A Comparison of Some Probability Distributions of Peak Combined Responses

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SUMMARY

The objective is to compare the exact probability distribution of the peak combined response with an approximation. The combined response is the sum of a Poisson Square wave and a Poisson shock process. The sizes of the square waves are independent and identically distributed random variables, and so are the shock sizes. The combined response models earthquake and operating load responses. The two stochastic processes are dependent because events in the processes may occur simultaneously. That is, shocks may cause changes in the square wave process, or changes in the square wave process may cause shocks.

The approximation for independent processes is accurate in the neighborhood of the mean of the peak combined load if the responses in one of the stochastic processes dominates the responses in the other or if shocks are rare. The approximation is not accurate if the processes have approximately the same means and if events occur frequently in the processes. This inaccuracy is shown in one example.

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1. The Probability Distribution of the Peak Combined Load From a Bivariate Poisson Shock and Square Wave Process

Renewal theory helps find the cumulative distribution function (cdf) of the peak combined load in a finite time interval from Poisson processes. The Poisson processes may be dependent. The cdf is given as a Laplace transform.

The combined load is the sum of a Poisson shock process and a dependent Poisson square wave process. (See Figure 1.) These load processes are typical of shocks and operating loads. Dependence means a change in the square wave causes a shock or a shock causes a change in the square wave.

The peak combined load cdf for independent Poisson shock and square wave processes has been approximated [1 - 4]. The exact cdf was given as a multiple integral in [5]. Jacobs and Gaver [6] wrote the renewal equation for the peak combined load cdf and obtained its Laplace transform.

Dependent load processes are common but have rarely been modeled. Wen [7] and Winterstein [8] modeled load processes with shocks of finite duration that follow a Poisson event after random delays. This paper models the sum of two different, dependent stochastic load processes. The Poisson square wave process represents a load process that is constant but changes randomly at event times in a Poisson process. The Poisson shock process represents a load process that is zero except for event times in another Poisson process. The two Poisson processes are dependent as in [9]. The magnitudes of the square wave and the shocks are independent random variables.

Definitions:

$X(t)$ = magnitude of square wave process at time t

$F(x)$ = $P[X(t) \leq x]$

$\lambda_2 + \lambda_{12}$ = the rate of changes of the square wave process

$Y(t)$ = magnitude of shock load process at time t

$\bar{G}(y)$ = $P[Y(t) > y \mid Y(t) > 0]$

$\lambda_1 + \lambda_{12}$ = the rate of shocks

$Z(t)$ = $X(t) + Y(t)$

$M(t)$ = $\sup_{s \leq t} Z(s)$ maximum load combination to occur in $[0,t]$.

$H_x(t)$ = $P[M(t) \leq x]$

$H_x(\xi)$ = $\int_0^\infty e^{-\xi t} H_x(t)dt$

$dL_x(z)$ = $(\lambda_1/(\lambda_1 + \lambda_{12}))\, dF(z) + (\lambda_{12}/(\lambda_1 + \lambda_{12}))\, G(x - z)\, dF(z)$

$L_x(\xi)$ = $\int_0^\infty e^{-\xi z} dL_x(z)$

Renewal theory gives the equation for $H_x(t)$.

The renewal equation is easily solved for the Laplace transform of $H_x(t)$. The solution is

$$H_x(\xi) = L_x(\xi)\ [1 - (\lambda_1 + \lambda_{12})\ L_x(\xi)]^{-1}. \qquad (1)$$

This is derived in [10]. The transform will be inverted by subroutine FLINV [11] for comparison with approximations.

2. Comparison of the Exact Load Cdf for Independent Processes with Simulation

In this section, we compare the exact cdf obtained from inversion of the Laplace transform with the empirical cdf from simulation. We compare the cdf for independent Poisson shock and square wave processes. The comparison is good, and the computation time is about the same.

