A NOTE ON THE ASYMPTOTIC BEHAVIOR OF CONDITIONAL EXTREMES

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ABSTRACT

Suppose $(X_1,Z_1),\dots,(X_n,Z_n)$ are i.i.d. as (X,Z). Let $M_{n,k}$ be the maximum of all Z_i's for which X_i is among the k–nearest neighbors of $X=x_0$. It is shown that $M_{n,k}$ has the same asymptotic distribution as the maximum of k i.i.d. copies of the random variable Z given $X=x_0$. Hence, the asymptotic results for i.i.d. observations are applicable to this situation as well.

Key words: nearest neighbor, conditional extremes.

1. INTRODUCTION AND FORMULATION OF THE PROBLEM:

In recent years, the extreme value theory has generated considerable interest. Not only that the classical results for i.i.d. observations are revisited, but also these results are extended to handle some types of dependent observations as well. A review of the classicial and current results in this area may be found in Leadbetter, Lindgren and Rootzen (1983).

Our objective, however, is to study the asymptotic behaviour of local extremes of a conditional distribution. More specifically, suppose $(X_1,Z_1),\ldots,(X_n,Z_n)$ are i.i.d. as (X,Z). Let f and F be the density and the distribution of the random variable X, and let g and G be the conditional density and the conditional distribution of Z given X.

Define, at $X=x_0$

$$(1.1) \qquad Z_F = Z_F(x_0) = \sup\{z \mid G(z \mid x_0) < 1\} \leq \infty ,$$

Z_F is called, the right end point of the conditional distribution of Z given $X=x_0$. Also denote $G(z \mid x_0)$ simply by $G(z)$.

Now, choose $k=k(n)=0(n^{\alpha})$ where $\frac{1}{2} \leq \alpha < \frac{2}{3}$. More specifically, let $k=k(n)=[\varphi(n)b]$, where $\varphi(n)=n^{\alpha}$ for $\frac{1}{2} \leq \alpha < \frac{2}{3}$ and b is a constant, $0 < b < \infty$. For simplicity of argument, assume that $\varphi(n)b$ is a constant. Furthermore, define $Y_i = |X_i - x_0|$, $i=1,2,\ldots,n$ and $Y = |X-x_0|$. Also let $0 \leq Y_{n,1} < Y_{n,2} < \ldots Y_{n,n} < \infty$ denote the order statistics of $Y_1,\ldots,Y_n$ and let $Z_{n,1},\ldots,Z_{n,n}$ be the induced order statistics of $(Y_1,Z_1),\ldots,(Y_n,Z_n)$; i.e., $Z_{n,i} = Z_j$ if $Y_{n,i} = Y_j$, $i=1,2,\ldots,n$.

Now, define

$$M_{n,k} = \max\{Z_{n,1},\ldots,Z_{n,k}\} .$$

Our objective is to study the asymptotic behavior of $M_{n,k}$.

Note that if $Z_F < \infty$, then $M_{n,k}$ may be thought of as a k–nearest neighbor estimate of Z_F. However, as far as the theoretical treatment of the problem is concerned, we don't need to assume that $Z_F < \infty$.

At this point, we make the following assumptions:

1. (a) $f(x_0) > 0$ and f'' is continuous at $X=x_0$.

   (b) $f'(x)$ is bounded at $X=x_0$.

2. (a) $G_x(z|x)$ and $G_{xx}(z|x)$ exist and are bounded for (x,z) lying in a neighborhood of (x_0,Z_F).

   (b) $G_{xx}(z|x)$ is equicontinuous in x at $X=x_0$ for z lying in a neighborhood of Z_F, i.e., $\lim_{x \to x_0} G_{xx}(z|x) = G_{xx}(z|x_0)$ hold uniformly for z lying in a neighborhood of Z_F.

2. ASYMPTOTIC DISTRIBUTION OF $M_{n,k}$

Let $\tilde{Z}_1, \tilde{Z}_2,\dots,\tilde{Z}_k$ be k i.i.d. observations having the distribution $G(z|x_0)$; i.e., $\tilde{Z}_1,\dots,\tilde{Z}_k$ are i.i.d. as Z given $X=x_0$. Define

$$\tilde{M}_{n,k} = \max(\tilde{Z}_1,\dots,\tilde{Z}_k)$$

The following theorem describes the relationship between the asymptotic distributions of $M_{n,k}$ and $\tilde{M}_{n,k}$.

