Instability of Members with End Flexibility

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ABSTRACT

Numerous structural frameworks are necessary in a power plant for Seismic Category I commodity supports, etc. Given the intricate framing arrangements and the partial end restraints offered by the restraining members to the primary load resisting elements, the complex member end boundary conditions are very critical to the support framework's overall strength and stability. In order to accurately assess the true strength of a structural framework, elastic stability of a member with end flexibility is formulated with due consideration of its actual end conditions and stiffness relative to the restraining members. Effects of stiffness losses occurring as a result of yielding along the member or at the supports due to end relaxation is also addressed and conclusions are drawn. Application of the results will lead to efficient design and significant savings by eliminating member oversizing.

1. INTRODUCTION

The stable configuration of framing members in a structural joint produces flexible spring type supports that are capable of resisting member axial load and shear forces, in addition to bending and torsional moments. The end restraints, along with the magnitude and direction of residual stresses and initial crookedness, has a dominant effect on member maximum strength and stability.

As Shanley (1950) in his paper on applied column theory suggests, the theoretical effect of end restraint is very simple; it merely changes the effective length, i.e., the bifurcation. Afterwards, it affects the load versus displacement curve. Therefore, any end relaxation or yielding will modify the end restraint and hence the member effective length. However, an accurate determination of these effects is a difficult problem for any member other than a simply supported column, because of the evaluation of the true degree of end restraint in actual framed structures.

Herein, elastic stability of a compression member with end flexibility is formulated exactly. Procedures are introduced to assess and improve the effective length if stiffness losses occur as a result of yielding along the member or at the supports due to end relaxation or yielding.

SMiRT 11 Transactions Vol. K (August 1991) Tokyo, Japan, © 1991

For a review of earlier works, references are made to Chen (1980), Stoman (1982) and Razzaq (1983).

2. METHODOLOGY

Consider the prismatic member shown in Figure 1, with rotational restraints of stiffness β_1 and β_2. Equating the external and internal moments at a section a distance x from the origin,

$$EIy'' + Py + Vx - \beta_1\Theta_1 = 0 \tag{1}$$

with

$$V = \frac{\beta_1\Theta_1 - \beta_2\Theta_2}{\ell} \tag{2}$$

The general solution to this differential equation is

$$y = C_1 \sin kx + C_2 \cos kx + \frac{\beta_2\Theta_2}{P\ell}x - \frac{\beta_1\Theta_1}{P\ell}(x-\ell) \tag{3}$$

which upon the application of the geometric boundary conditions:

1. $y(x) = 0$ at $x = 0$
2. $y'(x) = \Theta_1$ at $x = 0$
3. $y(x) = 0$ at $x = \ell$
4. $y'(x) = -\Theta_2$ at $x = \ell$

results in a transcendental equation that is relevant to instability of members with end flexibility. That is,

$$\cos k\ell \left[\beta_1 + \beta_2 + 2\frac{\beta_1\beta_2}{P\ell}\right] - \frac{\sin k\ell}{k\ell}\left[\beta_1 + \beta_2 - (k\ell)^2\frac{\beta_1\beta_2}{P\ell} + P\ell\right] - 2\frac{\beta_1\beta_2}{P\ell} = 0 \tag{4}$$

where, $k^2 = P/EI$. Let $\beta_1 = mEI/\ell$ and $\beta_2 = nEI/\ell$, where m and n are numerical constants; then Equation 4 can be simplified to

$$\cos k\ell \left[(m+n) + \frac{2mn}{(k\ell)^2}\right] - \frac{\sin k\ell}{k\ell}\left[(m+n) - mn + (k\ell)^2\right] - \frac{2mn}{(k\ell)^2} = 0 \tag{5}$$

For given m and n, $k\ell$ and, hence, P_{cr} can be directly evaluated from this equation. For certain values of m and n Table 1 gives associated values of $k\ell$ and P_{cr}/P_E, where P_E is the Euler load for a pin-ended column of length ℓ.

Although special cases such as m or n equal to zero can be developed from Equation 5, a good appreciation of the behavior can be attained from the preceding equations. Consider the case of $\beta_1 = \beta_2 = \beta$ (i.e. m = n) and, hence, $\Theta_1 = \Theta_2 = \Theta$. The solution to Equation 1 reduces to

$$y = \Theta\left[\frac{\sin kx}{k} + \frac{\beta}{P}(1-\cos kx)\right] \tag{6}$$

Substituting for ß from above, Equation 6 can alternatively be written as

$$y = \Theta \frac{1}{k^2 \ell} [k\ell \sin kx + m (1 - \cos kx)] \qquad (7)$$

which upon the application of the boundary condition y = 0 at x = ℓ reduces to

$$(k\ell) \sin k\ell + m (1 - \cos k\ell) = 0 \qquad (8)$$

This simple equation results in critical load for a column with identical flexible supports at the two ends. Table 2 displays values of $k\ell$ and P_{cr} for some selected values of m. Values of ß equal to $3EI/\ell$ and $4EI/\ell$ (i.e., m = 3 and 4) correspond to the rotational stiffness provided by the hinged and fixed columns of the respective frames, shown in Figure 2, to the compressed members. See Figure 3 for a plot of Equation 8.

