

Dynamic model of a simple supported RC rectangular plate for spreadsheet application – Part II: Motion in elasto-plastic domain

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1 ABSTRACT

This paper is intended to provide civil engineers with a simple modeling tool to be run on current spreadsheet code. Such a tool allows performing the resolution of the dynamic motion of an elasto-plastic simply supported thin rectangular plate, made of reinforced concrete, when submitted to impact loading.

The rupture by flexion is characterized by a criterion of Johansen's type with correction factors to avoid singularities and for keeping law consistency. With the limiting bending moments in x and y directions for face I and face II, \bar{M}_x^I , \bar{M}_y^I , \bar{M}_x^{II} and \bar{M}_y^{II} , the criterion is expressed by:

$$\left\{ \begin{array}{l} f^I(\vec{M}) = \frac{M_{xy}^2}{k_2^2} + k_1^2 - (\bar{M}_x^I - M_x) \cdot (\bar{M}_y^I - M_y) = 0 \\ \text{or} \\ f^{II}(\vec{M}) = \frac{M_{xy}^2}{k_2^2} + k_1^2 - (\bar{M}_x^{II} - M_x) \cdot (\bar{M}_y^{II} - M_y) = 0 \end{array} \right. \quad \text{with} \quad \left\{ \begin{array}{l} 0 < k_2^2 = \frac{(\bar{M}_x^I - \bar{M}_x^{II} + \bar{M}_y^I - \bar{M}_y^{II})^2}{4 \cdot (\bar{M}_x^I - \bar{M}_x^{II}) \cdot (\bar{M}_y^I - \bar{M}_y^{II})} \\ \text{and} \\ 0 \leq k_1 \leq \sqrt{(\bar{M}_x^I - \bar{M}_x^{II}) \cdot (\bar{M}_y^I - \bar{M}_y^{II})} \end{array} \right.$$

When $f^I(\vec{M})$ or $f^{II}(\vec{M}) > 0$, a correction $\Delta\vec{M}$ is to be applied to the bending moment vector $\vec{M} = {}^t(M_x, M_y, M_{xy})$ whose value shall be considered as the trial value. The plastic curvature tensor increment is deduced from the consumption of the incremental elastic energy into energy dissipation due to plastic deformation, the plastic displacement is computed by double integration of the curvatures.

The motion of the plate is based on:

$$\ddot{n} \cdot h \cdot \left(\frac{\partial^2 w}{\partial t^2} + 2 \cdot \dot{h} \cdot \dot{w} \right) + \frac{\partial^2 M_x}{\partial x^2} + 2 \cdot \frac{\partial^2 M_{xy}}{\partial x \cdot \partial y} + \frac{\partial^2 M_y}{\partial y^2} - Q = 0$$

The computation is performed by finite differences on displacements decomposed on the modes space. The plate discretization into 30x30 elements gives sufficient accuracy for pre-sizing RC plates against impact loading and for checking results from more sophisticated non linear code.

2 INTRODUCTION

This paper deals with the modeling of the dynamic behavior of a rectangular reinforced concrete slab, which is simply supported and submitted to a variable loading. It complements the part I [Rambach, 2009] paper that was restricted to the elastic domain: in this paper an elasto-plastic law is introduced. The proposed yield limit for the elastic domain is of Johansen's type with a little adaptation in order to be locally and globally consistent and in order to get a yielding surface without singularities, as "smooth" as possible. The classical method of plastic correction by cutting plane is presented and adapted to the spreadsheet code possibilities, for static case then for dynamic case. An application is proposed in order to demonstrate the efficiency of the method.

3 THEORETICAL BACKGROUND

3.1 Fundamental equation of motion of a thin plate

The fundamental equation of motion of a thin plate is expressed by:

$$\tilde{n}.h.\left(\frac{\partial^2 w}{\partial t^2} + 2.\hat{i}.\dot{u}.\frac{\partial w}{\partial t}\right) + \frac{\partial^2 M_x}{\partial x^2} + 2.\frac{\partial^2 M_{xy}}{\partial x.\partial y} + \frac{\partial^2 M_y}{\partial y^2} - Q = 0 \quad (1)$$

with usual notation:

ρ : mass to volume ratio of the constitutive material of the plate,

ξ : critical damping ratio,

ω : 1st mode pulsation

h : plate thickness supposed constant

M_x, M_y and M_{xy} : components of the bending moment tensor

$Q=Q(x,y,t)$: loading distribution

The expression (1) can be considered as the balance of the loading Q by the sum of the inertial force density (part involving $\frac{\partial^2 w}{\partial t^2}$), the viscosity force density (part involving $\frac{\partial w}{\partial t}$), and the resisting force density

$$\text{exerted by the structure } F_s = \frac{\partial^2 M_x}{\partial x^2} + 2.\frac{\partial^2 M_{xy}}{\partial x.\partial y} + \frac{\partial^2 M_y}{\partial y^2} \quad (1')$$

For a simply supported rectangular plate in the elastic domain, a method of resolution of this equation is proposed in [Rambach, 2009], by recurrence and modal decomposition. The propagation of the solution with time is made, in the modal space, according to the following set of 2 equations:

$$\begin{cases} (1 + \hat{i}.\dot{u}.\ddot{A}t + \frac{\ddot{A}t^2}{4.\rho.h} \cdot D_{mn}).V_{mn}^{t+\ddot{A}t} = (1 - \hat{i}.\dot{u}.\ddot{A}t - \frac{\ddot{A}t^2}{4.\rho.h} \cdot D_{mn}).V_{mn}^t - \frac{\Delta t}{\tilde{n}.h} \cdot (F_s^t - \frac{Q_{mn}^{t+\ddot{A}t} + Q_{mn}^t}{2}) \\ (1 + \hat{i}.\dot{u}.\ddot{A}t + \frac{\ddot{A}t^2}{4.\rho.h} \cdot D_{mn}).F_s^{t+\ddot{A}t} = (1 + \hat{i}.\dot{u}.\ddot{A}t - \frac{\ddot{A}t^2}{4.\rho.h} \cdot D_{mn}).F_s^t + \Delta t.D_{mn}.V_{mn}^t + \frac{\ddot{A}t^2}{4.\rho.h} \cdot \frac{Q_{mn}^{t+\ddot{A}t} + Q_{mn}^t}{2} \end{cases} \quad (2)$$

$$\text{with } D_{mn} = D.\left(\left(\frac{m.\partial}{a}\right)^2 + \left(\frac{n.\partial}{b}\right)^2\right)^2$$

