lattice, as do any edges between degree 8 vertices. So it is clear we do not have the label preserving bijection we need here to allow us to label the edges of the graph in a naive way and use that in our embedding.

However, even when we used edges labels corresponding to the fundamental weights, we cannot embed Web(SP_4) into the graph planar algebra of this graph. The reason for this, is to prove the existence of the functor needed for the embedding, we must show that the relations hold generically for any choice of boundary vertices. Because we have two different "types" of vertices (that is degree 4 and degree 8 vertices), there are several cases to consider for even the most basic vertices. For example, we must consider the case in equation 4.4 where the outside vertex has degree 4 and when it has degree 8. The issue with this is that there are no edges with label two from the degree 4 vertex, so the morphism is 0, while the degree 8 vertex gives us a value of 4 (or 5 if we add self-loops). So we can never have a relation on a circle with label 2 for every choice of boundary vertices. Doing the partial completion operation in Definition 4.10 regains the symmetry that allows us to treat every case of boundary vertices similarly and prove the embedding above. This of course is unnecessary in type A, as every vertex has the same degree from the start.

Another major difference between our work here in type C and the results of Chapter 3 is the nature of the web category that we are choosing. Recall that the [Bod22] is a modification of Kuperberg's B_2 spider that is then extended in [BERT21]. These categories corresponds to the [CKM14] web categories in type A. However, our results in Chapter 3 use a version of SL_n that have been adapted for positive characteristic (and no longer take q-deformation into account) in [BEAEO20] and [Jon21]. A major difference between the two type A characterization is that [Jon21] indirectly constructs cup and cap morphisms instead of defining them as generators. It is possible that there is a similar positive characteristic version of the type C webs which will be the "correct" category to build our module categories.

Finally, we must address the issues caused by the non-special vertices present in type C when we try to extend this embedding to buildings. It is possible (and we believe it is likely) that there is a canonical way to add extra edges to the graph derived from a building in a similar way to Definition 4.10. But extending this construction beyond $n = 2$ gets messy quickly. It would be better to find a way to describe the links of non-special vertices as some kind of direct product of C_n Coxeter complexes (and thus buildings) and to use this definition to do the appropriate combinatorics, but initial attempts to do this have been unsuccessful.

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APPENDIX

APPENDIX

A

LIST OF SYMBOLS AND NOTATION

Table A.1: A collection of important notation used throughout

Notation	**Brief Definition**	**First Appearance**
(W, S)	Coxeter System	Definition 2.32
$\Sigma(W, S)$	The Coxeter complex of (W, S)	Definition 2.46
Σ	Shorthand for a Coxeter complex	Definition 2.46
ℓ	The canonical labeling on a Coxeter complex or building	Def. 2.42 Prop. 2.50
$lk_\Sigma A$	The link of A in Σ	Def. 2.43
Δ	A building	Definition 2.53
$\mathcal{A}$	A system of apartments for a building Δ	Definition 2.53
$Web(SL_n^-)$	Positive characteristic type A web category	Definition 3.2.3
$\mathcal{T}$	A triangle presentation (in Ch. 3, the degenerate one)	Def. 2.66 Def. 3.15
Λ_W	The weight lattice of a Lie algebra (here, $\mathfrak{sl}_n$)	Section 3.3.2
$[k]_q$	the quantum integer k	Section 3.2.1
Γ_Δ	The graph associated to a building Δ	Section 3.2.2
$type(p)$	The list of labels of the edges in a path p	Definition 3.10

$G(\Gamma)$	The graph planar algebra of a graph Γ	Definition 3.11
$G(\Delta)$	$G(\Gamma_\Delta)$	Section 3.2.2
L_i	Linear functional selecting the ith entry of a diagonal matrix	Section 3.17
σ	Involution associated to a triangle presentation	Definition 3.15
$Vec(\Delta)$	Vector bundles over the vertices of Δ	Definition 3.25
$End(Vec(\Delta))$	Functors from $Vec(\Delta)$ to $Vec(\Delta)$	Section 3.4.1
F_m	For $m \in \{1, ..., n\}$ a functor in $End(Vec(\Delta))$ that selects edges of label m	Definition 3.26
$E(m, i)$	Vertices k in Γ_Δ with an edge $i \to k$ with label m	Definition 3.26
$\mathbb{k}_a$	The object in $Vec(\Delta)$ whose only non-zero component is $\mathbb{k}$ at index a	Definition 3.27
A_g	Functor describing the action of $g \in G$ on $\mathcal{C}$	Definition 3.32
$\mathcal{C}^G$	The equivariantization of $\mathcal{C}$ under the action of G	Definition 3.34
F_m^G	Equivariant version of F_m functors	Definition 3.36
$Web(SP_4^-)$	$q = -1$ version of Bodish's $\mathfrak{sp}_4$ webs	Definition 4.6
$\bar{\Gamma}_{\tilde{C}_2}$	partial completion of $\tilde{C}_2$ Coxeter complex as a graph	Definition 4.10