Table 1. Critical Loads for Members with End Flexibility

m	n	$k\ell$	P_{cr}/P_E
0	0	π	1.000
0	400	4.482	2.036
3	3	4.350	1.917
3	4	4.463	2.018
3	400	5.180	2.719
4	4	4.578	2.124
8	8	5.141	2.678
400	400	6.252	3.960

Table 2. Critical Load for a Member with Identical Flexible Supports

m	$k\ell$	P_{cr}/P_E	m	$k\ell$	P_{cr}/P_E
0.0	π	1.000	4.0	4.578	2.123
0.1	3.204	1.040	8.0	5.141	2.678
0.2	3.264	1.079	10.0	5.307	2.854
0.3	3.322	1.118	100.0	6.160	3.845
0.4	3.377	1.155	200.0	6.220	3.920
0.5	3.431	1.193	400.0	6.252	3.960
1.0	3.673	1.367	1,000.0	6.271	3.984
3.0	4.349	1.917	10,000.0	6.280	3.996
3.5	4.470	2.024	∞	6.283	4.000

Pinned–Flexible members: The instability of Pinned–Flexible members can be studied from Equation 5 if either m or n is set equal to zero. Assuming n = 0, the equation reduces to

$$m \cos k\ell - [m + (k\ell)^2] \frac{\sin k\ell}{k\ell} = 0 \qquad (9)$$

However, if the instability of the Pinned-Flexible member is directly formulated, the deflected shape will be found as

$$y = \frac{\beta\Theta}{P_E}\left[\frac{x}{\ell} - \frac{\sin kx}{\sin k\ell}\right] \tag{10}$$

where $\beta = mEI/\ell$. Equation 10 has a maximum at a distance x away from the pinned end, i.e. from the origin, such that

$$x = Arc\ \cos\left(\frac{\sin k\ell}{k\ell}\right) \tag{11}$$

With the application of the boundary condition $y'(x) = -\Theta$ at $x = \ell$ on Equation 10, the applicable transcendental equation is obtained. That is

$$1 + \frac{m}{k\ell}\left[\frac{1}{k\ell} - \frac{\cos k\ell}{\sin k\ell}\right] = 0 \tag{12}$$

This equation is the same as Equation 9 but written in a different from. Normalized values of P_{cr} as evaluated from Equation 12 are plotted in Figure 4 versus m.

3. RESULTS

It is evident from Table 2 or Figure 3, the plot of Equation 8, that a column with flexible supports whose value of m is about 12 can carry up to 75% of the critical load of a rigid or fixed supported column. With further increases in m the capacity increases asymptotically to a maximum of $4.0P_E$. The flat slope of the curve for large values of m indicates that partial end stiffness losses or gains have negligible effect on the critical load. For instance, for a support with an m value of 200, a 50% loss in stiffness reduces the axial load capacity by less than 2%.

In the intermediate range, i.e. for values of m around 20, end effects are more pronounced, such that a 50% loss of stiffness lowers the critical load of the member by about 14%. Whereas in the lower ranges of the curve, these effects can lower member capacity by about 21%.

If the member loses stiffness due to yielding along its span, the end stiffness relative to member stiffness increases and, therefore, has a stabilizing effect on the member by reducing its effective length. This would raise the critical load capacity, but since the effective modules is also reduced, the increase is somewhat mitigated. Thus, an accurate evaluation of ß and member stiffness are necessary in describing true behavior. For members in structural frameworks this can be accomplished by an analysis that considers stress-strain distribution as yielding progresses (see Razzaq 1983).

Likewise, Figure 4 reveals that a small rotational restraint can provide the Pinned-Flexible member with the necessary end stiffness to develop almost its full capacity. Moreover, it illustrates that some losses in rotational stiffness of the connecting beams and columns will not significantly affect member critical load if $m \geq 40.0$. A similar curve is also provided by Galambos (1968).

Improved values of the effective length can be obtained by repeating the elastic stability solution for an adjusted value of the rotational restraint, ß. That is,

$$\beta' = m' \frac{EI}{\ell} \tag{13}$$

where

$$m' = m \frac{(EI)elastic}{(EI)inelastic} \approx m \frac{(P_{cr})elastic}{(P_{cr})inelastic} \tag{14}$$

Then, P_{cr} is evaluated with the improved relative stiffness values – see Yura (1971).

4. CONCLUSIONS

Elastic stability of members with end flexibility stiffness is formulated. The formulation considers the effects of member yielding and/or support stiffness losses due to end relaxation or yielding. Procedures are introduced for evaluating improved values of the effective length, using adjusted values of end restraint that account for inelasticity. Application of the methodology will result in substantial savings due to design efficiency and due to understanding the behavior of members with end flexibility.