It is necessary to come back from the modal space into the physical space, at each time step, in order to get the successive values of the displacement and bending moments.

3.2 Bending moment - Curvature law in elasto-plastic domain

For a beam, the bending moment – curvature law is simply expressed, for an isotropic elastic - perfectly plastic material, by $M=EI\chi$ (χ being the curvature and EI being the rigidity) for $|\chi| \leq Mpl/EI$ and $M=Mpl$ for $|\chi| \geq Mpl/EI$, Mpl being the limit bending moment.

For a plate, the bending moment – curvature law in the elasto-plastic domain is less easily expressed, owing to the fact that it involves relation between bending moment tensor and curvature tensor.

The bending moment – curvature law between the tensors components in elastic domain is as follows:

$$M_x = D.\left(\frac{\partial^2 w}{\partial x^2} + \hat{i}.\frac{\partial^2 w}{\partial y^2}\right), \quad M_y = D.\left(\frac{\partial^2 w}{\partial y^2} + \hat{i}.\frac{\partial^2 w}{\partial x^2}\right) \quad \text{and} \quad M_{xy} = D.(1 - \hat{i}).\frac{\partial^2 w}{\partial x.\partial y}, \quad (3)$$

$$\text{with } D = \frac{E.h^3}{12.(1 - \hat{i}^2)} \quad \text{the plate stiffness (E: Young's modulus and } \nu : \text{Poisson's ratio)}. \quad (4)$$

In the plastic domain, the limitation of the bending moment components is generally expressed by a yield surface, in the bending moment space, on which the point (M_x, M_y, M_{xy}) representative of the stress state will be located. A first attempt has been proposed by [Nahas, 1983] in SMiRT 7 Proceedings.

4 YIELD SURFACE

4.1 Original Johansen's criterion

The reinforcement of a rectangular concrete slab is generally made by rebars parallel to the edges and arranged in meshes parallel to the upper and to the lower face. It is convenient to express the reinforcement

capacity into resisting bending moment: there are 4 resisting moments that can be called \bar{M}_x^I , \bar{M}_y^I , \bar{M}_x^{II} and \bar{M}_y^{II} , the index x (y) corresponds to the direction of the acting rebars, the exponent I or II corresponds to the conventional sign for the bending moment, I (II) put in tension the lower (upper) face.

The criterion is said of Johansen's type when the relation between the bending moments of the slab M_x , M_y and M_{xy} and the resisting moments at plasticity is:

$$\left\{ \begin{array}{l} f^I(\vec{M}) = M_{xy}^2 - (\bar{M}_x^I - M_x)(\bar{M}_y^I - M_y) = 0 \\ \text{or} \\ f^{II}(\vec{M}) = M_{xy}^2 - (\bar{M}_x^{II} - M_x)(\bar{M}_y^{II} - M_y) = 0 \end{array} \right. \quad (5a)$$

$$\left\{ \begin{array}{l} f^I(\vec{M}) = M_{xy}^2 - (\bar{M}_x^I - M_x)(\bar{M}_y^I - M_y) = 0 \\ \text{or} \\ f^{II}(\vec{M}) = M_{xy}^2 - (\bar{M}_x^{II} - M_x)(\bar{M}_y^{II} - M_y) = 0 \end{array} \right. \quad (5b)$$

With $\vec{M} = {}^t(M_x, M_y, M_{xy})$

In the (M_x, M_y, M_{xy}) space, the Johansen criterion is represented by two coaxial and opposed cones, see Figure 1 below representing the surface for $M_{xy} \geq 0$.

The yield surface envelops the elastic domain: when the representative point (M_x, M_y, M_{xy}) of the stress state is strictly inside the yield surface (and not located on the surface), the representative point is said in the elastic domain, when the point is on the yield surface, the point is said in the plastic domain. The yield surface is supposed constant (there is no hardening nor softening: the plasticity is supposed perfect).

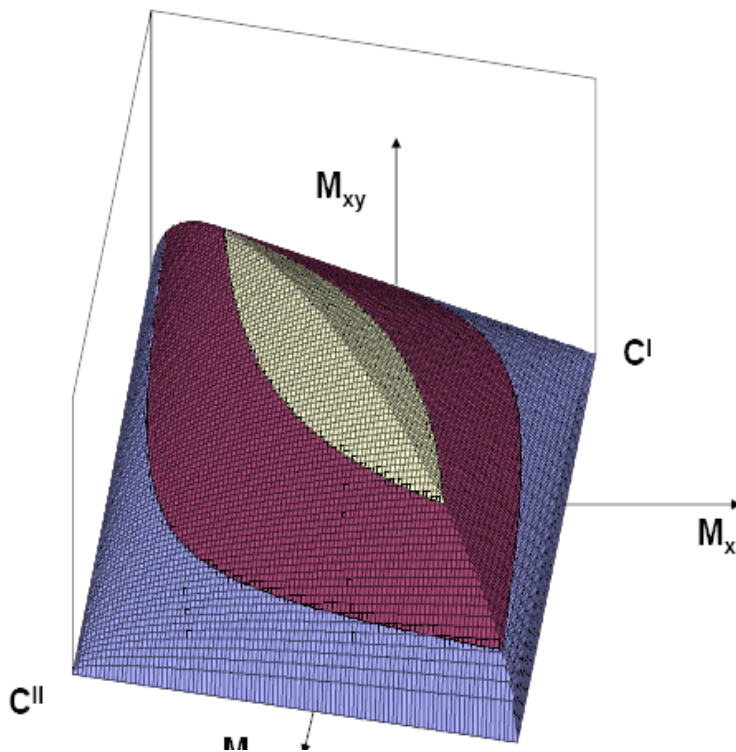


Figure 1 – Yield surface according to Johansen's criterion in bending moment space (for $M_{xy} > 0$)