When the shock and square wave processes are independent, $\lambda_{12} = 0$, and the Laplace transform is

$$H_x(\xi) = M_x(\xi)\ [1 - \lambda_2\ M_x(\xi)]^{-1} \qquad (2)$$

where [6]

$$M_x(\xi) = {}_0\int^{\infty} [\xi + \lambda_2 + \lambda_1\ \overline{G}(x - y)]^{-1} dF(y). \qquad (3)$$

If shock sizes have an exponential cdf, $\overline{G}(x) = e^{-x}$, and the square wave is 1.0 or 2.0 with equal probability,

$$F(x) = \begin{cases} 0 & x < 1 \\ 1/2 & 1 \leq x < 2 \\ 1 & x \geq 2, \end{cases} \qquad (4)$$

then from (3)

$$M_x(\xi) = \begin{cases} 0 & x < 1 \\ (1/2)\ [\xi + \lambda_2 + \lambda_1 e^{-x+1}]^{-1} & 1 \leq x < 2 \\ (1/2)\ \{[\xi + \lambda_2 + \lambda_1\ e^{-x+1}]^{-1} + [\xi + \lambda_2 + \lambda_1\ e^{-x+2}]^{-1}\} & x \geq 2 \end{cases} \qquad (5)$$

In order to invert $H_x(\xi)$, we need the singularities of (2), the solutions of

$$1 - \lambda_2\ M_x(\xi) = 0. \qquad (6)$$

Assume $\lambda_1 = \lambda_2 = 1$. The singularities are

$$\xi_o = \frac{[1 + e^{-x+1} + e^{-x+2}] \pm \sqrt{[1 + e^{-x+1} + e^{-x+2}]^2 - 4(e^{-2x+3} - 1)}}{2} \qquad (7)$$

for $x \geq 2$. These singularities are more interesting because the square wave is 2.0 half the time, so the peak load has probability 1/2 of being at least as large as 2.0.

Table 1 compares the exact cdf of M(t) with the simulated empirical cdf. The simulation was for several t values and 10000 replications were used for each simulation. The same seed was used for every replication, and this probably explains the simulation bias.

Table 1. A Comparison of Exact and Simulated Peak Load Cdfs

Load x	Time t	Exact $H_x(t)$	Empirical cdf of $H_x(t)$
2	1	0.5233	0.5290
3	1	0.7814	0.7772
4	1	0.9122	0.9136
5	1	0.9666	0.9676
10	1	0.9998	0.9999
2	5	0.0477	0.0526
3	5	0.2999	0.3023
4	5	0.6341	0.6362
5	5	0.8443	0.8404
10	5	0.9988	0.9985
2	10	0.00247	0.0038
3	10	0.0911	0.0950
4	10	0.4029	0.4050
5	10	0.7130	0.7166
10	10	0.9977	0.9977

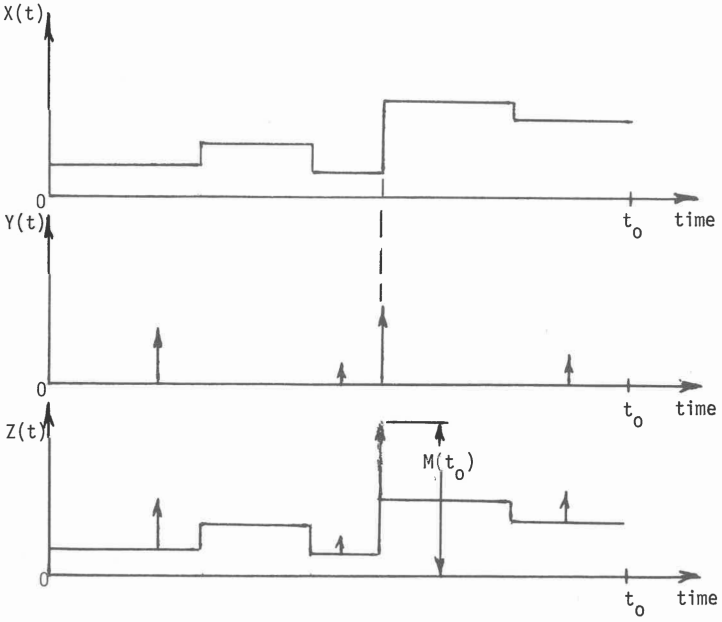

FIGURE 1. THE SUPERPOSITION OF A POISSON SQUARE WAVE AND A POISSON SHOCK PROCESS.

3. Comparison of the Exact Load Cdf for Independent Processes with an Approximation

In this section, the exact load cdf for independent processes is compared with an analytical approximation. (We don't compare the exact cdf for dependent processes with an approximation because there is no approximation.) The comparison is bad.