REFERENCES

Chen, W. F. (1980). End Restraint and Column Stability, *Journal of Structural Engineering,* ASCE, Vol. 106, No. ST11, pp. 2279-2295.

Galambos, T. V. (1968). *Structural Members and Frames,* Prentice Hall, Inc., Englewood Cliffs, New Jersey

Razzaq, Z. (1983). End Restraint Effect on Column Strength, *Journal of Structural Engineering,* ASCE, Vol. 109, No. 2, pp. 2279-2295.

Shanley, F. R. (1950). Applied Column Theory, *Transactions,* ASCE, Vol. 115, pp. 698-750.

Stoman, S. H. (1982). Nonlinear Collapse Load Analysis of Braced, Framed Structures, Thesis presented to Case Western Reserve University, at Cleveland, Ohio, in partial fulfillment of the requirements for the degree of doctor of philosophy.

Yura, J. A. (1971). The Effective Length of Columns in Unbraced Frames, *AISC Engineering Journal,* Vol. 8, No. 2, pp. 37-42.

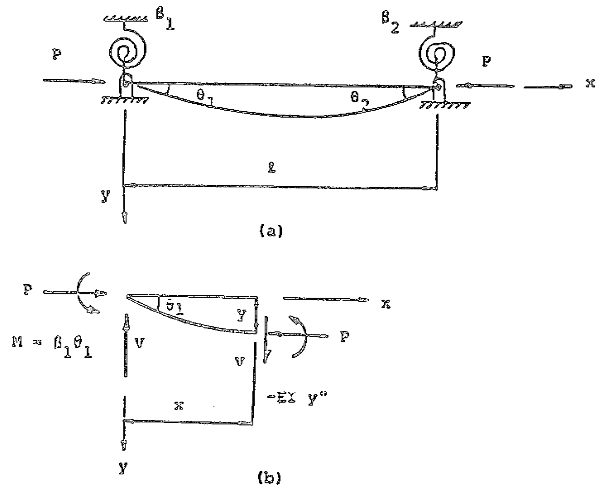

Fig. 1 Member with End Flexibility

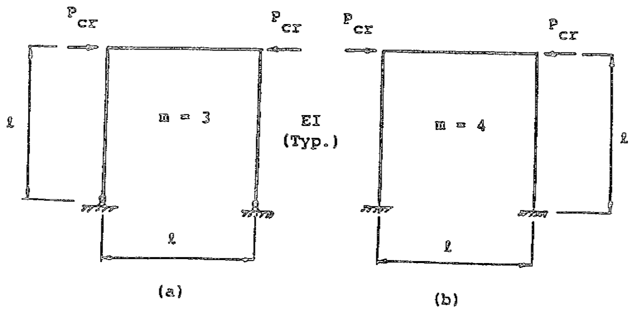

Fig. 2 Typical Support Frames

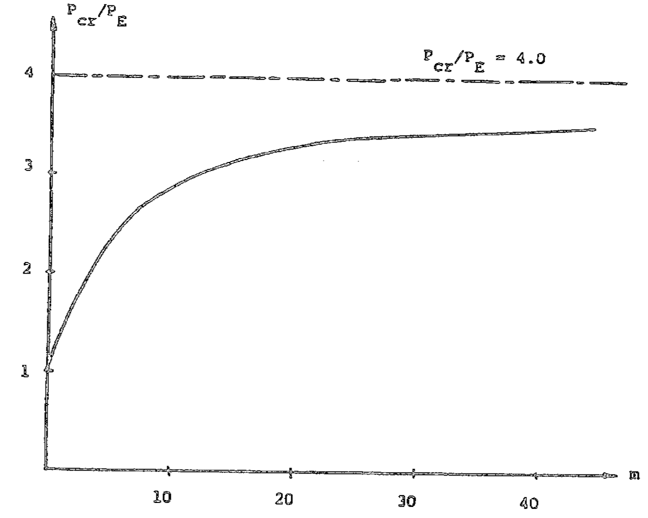

Fig. 3 Plot of End Effects for a Member with Identical Flexible Supports

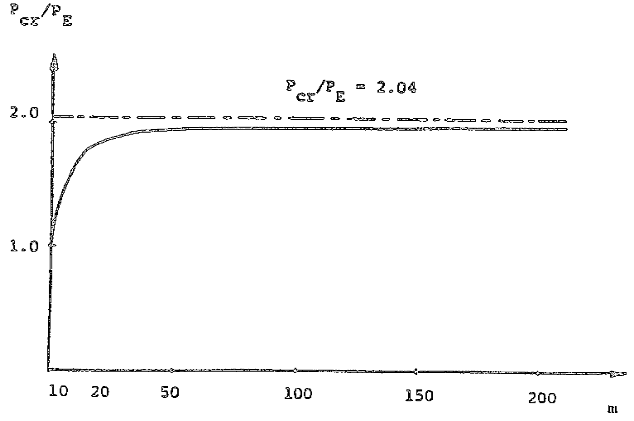

Fig. 4 Plot of End Effects for Pinned-Flexible Members