The projection of (M_x, M_y, M_{xy}) on the (M_x, M_y) plane shall be located inside a rectangle whose corners are defined by the limiting bending moments: \bar{M}_x^I , \bar{M}_y^I , \bar{M}_x^{II} and \bar{M}_y^{II} i.e. $(\bar{M}_x^I, \bar{M}_y^I)$, $(\bar{M}_y^I, \bar{M}_x^I)$, $(\bar{M}_x^{II}, \bar{M}_y^{II})$ and $(\bar{M}_y^{II}, \bar{M}_x^{II})$, see Figure 2.

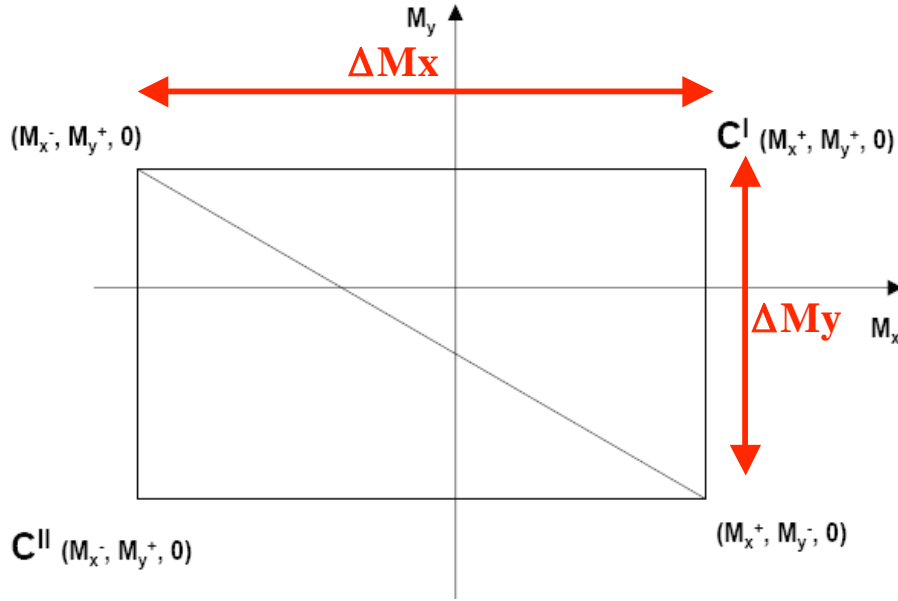


Figure 2 – Projection of the yield surface on the (M_x, M_y) plane, the projection of the bending moment triplet is located inside the rectangle defined by the cone summits C^I and C^{II} .

A criterion of this type has been used by Koechlin in his PhD thesis [Koechlin, 2007], by taking into account, in addition, the membrane force.

4.2 Modified Johansen's criterion

Two modifications are introduced with the two following parameters: the first one, k_1 , for regularization of the surface in the vicinity of the cone summits and the second one, k_2 , for the consistency of the criterion in its global and local expression.

Regularization

The Johansen's yield surface has 2 punctual singularities (the summits C^I and C^{II}) and a linear singularity (the intersection of the 2 cones). It is convenient to regularize the yield surface near the summits, in order to get a continuous normal in the vicinity of the summit. The coefficient k_1 is then introduced and its physical meaning is discussed below.

Local and global consistency of the criterion

The Johansen's criterion (5a and 5b) may be considered as expressed under its global form. In Mohr's representation, the bending moments M_x or M_y are along the abscissa axis and the twisting moment M_{xy} along the ordinate axis, and a triplet (M_x, M_y, M_{xy}) defines a circle whose centre is on the abscissa axis at

the position $\frac{(M_x + M_y)}{2}$ and whose radius is equal to $R = \sqrt{M_{xy}^2 + \frac{(M_x - M_y)^2}{4}}$.

The maximum principal value of the bending moment, M_1 is equal to $\frac{M_x + M_y}{2} + \sqrt{M_{xy}^2 + \frac{(M_x - M_y)^2}{4}}$.

In this representation, the resisting bending moment \overline{M}_ϑ^I , for a section whose normal is oriented by the angle ϑ with respect to x direction, is given by:

$$\overline{M}_\vartheta^I = \overline{M}_x^I \cdot \cos^2 \vartheta + \overline{M}_y^I \cdot \sin^2 \vartheta$$

The plastic state is reached for the section oriented by ϑ (defined by the triplet $(M_x, M_y$ and $M_{xy})$) and the maximum principal value M_1 is then equal to the resisting moment \overline{M}_ϑ^I .