Theorem 1

Under the assumptions 1 and 2, we have

$$P(M_{n,k} \le z) = G^k(z|x_0) + R_{n,k} ,$$

where the bias term $R_{n,k}$ is such that

$$R_{n,k} \to 0 \text{ as } n \text{ (and, hence, } k) \to \infty$$

A detailed proof of the theorem is given in Section 3. The following corollary is a simple consequence of Theorem 1.

Corollary 1

Under the assumptions of Theorem 1

$$P(M_{n,k} \le z) - P(\tilde{M}_{n,k} \le z) \to 0.$$

Corollary 2

Under the assumptions of Theorem 1

$$M_{n,k} \xrightarrow{a.s.} Z_F$$

Proof

Let $\epsilon > 0$ and consider

$$P(M_{n,k} \le Z_F - \epsilon) = G^k(Z_F - \epsilon) + R_{n,k},$$

by theorem 1. So, by (1.1) we have

(2.1) $\qquad P(M_{n,k} \le Z_F - \epsilon) \to 0 \quad \text{as } n \text{ (and } k) \to \infty.$

Similarly;

(2.2) $\qquad P(M_{n,k} \ge Z_F + \epsilon) \to 0 \quad \text{as } n \to \infty.$

So, by (2.1) and (2.2), we have

$$M_{n,k} \to Z_F$$

in probability. However, $M_{n,k}$ is a monotone increasing function of n and k; hence

$$M_{n,k} \xrightarrow{a.s.} Z_F.$$

Q.E.D.

Theorem 2

Under the assumptions 1 and 2, if for some constants $a_k > 0$; b_k; we have

$$P(a_k(M_{n,k}-b_k) \leq Z) \xrightarrow{w} H(z|x_0)$$

for some non–degenerate H, then H is one of the following three extreme value type:

Type I

$$H(z|x_0) = \exp(-e^{-z}) \quad -\infty < z < \infty$$

Type II

$$H(z|x_0) = \begin{cases} 0 & z \leq 0 \\ \exp(-z^{-\alpha}) & \text{for some } \alpha > 0,\ z > 0 \end{cases}$$

Type III

$$H(z|x_0) = \begin{cases} \exp(-(-z)^{\alpha}) & \text{for some } \alpha > 0 \text{ and } z \leq 0 \\ 1 & z > 0 \end{cases},$$

Proof:

By theorem 1

$$P(a_k(M_{n,k}-b_k) \leq z) \xrightarrow{w} H(z|x_0)$$

if and only if

$$P(a_k(\tilde{M}_{n,k} - b_k) \leq z) \xrightarrow{w} H(z|x_0),$$

hence the result is an easy consequence of theorem 1.4.2 of (4) for i.i.d. sequence.

Q.E.D.

The following theorem gives a necessary and sufficient condition for the asymptotic distribution to belong to each of the three types.

Theorem 3

Necessary and sufficient conditions for the conditional distribution G of the r.v.'s of the i.i.d. sequence $\{\tilde{Z}_k\}$ given $X=x_0$ to belong to each of the three types are:

Type I

There exists some strictly positive function $g(t,x_0)$ such that

$$\lim_{t\uparrow Z_F} \frac{1 - G(t+z\,g(t,x_0))}{1 - G(t)} = e^{-z}$$

for all real z.

Type II

If $Z_F = \infty$ and

$$\lim_{t\to\infty} \frac{(1 - G(tz))}{1 - G(t))} = z^{-\alpha}, \quad a > 0 \ ,$$

for each $z > 0$.

Type III

$$Z_F < \infty \quad \text{and} \quad \lim_{h\downarrow 0} \frac{1 - G(Z_F - zh)}{1 - G(Z_F - h)} = z^{\alpha} ,$$

$\alpha > 0$, for each $z > 0$.

Proof of this theorem is exactly the same as that of theorem 1.6.2 in (4). A simpler sufficient condition for G to belong to each of the three possible types under additional conditions is given in theorem 1.6.1 of (4).

