ABSTRACT

AL-ASHHAB, SAMER SHAFIQ. The role of sh-Lie algebras in Lagrangian field theory. (Under the direction of Ron Fulp.)

The purpose of this dissertation is to study strongly homotopy Lie algebras (sh-Lie algebras) and their applications with primary emphasis on applications to field theory. Strongly homotopy Lie algebras are defined on graded vector spaces. They generally consist of an infinite sequence of mappings $l_1, l_2, l_3, \cdots$, which satisfy certain identities. We show that, in the presence of appropriate hypotheses, there always exists a simplified sh-Lie algebra structure with $l_n = 0$ for $n > 3$. This is a special case which has occured in several applications. While it is known that sh-Lie algebras arise in field theory as a homological resolution of a Poisson bracket defined on the space of local functionals, we show how these sh-Lie algebras transform in the event of canonical transformations on the space of local functionals. Additionally, it is shown how a group which acts via canonical transformations transforms the sh-Lie structure and eventually leads to reduction theorems. Two kinds of reduction are obtained corresponding to two different kinds of group action and, in each case it is shown how to obtain an induced sh-Lie algebra on a corresponding reduced graded vector space. Several applications of the theory are considered as well.

THE ROLE OF SH-LIE ALGEBRAS IN LAGRANGIAN FIELD THEORY

by

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Chair of Advisory Committee

To my parents.

Biography

The author had his undergraduate study at the University of Jordan, Amman, Jordan. He has joined the mathematics department at North Carolina State University as a graduate student in 1998. He has earned a Master's of Science in applied mathematics in August 2000.

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Contents

Chapter 1

Introduction

Strongly homotopy Lie algebras (sh-Lie algebras/structures) have recently been the focus of study in mathematics [6, 15, 16]. Their applications have appeared in mathematics [21], in mathematical physics [6, 12], and in physics [24]. In many applications, a simplified sh-Lie algebra structure has appeared where the only nonzero mappings were the lower order mappings on lower degrees, i.e., the lower order mappings were trivial on higher degrees. In Chapter 2 we show that, from the algebraic standpoint, one can always choose such a simplified structure in the presence of suitable hypotheses. Also included in Chapter 2: a presentation of sh-Lie algebras, their basic definition, and the main results relating to them that will be needed in this dissertation.

In Chapter 3 we focus on the application of sh-Lie structures to Lagrangian field theory. We extend the study by finding how these structures relate to canonical transformations of local functionals of the theory. One can then consider ideas of reduction which arise when a Lie group acts via canonical transformations on the space of local functionals. We leave reduction for Chapters 4 and 5.

1.1 Strongly homotopy Lie algebras and Lagrangian field theory

Recall that the dynamical "equations of motion" of a Lagrangian field theory are usually derived from a variational principle of "least action". Given a Lagrangian L, the action of L is the functional S defined by

$$S(\phi) = \int_M L((j^n\phi)(x))Vol_M$$

where M is a manifold, ϕ may be either a vector-valued function or a section of a vector bundle E over M, $j^n\phi$ is the prolongation of ϕ to J^nE, L is a real-valued function on some finite jet bundle J^nE, and Vol_M is a volume on M. More generally, if $\pi : E \to M$ is a vector bundle and $\pi^\infty : J^\infty E \to M$ is the corresponding prolongation of E, then a smooth function $P : J^\infty E \to \mathbf{R}$ is called a *local function* on E provided that for some positive integer n there is a smooth function $P_n : J^nE \to \mathbf{R}$ such that $P = P_n \circ \pi_n$ where π_n is the projection of $J^\infty E$ onto J^nE. Thus all Lagrangians are local functions on an appropriate bundle. To say that $\mathcal{P}$ is a *local functional* on E means that $\mathcal{P}$ is a mapping from a subspace of compactly supported sections of $E \to M$ into $\mathbf{R}$ such that

$$\mathcal{P}(\phi) = \int_M (P \circ j^\infty\phi)(x)Vol_M$$

for some local function P and for all such sections ϕ of E.

In imitation of Hamiltonian mechanics one postulates the existence of a "Poisson bracket" on the space $\mathcal{F}$ of local functionals and then uses it to develop a Hamiltonian theory of fields. This bracket is assumed to satisfy the Jacobi identity and so defines a Lie algebra structure on $\mathcal{F}$. On the other hand there is no obvious commutative multiplication of such functionals and consequently $\mathcal{F}$ is not a Poisson algebra. This is such a well-known development that we may refer to standard monographs on the subject. In particular we call attention to [11] and [20] for classical expositions, and to [13] for a quantum field theoretic development.

It was shown in [6] that a Poisson bracket on the space of local functionals induces an sh-Lie structure on a part of the variational bicomplex which we refer to as the "de Rham complex" on $J^\infty E$. This sh-Lie structure is given by three mappings l_1, l_2, l_3 defined on this complex. The mapping l_2 is skew-symmetric and linear, and it may be regarded as defining a "bracket" but one which generally fails to satisfy the Jacobi identity. In fact, l_2 satisfies the Jacobi identity if $l_3 = 0$.