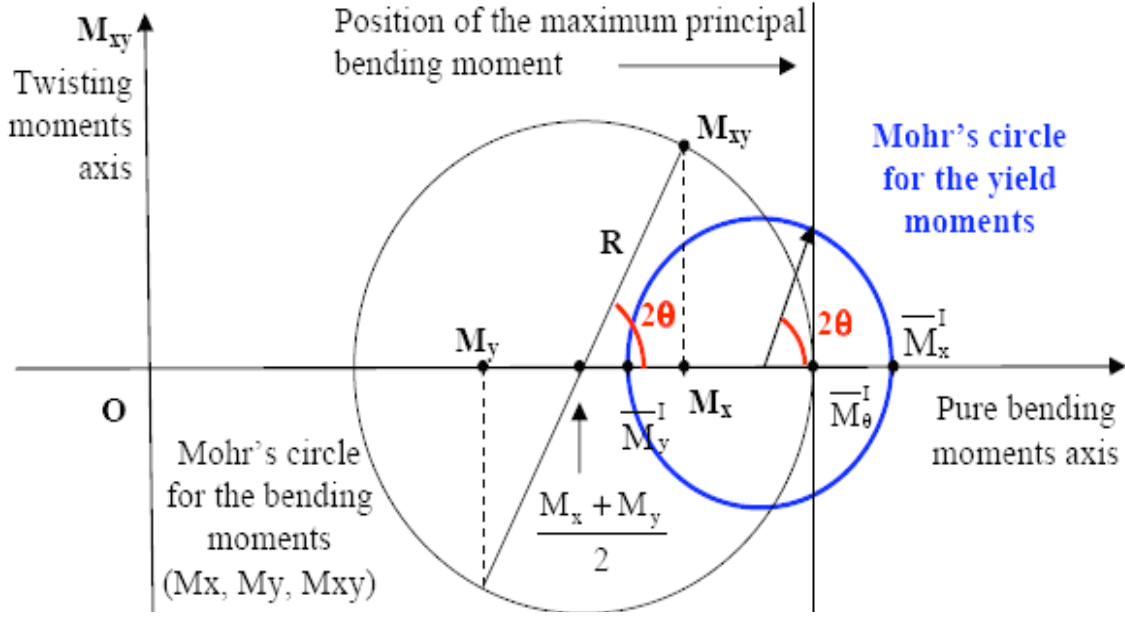


Figure 3 – Mohr's circle representation of Johansen's criterion

When the reinforcement is the same on each face of the plate, the bending moments \bar{M}_x^I and \bar{M}_x^{II} (resp. \bar{M}_y^I and \bar{M}_y^{II}) are opposed: at the centre of the rectangle, $M_x=0$ and $M_y=0$ and the value of $(M_{xy})^2$, according to Johansen's criterion, shall be $(\bar{M}_x^I) \cdot (\bar{M}_y^I) = (\bar{M}_x^{II}) \cdot (\bar{M}_y^{II})$. The plasticity is reached by the twisting moment M_{xy} at its extreme value, and the orientation of the rupture line is at 45° with respect to the principal directions x, y : it means that $(M_{xy})^2$ shall be equal to $[(\bar{M}_x^I) + (\bar{M}_y^I)]^2/4$ or $[(\bar{M}_x^{II}) + (\bar{M}_y^{II})]^2/4$. This is the reason why the M_{xy} value has to be corrected by dividing it by the coefficient $k_2 > 0$ in order to ensure the consistency of the Johansen's criterion expressed in its global form and in its local form.

The modified expression of the criterion after regularization and consistency is then as follows:

$$\left\{ \begin{array}{l} f^I(\bar{M}) = \frac{M_{xy}^2}{k_2^2} + k_1^2 - (\bar{M}_x^I - M_x) \cdot (\bar{M}_y^I - M_y) = 0 \\ \text{or} \\ f^{II}(\bar{M}) = \frac{M_{xy}^2}{k_2^2} + k_1^2 - (\bar{M}_x^{II} - M_x) \cdot (\bar{M}_y^{II} - M_y) = 0 \end{array} \right. \quad (6a) \quad (6b)$$

$$\text{With } 0 < k_2^2 = \frac{(\ddot{A}M_x + \ddot{A}M_y)^2}{4 \cdot \ddot{A}M_x \cdot \ddot{A}M_y} \quad (7)$$

$$\ddot{A}M_x = \bar{M}_x^I - \bar{M}_x^{II} \text{ and } \ddot{A}M_y = \bar{M}_y^I - \bar{M}_y^{II} \text{ (see figure 2)}$$

It can be checked that $k_2=1$ when $\bar{M}_x^I = -\bar{M}_x^{II} = \bar{M}_y^I = -\bar{M}_y^{II}$ i.e. when the ratio $\Delta M_x/\Delta M_y = 1$. When the ratio $\Delta M_x/\Delta M_y$ is comprised between 0.5 and 2, the coefficient k_2 is then comprised between 1 and 1.25.

Equations (6a) and (6b) are the equations of the 2-sheeted hyperboloids (H^I) and (H^{II}) having for asymptotic surfaces the preceding cones (C^I) and (C^{II}). The trace of the hyperboloids (H^I) and (H^{II}) on the plane (M_x, M_y) are the hyperbole branches (h^I) and (h^{II}) located inside the rectangle (axb), see Figure 4. The equations of the hyperbole branches are:

$$k_1^2 - (\bar{M}_x^I - M_x) \cdot (\bar{M}_y^I - M_y) = 0 \text{ and } k_1^2 - (\bar{M}_x^{II} - M_x) \cdot (\bar{M}_y^{II} - M_y) = 0$$

Geometrically, k_1^2 is the expression of the area of the rectangle having for corners (C^I) or (C^{II}) and the current point (M_x, M_y) of the hyperbole branch (h^I) or (h^{II}). In other words, k_1 can be considered as the side length of the largest square located between the asymptotes and the hyperbole branch: k_1 is then the maximum closest distance of the hyperboloid to the cone surface, it then characterizes the approximation of the cone by the hyperboloid.