Example:

Let (X,Z) have a bivariate normal distribution with $\mu_X=0$, $\mu_Z=0$, $\sigma_X^2=1$, $\sigma_Z^2=1$ and $\mathrm{Corr}(X,Z) = \rho$. Then the conditional distribution of Z given $X=x_0$ is normal with mean ρx_0 and variance $(1-\rho^2)$.

So, by Corollary 1 and theorem 1.5.3 of (4), we have

$$P(a_k(M_{n,k}-b_k) \le z) \to \exp(-e^{-z})$$

where

$$a_k = \frac{(2\ \log\ k)^{\frac{1}{2}}}{\sqrt{1-\rho^2}}$$

and

$$b_k = \rho\, x_0 + [(2 \log k)^{\frac{1}{2}} - \tfrac{1}{2}(2 \log k)^{-\frac{1}{2}} (\log \log k + \log 4\pi)]\sqrt{1-\rho^2}\ .$$

3. PROOF OF THEOREM 1

First, we will establish two important results in lemma 1 and lemma 2. Lemma 1 deals with the order statistics of the random variable $Y = |X-x_0|$ and lemma 2 gives a representation of the conditional distribution of Z given $Y=Y_{n,i}$ (for $i=1,2,\dots,k$) in terms of the conditional distribution of Z given $X=x_0$.

Lemma 1

Let $\varphi(n) \to \infty$ and $n^{-1}\varphi(n) \to 0$ as $n \to \infty$. Also assume $f(x_0) > 0$. Then for $B \ge \frac{b}{f(x_0)}$ and $C \ge 0$, we have

$$P(Y_{n,\varphi(n)b} > n^{-1}\, \varphi(n)B + C) \le \exp[-2\, n^{-1}\, \varphi^2(n)\, (B\, f(x_0)-b)^2$$
$$-2\, n\, C^2(f(x_0))^2 - 4C\, \varphi(n)f(x_0)(B\, f(x_0)-b)]$$

Proof: Bhattacharya and Mack (1987) proved the result for $C = 0$. Exactly the same line of argument may be used to obtain the result for any $C \geq 0$.

Q.E.D.

Now, let f_Y and F_Y denote the pdf and cdf of Y, $G^*(\cdot|y)$ is the conditional cdf of Z given $Y=y$. Note that $G^*(\cdot|0) = G(\cdot|x_0) = G(\cdot)$. Define $G^*(\cdot|Y_{n,i})$ (i.e., the conditional distribution of Z given $Y=Y_{n,i}$) by $G_{n,i}(\cdot)$.

Lemma 2

Under the assumptions 1 and 2

$$G_{n,i}(z) = G(z) + \delta_{n,i}\,(Q(z) + R(z,\delta_{n,i})) \qquad i=1,2,\ldots,n$$

where

$$\delta_{n,i} = \frac{1}{2} Y_{n,i}^2$$

$$Q(z) = G_{xx}(z|x_0) + 2 \cdot \frac{f'(x_0)\; G_x(z|x_0)}{f(x_0)}$$

and

$$\lim_{\delta \downarrow 0} R(z,\delta) = 0$$

Uniformly in z lying in a neighborhood of Z_F.

Proof:

This result is very similar to lemma 2.3 in Gangopadhyay (1987). Here we will sketch the line of argument.

Since $Y = |X-x_0|$

$$P(Y \leq y) = F(x_0+y) - F(x_0-y) = F_Y(y)$$

and

$$f_Y(y) = f(x_0+y) + f(x_0-y).$$

Also;

$$G_{Z|Y}(z|y) = P(Z \le z | Y = y) = \frac{f(x_0+y)\ G(z|x_0+y)\ + f(x_0-y)\ G(z|x_0-y)}{f(x_0+y)\ +\ f(x_0-y)}$$

Expand $f(x_0+y)$, $f(x_0-y)$, $G(z|x_0+y)$ and $G(z|x_0-y)$ around x_0 to obtain

$$f(x_0+y) + f(x_0-y) = 2\, f(x_0) + \tfrac{1}{2}\, y^2\{f''(x_0+\theta_1 y) + f''(x_0+\theta_2\, y)\}$$

and

$$f(x_0+y)\ G(z|x_0+y) + f(x_0-y)\ G(z|x_0-y)$$

$$= 2\, f(x_0)\ G(z|x_0) + \tfrac{1}{2}\, y^2[\{G_{xx}(z|x_0+\theta_3 y) + G_{xx}(z|x_0+\theta_4 y)\}f(x_0)$$

$$+ G(z|x_0)\ \{f''(x_0+\theta_1 y) + f''(x_0+\theta_2 y)\} + 4\, f'(x_0)G_x(z|x_0)]$$

where $|\theta_i| \le 1$, $i=1,2,3,4$.