The approximation is [2]

$$H_x(t) \simeq \exp\{-\lambda_2 \; t \; P[X(t) + U > x]\} \qquad (8)$$

where U is a random variable that approximates the shock process. Its cdf is

$$P[U \leq u \; = \{1 + \frac{\lambda_1}{\lambda_2} \; \overline{G}(u)\}^{-1} \qquad (9)$$

Since the square wave process has equal probability of being 1.0 or 2.0, the convolution cdf (assuming $\overline{G}(x) = e^{-x}$) is

$$P[X(t) + U \leq x] = \begin{cases} 0 & x < 1 \\ (1/2)\;[1 + \frac{\lambda_1}{\lambda_2}\; e^{-x+1}]^{-1} & 1 \leq x < 2 \\ (1/2)\;\{[1 + \frac{\lambda_1}{\lambda_2}\, e^{-x+1}]^{-1} + [1 + \frac{\lambda_1}{\lambda_2}\; e^{-x+2}]^{-1}\} & x \geq 2. \end{cases} \qquad (10)$$

Table 2 gives the exact cdf and the approximation for the same parameters as in Section 2, $\lambda_1 = \lambda_2 = 1$ and $\lambda_{12} = 0$. The approximation is, for $x \geq 2$,

$$H_x(t) \simeq \exp\;\{-t[1-(2(1+e^{-x+1}))^{-1} \; -(2(1+e^{-x}e+2))^{-1}]\}. \qquad (12)$$

The cdf $H_x(t)$ tells the probability peak combined response during a time interval of length t is less than or equal to x. Since the approximation for $H_x(t)$ is greater than the exact, the approximation is not conservative.

Table 2. A Comparison of Exact and Approximate Peak Load Cdfs

Load x	Time t	Exact $H_x(t)$	Approximate $H_x(t)$
2	1	0.5233	0.8236
3	1	0.7814	0.9201
4	1	0.9122	0.9678
5	1	0.9666	0.9877
10	1	0.9998	0.9999
2	5	0.0477	0.3790
3	5	0.2999	0.6593
4	5	0.6341	0.8491
5	5	0.8443	0.9402
10	5	0.9988	0.9996
2	10	0.00247	0.1436
3	10	0.0911	0.4347
4	10	0.4029	0.7210
5	10	0.7130	0.8839
10	10	0.9977	0.9992

References

1. Bosshard, W., "On Stochastic Load Combination," Earthquake Engineering Center, Stanford University, June, 1975.

2. Wen, Y.-K., "Statistical Combination of Extreme Loads," J. Structural Division, ASCE, May, 1977.

3. McGuire, R. K. and C. A. Cornell, "Live Load Effects on Office Buildings," J. Structural Division, ASCE, July 1974.

4. Der Kiureghian, A., "Reliability Analysis Under Combinations of Stochastic Loads," J. Structural Division, ASCE, February, 1980.

5. Hasofer, A. M., "Time Dependent Maximum of Floor Live Loads," J. of Engineering Mechanics Division of ASCE, Vol. 100, No. EM5, pp 1086-1091, 1974.

6. Jacobs, P. A. and D. P. Gaver, "On Combinations of Random Loads," SIAM J. Appl. Math., to appear.

7. Wen, Y.-K., "A Clustering Model for Combination of Correlated Load Processes," Tech. Report for NSF Grant CME 79-18053, Dept. of Civil Engineering, University of Illinois, Urbana, Illinois, April, 1980.

8. Winterstein, S. R., "Stochastic Dynamic Response Combinations," M. S. Thesis, Dept. of Civil Engineering, M.I.T., Cambridge, Massachusetts, 1980.

9. Teicher, H., "On the Multivariate Poisson Distribution," Skandinavisk Adurietidskrift, Vol. 1, pp 1 - 9, 1954.

10. George, L. L., "Peak Combined Load From a Bivariate Shock and Square Wave Process," ORSA/TIMS Meeting, Colorado Springs, CO, Lawrence Livermore National Laboratory UCRL-84150, Livermore, California, November 1980.

11. IMSL Library Reference Manuals I and II (Version 8), International Mathematics and Statistics Library, Inc., Suite 510, Hillcroft, Houston, Texas, 1979.