The intersection of the hyperboloids (H^I) and (H^{II}) occurs along an ellipse located in a plane perpendicular to (M_x, M_y) plane in (M_x, M_y, M_{xy}) space, and whose trace on this plane is the diagonal (**D2**) other than that (**D1**) containing the corners C^I and C^{II} of the rectangle (\mathbf{axb}). It can be demonstrated that the major axis of above ellipse is the part of preceding diagonal comprised between the intersections with the hyperbole branches, as shown on Figure 4 and whose length is equal

to: $\sqrt{\Delta M_x^2 + \Delta M_y^2} \cdot \sqrt{1 - \frac{k_1^2}{\Delta M_x \cdot \Delta M_y}}$; the minor axis is equal to $\left(\frac{\Delta M_x + \Delta M_y}{2}\right) \cdot \sqrt{1 - \frac{k_1^2}{\Delta M_x \cdot \Delta M_y}}$. It can be noted that the lower the value of $\frac{k_1^2}{\Delta M_x \cdot \Delta M_y}$ is, the better

the approximation of the cones by the hyperboloids is. A practical value $\frac{k_1^2}{\Delta M_x \cdot \Delta M_y} = 10^{-2}$ is sufficient.

The normals to the hyperboloids are discontinuous when crossing the above ellipse; nevertheless, it can be admitted that the normal for a current point from this ellipse is carried by the intersection of the plane of ellipse with the plane pivoting around diagonal D1 and containing the current point. In other words, the normal to the surface along this ellipse is equal to the half-sum of the normals deduced from each hyperboloid.

The expression of the equation of the yield surface formed by the two intersecting sheets from those two hyperboloids may be simply written in a compact form. The projection of the whole yield surface on the plane (M_x, M_y) is limited by the two hyperbola branches (h^I) and (h^{II}) having for asymptotes the sides of the above-mentioned rectangle and intersecting on diagonal (**D2**). The position of the current projection point (M_x, M_y) with respect to diagonal (**D2**) defines the hyperboloid to which the current stress representative point M on the yield surface belongs.

Let us define φ by $\varphi = \frac{M_x - \bar{M}_x^{II}}{\bar{M}_x^I - \bar{M}_x^{II}} + \frac{M_y - \bar{M}_y^I}{\bar{M}_y^I - \bar{M}_y^{II}}$: the sign of φ characterizes the location of the

projection of the representative point M(M_x, M_y, M_{xy}) on the projection of the yield surface on (M_x, M_y) plane. $\varphi > 0$ means $M \in (H1)$, $\varphi < 0$ means $M \in (H2)$ and $\varphi = 0$ means $M \in (H1)$ and $M \in (H2)$, i.e. M belongs to the ellipse (**E**) that seams the hyperboloids (**H1**) and (**H2**) sheets, the projection of (**E**) being diagonal **D2**.

Let us define Ψ : $\Psi=1$ when $\varphi > 0$, $\Psi=-1$ when $\varphi < 0$ and $\Psi=0$ when $\varphi = 0$. With this notation, the expression of the yield surface is:

$$f = \frac{(1 + \Psi)}{2} \cdot f^I + \frac{(1 - \Psi)}{2} \cdot f^{II} = 0 \quad (8a)$$

$$\text{with } f^I(M) = \frac{M_{xy}^2}{k_2^2} + k_1^2 - (\bar{M}_x^I - M_x) \cdot (\bar{M}_y^I - M_y) , \quad (8b)$$

$$f^{II}(M) = \frac{M_{xy}^2}{k_2^2} + k_1^2 - (\bar{M}_x^{II} - M_x) \cdot (\bar{M}_y^{II} - M_y) \quad (8c)$$

$\emptyset = +1$ when $\varphi > 0$, $\emptyset = -1$ when $\varphi < 0$, $\emptyset = 0$ when $\varphi = 0$

$$\text{with } \varphi = \frac{M_x - \bar{M}_x^{II}}{\bar{M}_x^I - \bar{M}_x^{II}} + \frac{M_y - \bar{M}_y^I}{\bar{M}_y^I - \bar{M}_y^{II}} , \quad \bar{M}_x^{II} \leq M_x \leq \bar{M}_x^I \text{ and } \bar{M}_y^{II} \leq M_y \leq \bar{M}_y^I$$

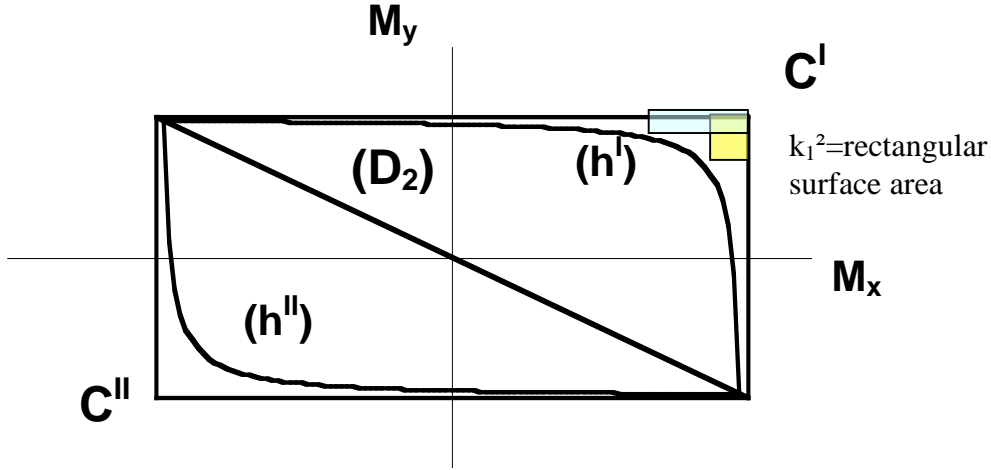


Figure 4 – Intersection of the hyperboloid sheets with (M_x, M_y) plane: hyperbole branches (h^I) and (h^{II})

5 INCURSION IN THE PLASTIC DOMAIN

5.1 Bending moment tensor in plastic domain

When $f(\vec{M}) < 0$ the representative bending moment vector \vec{M} issued from the origin of the bending moments space is in the elastic domain and when $f(\vec{M}) = 0$, the extremity of the representative vector \vec{M} is on the yield surface and therefore in the plastic domain; there is no possibility for a representative vector \vec{M} to cross the yield surface, i.e. such as $f(\vec{M}) > 0$.