Since by virtue of our regularity conditions

$$\lim_{y\to 0}\ [f''(x_0+\theta_i y) - f''(x_0)] = 0$$

and

$$\lim_{y\to 0}\ [G_{xx}(z|x_0+\theta_i y) - G_{xx}(z|x_0)] = 0$$

uniformly in z lying in a neighborhood of Z_F, we can write

$$[f(x_0+y) + f(x_0-y)]^{-1} = \{2\, f(x_0)\}^{-1}\ [1 + \tfrac{1}{2}\, y^2\{f''(x_0)/f(x_0)\} + y^2\ \epsilon_1(y)]^{-1}$$

$$= \{2\, f(x_0)\}^{-1}\left[1 - \tfrac{1}{2}\, y^2\left[\frac{f''(x_0)}{f(x_0)}\right] + y^2\ \epsilon_2(y)\right]$$

where

$$\epsilon_1(y) = \frac{1}{4\, f(x_0)}\ \sum_{i=1}^{2}\ \{f''(x_0+\theta_i y) - f''(x_0)\}\ =\ o(1)$$

as $y \to 0$, so that $\epsilon_2(y)$ is also $o(1)$ as $y \to 0$, and

$$f(x_0+y)\,G(z|x_0+y) + f(x_0-y)\,G(z|x_0-y)$$

$$= 2\,f(x_0)[G(z|x_0) + \tfrac{1}{2}\,y^2\{f(x_0)G_{xx}(z|x_0) + G(z|x_0)f''(x_0)$$

$$+ 2\,f'(x_0)\,G_x(z|x_0)\} \,/\, f(x_0) + y^2\,\epsilon_3(z,y)]\,,$$

where

$$\epsilon_3(z,y) = (\tfrac{1}{4})\,\frac{g(z|x_0)}{f(x_0)}\,\sum_{i=1}^{2}\,\{f''(x_0+\theta_i y) - f''(x_0)\}$$

$$+ (\tfrac{1}{4})\sum_{i=3}^{4}\,\{G_{xx}(z|x_0 + \theta_i y) - G_{xx}(z|x_0)\}$$

is $o(1)$ as $y \to 0$, uniformly in z lying in a neighborhood of Z_F.

Multiplying the last two expressions, we now have

$$G_{Z|Y}(z|y) = [1 - (\tfrac{1}{2})y^2\,\{f''(x_0)/f(x_0)\} + y^2\epsilon_2(y)]$$

$$+ [G(z|x_0) + \tfrac{1}{2}\,y^2\{f(x_0)\,G_{xx}(z|x_0) + G(z|x_0)f''(x_0)$$

$$+ 2\,f'(x_0)\,G_x(z|x_0)\}/f(x_0) + y^2\,\epsilon_3(z,y)]$$

$$= G(z|x_0) + \tfrac{1}{2}\,y^2\{Q(z) + R(z, \tfrac{1}{2}\,y^2)\}$$

$$= G(z|x_0) + \delta\{Q(z) + R(z,\delta)\}$$

where

$$\delta = \tfrac{1}{2}\,y^2$$

$$Q(z) = G_{xx}(z|x_0) + 2\;\frac{f'(x_0)\;G_x(z|x_0)}{f(x_0)}\;,$$

and

$$\delta\,R(z,\delta) = \tfrac{1}{2}\,y^2\;R(z,\tfrac{1}{2}\,y^2)$$

stands for the remainder term; i.e., $R(z,\frac{1}{2}\,y^2)$ is of the form

$$R(z,\tfrac{1}{2}y^2) = A(z)\,\epsilon_2(y) + B\,\epsilon_3(z,y) + 2\,y^2\,\epsilon_2(y)\,\epsilon_3(z,y)$$

where $A(z)$ is a linear combination of $G(z|x_0)$, $G_x(z|x_0)$ and $G_{xx}(z|x_0)$ which are bounded in a neighborhood of Z_F. Together with the convergence properties of $\epsilon_2(y)$ and $\epsilon_3(z,y)$, as observed earlier, this implies that

$$\lim_{\delta\to 0} R(z,\delta) = \lim_{y\to 0} R(z,\tfrac{1}{2}y^2) = 0$$

uniformly in z lying a neighborhood of Z_F.