Let us define the elastic increment vector $\ddot{\Delta}\vec{M}^e = {}^t [\ddot{\Delta}M_x^e \ \ddot{\Delta}M_x^e \ \ddot{\Delta}M_{xy}^e]$ of the representative bending moment vector \vec{M}_t (t : at the time t) during the time increment Δt . Let us define the trial value \vec{M}^{trial} : $\vec{M}^{\text{trial}} = \vec{M}_t + \ddot{\Delta}\vec{M}^e$. If the trial value $\vec{M}^{\text{trial}} = \vec{M}_t + \ddot{\Delta}\vec{M}^e$ is such that $f(\vec{M}^{\text{trial}}) > 0$ (i.e. \vec{M}^{trial} outside the yield surface), there is a coefficient $0 \leq \mu < 1$ such that $f(\vec{M} + \mu \cdot \ddot{\Delta}\vec{M}^e) = 0$ ($\mu = 0$ if M is already on the yield surface, $\mu \neq 1$ because $f(\vec{M}^{\text{trial}}) > 0$). Let us define M^* the point where the elastic increment vector $\ddot{\Delta}\vec{M}^e$ crosses the yield surface: the normal to the yield surface at the point M^* is parallel to $\overrightarrow{\text{grad}}(f(\vec{M}^*))$, \vec{M}^* being the representative bending moment vector associated to the point M^* .

5.2 Cutting plane method

The cutting plane method is a way to force to the representative point of the bending moment tensor to rest on the yield surface: a corrective bending moment vector $\Delta\vec{M}_c^p$ must be added to \vec{M}^{trial} in order to locate $\vec{M}_{t+\Delta t} = \vec{M}^{\text{trial}} + \Delta\vec{M}_c^p$ on the yield surface, i.e. $f(\vec{M}^{\text{trial}} + \Delta\vec{M}_c^p) = 0$

The Hill's theorem implies that the vector involving the plastic curvature increment $\Delta\vec{\Phi}^p$ shall be normal to the yield surface of corresponding bending moments, i.e. there is a coefficient $\lambda > 0$ such as $\ddot{\Delta}\vec{O}^p = \ddot{e} \cdot \overrightarrow{\text{grad}}(f(\vec{M}^*))$. Let us assume that the relationship between $\ddot{\Delta}\vec{M}_c^p$ and $\Delta\vec{\Phi}^p$ is of the same type as the one supposed in elastic domain: $\ddot{\Delta}\vec{M}_c^p = -[\mathbf{K}^p] \Delta\vec{\Phi}^p$ ($[\mathbf{K}^p]$ is the tangent incremental elasticity matrix with possibly reduced value, $[\mathbf{K}^p]$ is supposed invertible, the sign $-$ is introduced in order to mean that the “direction” of $\ddot{\Delta}\vec{M}_c^p$ is opposite to that of $\Delta\vec{\Phi}^p$).

Owing that $\ddot{\Delta}\vec{M}^e$ is small with respect to \vec{M}^{trial} , it can be demonstrated that the coefficient λ is given by:

$$\ddot{e} = \frac{(1-\dot{\lambda}) \cdot \overrightarrow{\ddot{A}M^e} \cdot \overrightarrow{\text{grad}}(f(\overrightarrow{M}^*))}{\overrightarrow{\text{grad}}(f(\overrightarrow{M}^*)) \cdot [\mathbf{K}^p] \cdot \overrightarrow{\text{grad}}(f(\overrightarrow{M}^*))} = \frac{f(\overrightarrow{M}^{\text{trial}})}{\overrightarrow{\text{grad}}(f(\overrightarrow{M}^*)) \cdot [\mathbf{K}^p] \cdot \overrightarrow{\text{grad}}(f(\overrightarrow{M}^*))} \quad (9)$$

The symbol \cdot corresponds to the scalar product. The bending moment representative vector $\overrightarrow{M}_{t+\Delta t}$, after correction and at the end of the time increment Δt , is given by:

$$\overrightarrow{M}_{t+\Delta t} = \overrightarrow{M}^{\text{trial}} + \overrightarrow{\ddot{A}M_c^p} = \overrightarrow{M}^{\text{trial}} - \frac{f(\overrightarrow{M}^{\text{trial}}) \cdot [\mathbf{K}^p] \cdot \overrightarrow{\text{grad}}(f(\overrightarrow{M}^*))}{\overrightarrow{\text{grad}}(f(\overrightarrow{M}^*)) \cdot [\mathbf{K}^p] \cdot \overrightarrow{\text{grad}}(f(\overrightarrow{M}^*))} \quad (10)$$

The expression (10) shows that the correction $\overrightarrow{\ddot{A}M_c^p}$ is weakly dependent on the tangent incremental elasticity matrix $[\mathbf{K}^p]$: if $[\mathbf{K}^p]$ is reduced to the unit matrix, the formula indicates that the representative point $\overrightarrow{M}_{t+\Delta t}$ is obtained as the normal projection on the yield surface of the extremity of the representative point $\overrightarrow{M}^{\text{trial}}$ (see Figure 5), $\overrightarrow{\ddot{A}M_c^p}$ is parallel to $\overrightarrow{\text{grad}}(f(\overrightarrow{M}^*))$ and its norm is equal to $f(\overrightarrow{M}^{\text{trial}}) / \left\| \overrightarrow{\text{grad}}(f(\overrightarrow{M}^*)) \right\|$.

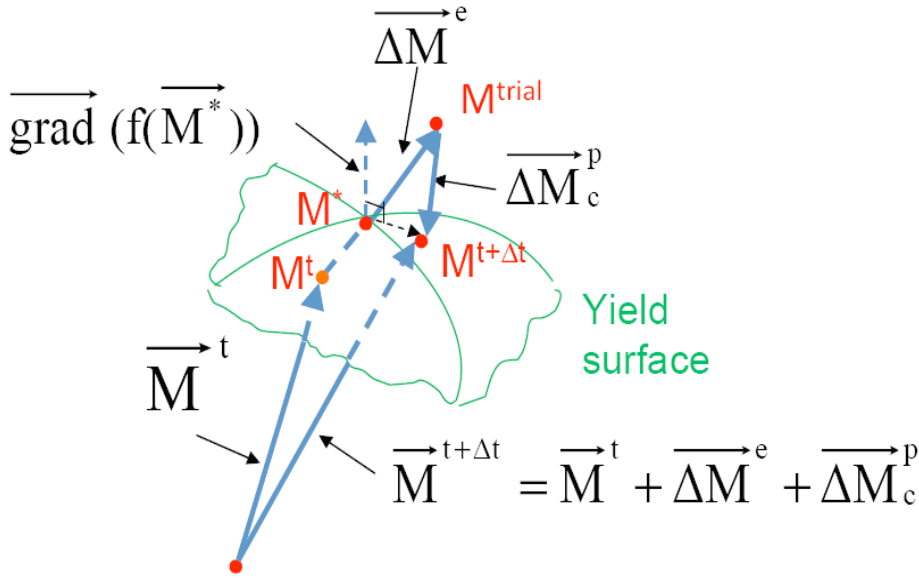


Figure 5 – Scheme showing the correction to the bending moments on the yield surface

5.3 Structural response force Fs

Once the bending moments are corrected, the structural response force F_s is computed by second derivation of the bending moments, according to (1'). The obtained value of F_s is to be decomposed on the modal basis and has to be substituted to the value proposed by equations (2).

5.4 Plastic displacement

The plastic curvature increment $\Delta \overrightarrow{\Phi}^p$ is not deduced from the preceding relation $\overrightarrow{\ddot{A}O}^p = \ddot{e} \cdot \overrightarrow{\text{grad}}(f(\overrightarrow{M}^*))$, but from the hypothesis of plastic deformations concentrated along hinges: the curvature increment reaches its maximum value $\Delta \Phi_1$ in the same principal direction θ where the bending moment reaches its maximum absolute value M_1 (which is also the yield value), whereas in the perpendicular direction, the curvature increment is nil. According to these assumptions, the vector of curvature plastic increment $\overrightarrow{\ddot{A}O}^p = {}^t [\ddot{A}O_x^p \ \ddot{A}O_y^p \ \ddot{A}O_{xy}^p]$ is defined by:

$$\ddot{A}\ddot{O}_x^p = \frac{\ddot{A}\ddot{O}_1}{2} \cdot [\bar{l} + \cos(2 \cdot \dot{\epsilon})] , \quad \ddot{A}\ddot{O}_y^p = \frac{\ddot{A}\ddot{O}_1}{2} \cdot [\bar{l} - \cos(2 \cdot \dot{\epsilon})] \text{ and } \ddot{A}\ddot{O}_{xy}^p = \frac{\ddot{A}\ddot{O}_1}{2} \cdot \sin(2 \cdot \dot{\epsilon}) . \quad (11)$$

If the direction θ cannot be determined, i.e. when $M_2 = M_1$, $\ddot{A}\ddot{O}_x^p = \ddot{A}\ddot{O}_y^p = \frac{\ddot{A}\ddot{O}_1}{2}$ and $\ddot{A}\ddot{O}_{xy}^p = 0$

The energy dissipation density due to the plastic deformation ΔE^p is equal to:

$$\ddot{A}E^p = M_x \cdot \ddot{A}\ddot{O}_x^p + M_y \cdot \ddot{A}\ddot{O}_y^p + 2 \cdot M_{xy} \cdot \ddot{A}\ddot{O}_{xy}^p \quad (12)$$

After some algebraic manipulations, it can be demonstrated that $\ddot{A}E^p = M_1 \cdot \Delta\Phi_1$. The intensity of the curvature increment $\Delta\Phi_1$ is obtained by equating the energy dissipation density (due to plastic deformation) to the difference between the elastic energy density $E^e(\vec{M}^{\text{trial}})$ of the stress state represented by \vec{M}^{trial} and the elastic energy density $E^e(\vec{M}_{t+\dot{A}t})$ of the stress state represented by $\vec{M}_{t+\dot{A}t}$, after correction.