Hence,

$$G_{n,i}(z) = G^*(z|Y_{n,i}) = G_{Z|Y}(z|Y_{n,i})$$

$$= G(z|x_0) + \delta_{n,i}(Q(z) + R(z,\delta_{n,i}))$$

where

$$\delta_{n,i} = \frac{1}{2}Y_{n,i}^2 .$$

This completes the proof of the lemma 2.

Proof of theorem 1

Let $\mathscr{A} = \sigma(Y_1,\ldots,Y_n)$. We use the fact that the induced order statistics $Z_{n,1},\ldots,Z_{n,n}$ are conditionally independent given $Y_1,\ldots,Y_n$ with the conditional cdf $G_{n,1},\ldots,G_{n,n}$ respectively (Bhattacharya, 1974) to obtain

$$P(M_{n,k} \le z)$$

$$= P(Z_{n,1} \le Z,\ldots,Z_{n,k} \le z)$$

$$= E(P(Z_{n,1} \le z,\ldots,Z_{n,k} \le z/\mathscr{A}))$$

$$= E\left[\prod_{i=1}^{k} G_{n,i}(z)\right]$$

$$= E\left[\prod_{i=1}^{k} \{G(z) + \delta_{n,i}(Q(z) + R(z,\delta_{n,i}))\}\right]$$

$$= E\left[\prod_{i=1}^{k} (G(z) + V_{n,k,i})\right]$$

$$= G^k(Z) + R_{n,k} ,$$

by lemma 2, where

$$V_{n,k,i} = \delta_{n,i}(Q(z) + R(z,\delta_{n,i}))$$

and

$$R_{n,k} = E\left[\prod_{i=1}^{k} (G(z) + V_{n,k,i})\right] - G^k(z) .$$

Objective is to show that $R_{n,k} \to 0$ as n (and hence k) $\to \infty$.

First, note

$$R_{n,k} \leq E\left[\prod_{i=1}^{k} (G(z) + |V_{n,k,i}|)\right] - G^k(Z)$$

(3.1) and

$$R_{n,k} \geq E\left[\prod_{i=1}^{k} (G(z) - |V_{n,k,i}|)\right] - G^k(Z) .$$

So, it is enough to show

$$E\left[\prod_{i=1}^{k} (G(Z) + |V_{n,k,i}|)\right] - G^k(z) \to 0 \tag{3.2}$$

and

$$E\left[\prod_{i=1}^{k} (G(z) - |V_{n,k,i}|)\right] - G^k(z) \to 0 \tag{3.3}$$

as n (and hence k) $\to \infty$.

We will prove (3.1). Proof of (3.2) is similar. We start by defining

$$W_{n,k} = E\left[\prod_{i=1}^{k} (G(z) + |V_{n,k,i}|)\right] - G^k(z) .$$

Furthermore, define

$$\begin{aligned} W^*_{n,k} &= W_{n,k} \quad \text{if } Y_{n,\varphi(n)b} \le n^{-1}\varphi(n)B \\ &= 0 \qquad\;\; \text{if } Y_{n,\varphi(n)b} > n^{-1}\varphi(n)B \ . \end{aligned}$$

So,

(3.4) $$0 \le E(W^*_{n,k}) = E[W_{n,k}\, 1(Y_{n,\varphi(n)b} \le n^{-1}\varphi(n)B)]$$

However, on the set $S_n = \{Y_{n,\varphi(n)b} \le n^{-1}\varphi(n)B\}$

$$\begin{aligned} 0 \le W_{n,k} &\le \prod_{i=1}^{k} \{G(z) + \tfrac{1}{2} Y^2_{n,i}(|Q(z)| + |R(z,\delta_{n,i})|)\} - G^k(z) \\ &\le \prod_{i=1}^{k} \{G(z) + \tfrac{1}{2} n^{-2}\varphi^2(n)B^2(|Q(z)| + \sup_{0 \le \delta \le n^{-2}\varphi^2(n)B^2} |R(z,\delta)|)\} \\ &\quad - G^k(Z) \end{aligned}$$