The elastic energy density stored in a thin plate (when neglecting shear distortion) is given by:

$$\left\{ \begin{array}{l} E^e(\vec{M}) = \frac{1}{2 \cdot D \cdot (1 - \nu^2)} \cdot [M_x + M_y]^2 + 2 \cdot (1 + \nu) \cdot (M_{xy}^2 - M_x \cdot M_y) \\ \quad = \frac{1}{2 \cdot D \cdot (1 - \nu^2)} \cdot [M_1^2 + M_2^2 - 2 \cdot \nu \cdot M_1 \cdot M_2] , \\ M_1 \text{ and } M_2 \text{ being the principal bending moments} \end{array} \right. \quad (13)$$

Once the plastic curvature increment $\Delta\vec{\Phi}^p$ is determined, the plastic displacement increment ΔW^p is computed by direct double integration, according to the definition of the curvature and the limit conditions $W=0$ all along the 4 edges of the plate. It is thus possible to determine the corrected value of the velocity:

$$V^{t+\dot{A}t} = V^t + \frac{\dot{\lambda} \cdot \ddot{A}W^e + \ddot{A}W^p}{\ddot{A}t} , \quad (14)$$

ΔW^e being the elastic displacement increment associated to the bending moment increment $\Delta\vec{M}^e$. The value of $V^{t+\dot{A}t}$ is to be decomposed on the modal basis and to be substituted to the value proposed by equations (2). The displacement $W^{t+\dot{A}t}$, in physical space, is thus simply obtained by:

$$W^{t+\dot{A}t} = W^t + \frac{V^{t+\dot{A}t} + V^t}{2} \cdot \dot{A}t . \quad (15)$$

5.5 Practical Algorithm

The following Figure 6 proposes a structure of algorithm allowing the computation of the motion of a thin plate in elastic and beyond elastic domains.

The module called “propagator” is the module that ensures the progression of the motion: the recurrence formula allows the computation of the value, in the modal space, of V and F_s at the step time $t+\dot{A}t$ from the knowledge of their value at the time t and of the applied loading value Q . The values of V and F_s are translated in the physical space by modal summation at each time step, in order to compute the displacement increment and then the bending moment tensor $\ddot{A}\vec{M}^e$ increment.

At each time step, the location of the extremity of the representative vector $\vec{M}^{\text{trial}} = \vec{M}_t + \ddot{A}\vec{M}^e$ is tested with respect to the yield surface: if \vec{M}^{trial} is outside the yield surface, the results are processed by the “plastic module”, if not by the “elastic module”.

The module called “plastic module” is intended to correct the bending moment, to correct the structural force F_s , to compute the plastic curvature increment, the plastic displacement increment and then to correct the velocity.

The “elastic module” is intended to compute the displacements, whereas the velocity and the structural force are unchanged.

A test is done concerning the end of the computed motion.

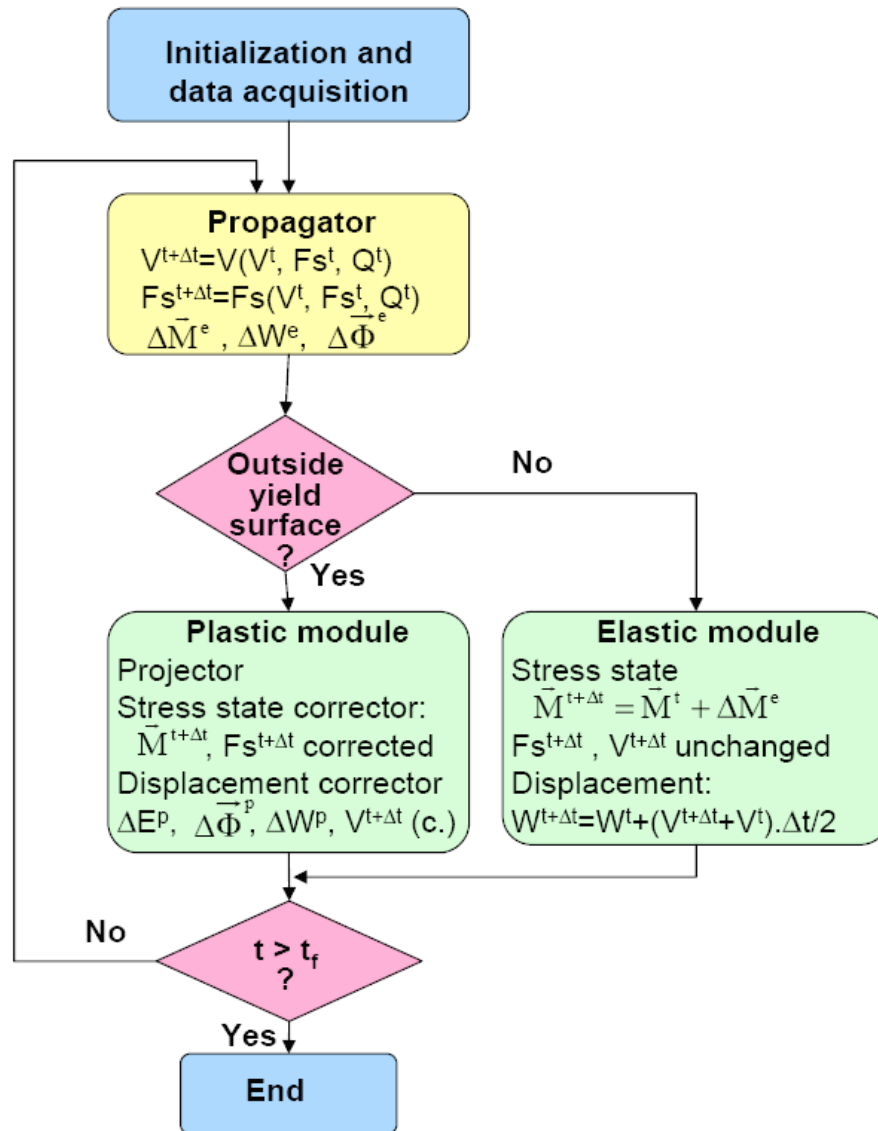


Figure 6 – Diagram showing the algorithm for resolution of the motion of a thin rectangular plate, in elasto-plastic domain, submitted to a variable loading

6 CONCLUSIONS

The motion of a simply supported elasto-plastic RC plate when submitted to a variable loading can be simulated on current spreadsheet software. The dominant rupture mode is by flexion. The accuracy is sufficient to pre-size a RC rectangular plate against impact loading and to check an order of magnitude of the maximum and of the permanent deflections coming from results of more sophisticated non linear computation code.

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