But $Q(z)$ is a function of $f'(x_0)$, $G_{xx}(z|x_0)$, $G_x(z|x_0)$ which are bounded in the neighborhood of Z_F. Now, using the fact that $\lim_{\delta \downarrow 0} R(z,\delta) = 0$ for z lying in a neighborhood of Z_F; we have for $n > N_1$

$$\sup_{0 \le \delta \le n^{-2}\varphi^2(n)B^2} |R(z,\delta)| \le |Q(z)| \ .$$

So, on the set S_n and for $n > N_1$

$$0 \le W_{n,k} \le \{G(z) + n^{-2}\varphi^2(n)B^2\, |Q(z)|\}^k - G^k(z)$$

(3.5)
$$= \sum_{\ell=1}^{k} \binom{k}{\ell} [n^{-2}\, \varphi^2(n)B^2 |Q(Z)|]^{\ell}\, G^{k-\ell}(z)$$

which is $0(n^{-2}\varphi^3(n))$. Thus for any arbitrary constant $M > 0$, there exists N_2 such that for $n > N_2$

$$0 \le W_{n,k} \le M,$$

since $\varphi(n)$ is chosen to be n^{α} where $\frac{1}{2} \le \alpha < \frac{2}{3}$.

This implies that for $n \ge \max(N_1,N_2) = N$

$$W^*_{n,k} \le M.$$

Now, substituting (3.5) in (3.4), we have

$$0 \le E(W^*_{n,k}) \le \sum_{\ell=1}^{k} \binom{k}{\ell} \{n^{-2}\varphi^2(n)B^2\, |Q(z)|\}^{\ell}\, G^{k-\ell}(z)\,.$$

$$= 0(n^{-2}\, \varphi^3(n))$$

So; $n \to \infty$

(3.6)
$$E(W^*_{n,k}) \to 0.$$

Finally we will show that (3.6) implies

$$E(W_{n,k}) \to 0.$$

Choose $\epsilon_n = n^{-1}\varphi(n)B$, then

$$E|W_{n,k} - W^*_{n,k}| \le \epsilon_n\, P(0 \le |W_{n,k}-W^*_{n,k}| \le \epsilon_n)$$

$$+ \sum_{\ell=1}^{\infty} (\epsilon_n+\ell)\, P(\epsilon_n+\ell-1 \le |W_{n,k}-W^*_{n,k}| \le \epsilon_n+\ell)$$

$$\leq \epsilon_n P(0 \leq |W_{n,k} - W^*_{n,k}| \leq \epsilon_n)$$

$$+ \sum_{\ell=1}^{\infty} (\epsilon_n + \ell)[\exp\{-2\, n^{-1} \varphi^2(n)(B\, f(x_0) - b)^2$$

$$-2\, n(\ell-1)^2 (f(x_0))^2 - 4\varphi(n)(\ell-1) f(x_0)\, (B\, f(x_0) - b\})]$$

$$\leq \epsilon_n P(0 \leq |W_{n,k} - W^*_{n,k}| \leq \epsilon_n) + \sum_{\ell=1}^{\infty} f_n(\ell) \frac{1}{(\epsilon_n + \ell)^2},$$

Using lemma 1 (with $c = \ell - 1$) and where

$$f_n(\ell) = (\epsilon_n + \ell)^3 \exp[-2\, \epsilon_n\, \varphi(n) B^{-1} (B\, f(x_0) - b)^2$$

$$-2\, n(\ell-1)^2 (f(x_0))^2 - 4\varphi(n)(\ell-1)\, f(x_0)(B\, f(x_0) - b)]$$

So, except possibly for finitely many n, $f_n(\ell)$ is uniformly bounded for each ℓ.
Now, taking limit on both sides and noting that $\epsilon_n \to 0$ as $n \to \infty$, we have

$$\lim_n E|W_{n,k} - W^*_{n,k}| = 0,$$

showing

$$E(W_{n,k}) \to 0$$

So, by (3.1), (3.2) and (3.3), we have

$$R_{n,k} \to 0.$$

This completes the proof of Theorem 1.

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