Recall that, [19], if M is a Poisson manifold one says that a mapping from M to itself is a canonical transformation if, and only if, it preserves the Poisson bracket defined on $C^\infty(M)$. The space of local functionals is not a Poisson algebra and so there is no underlying Poisson manifold. The bracket l_2 is defined on the space of "top" forms of the "de Rham complex" which can be identified with the space of local functions on $J^\infty E$. This space is a commutative algebra under pointwise multiplication, but l_2 does not satisfy the Jacobi identity and so again one does not have a Poisson manifold.

The questions dealt with in this dissertation *relate more to the mathematical structures induced by Poisson brackets on the space of local functionals rather than to specific methods of solving dynamical field equations.* Moreover, we have restricted our attention to a class of theories in which the Poisson bracket is induced by a tensor ω which is scalar-valued rather than differential-operator valued. Once we understand this restricted case more fully we hope to extend these results to a larger class of theories for which ω is differential-operator valued.

Eventually, we also intend to expand our scope to include fermionic theories such as those in [13]. Indeed, the sh-Lie formalism is particularly well-suited to interact with super-field theories such as those needed to describe the Batalin-Vilkovisky approach to BRST cohomology. Once anti-fields are introduced, our vector bundle E can be modified in such a manner that both the bosonic fields studied here and the fermionic (anti-)fields become sections of the new bundle. In this context the Batalin-Vilkovisky anti-bracket is none other than our Poisson bracket of local functionals with appropriate grading. Thus we expect the modifications of this work to the latter case to be minimal. In fact, this work is motivated by both the classical field theories such as those described in [20] and the super-fields developed in [13]. This approach has proven its worth in investigations such as those found in [7] and [8].

Chapter 2

Strongly homotopy Lie algebras

In section 2.1 we present the general structure of sh-Lie algebras, then in section 2.2 we show that when one is given a mapping $\tilde{l}_2$, with appropriate properties, one can choose a simplified structure. Three examples are presented in section 2.3, two of which have appeared in two different applications. Finally, in section 2.4 we show how one may obtain an sh-Lie structure on a chain complex from another sh-Lie structure defined on another chain complex when there is a chain map between them. We will deal with the subject of sh-Lie structures on reduced complexes and their application to Lagrangian field theory in later chapters.

2.1 The general structure

Let X_* be a graded vector space with a differential (lowering the degree by 1) l_1, and where $X_n = 0$ for $n < 0$. Let H_* be the homology complex of the complex X_*. We assume that $H_n = 0$ for $n > 0$, while H_0 is generally non-trivial. We also assume the existence of maps $\eta : X_n \to H_n, \lambda : H_n \to X_n$, and $s : X_n \to X_{n+1}$ such that

$$\lambda \circ \eta - 1_{X_*} = l_1 \circ s + s \circ l_1. \tag{2.1}$$

Notice that (2.1) simplifies to $\lambda \circ \eta - 1_{X_*} = l_1 \circ s$ in degree 0, and $-1_{X_*} = l_1 \circ s + s \circ l_1$ in degree > 0. In this situation, $\eta : X_* \to H_*$ is a chain map with homotopy inverse $\lambda : H_* \to X_*$; i.e., $\eta \circ \lambda = 1_{H_*}$ and $\lambda \circ \eta \sim 1_{X_*}$ via the chain homotopy $s : X_* \to X_*$ which satisfies equation (2.1). The following is the formal definition of sh-Lie structures (see [6]):

Definition An sh-Lie structure on a graded vector space X_* is a collection of linear, skew-symmetric maps $l_k : \bigotimes^k X_* \to X_*$ of degree $k-2$ that satisfy the relation

$$\sum_{i+j=n+1} \sum_{unsh(i,n-i)} e(\sigma)(-1)^{\sigma}(-1)^{i(j-1)} l_j(l_i(x_{\sigma(1)}, \cdots, x_{\sigma(i)}), \cdots, x_{\sigma(n)}) = 0,$$

where $1 \leq i, j$.

Notice that in this definition $e(\sigma)$ is the Koszul sign which depends on the permutation σ as well as on the degree of the elements $x_1, x_2, \cdots, x_n$ (a minus sign is introduced whenever two consecutive odd elements are permuted, see [15]). Also observe that it is convenient in this context to suppress some of the notation and assume the summands are over the appropriate unshuffles with their corresponding signs (e.g. if $n = 3$ one writes $l_1 l_3 + l_2 l_2 + l_3 l_1 = 0$).

We assume the existence of a linear skew-symmetric map $\tilde{l}_2 : X_0 \otimes X_0 \to X_0$ satisfying conditions (i) and (ii) below so that an sh-Lie structure exists. The following theorem was proved in [6]:

Theorem 2.1 *A skew-symmetric linear map* $\tilde{l}_2 : X_0 \otimes X_0 \to X_0$ *that satisfies conditions (i) and (ii) below extends to an sh-Lie structure on the graded space* X_**;*

(*i*) $\tilde{l}_2(c, b_1) = b_2$

(*ii*) $\sum_{\sigma \in unsh(2,1)} (-1)^\sigma \tilde{l}_2(\tilde{l}_2(c_{\sigma(1)}, c_{\sigma(2)}), c_{\sigma(3)}) = b_3$

where c, c_1, c_2, c_3 *are cycles and* b_1, b_2, b_3 *are boundaries in* X_0*.*

Corollary 2.2 *The existence of a skew-symmetric linear map* $\tilde{l}_2 : X_0 \otimes X_0 \to X_0$ *that satisfies conditions (i) and (ii) as in Theorem* 2.1 *is equivalent to the existence of an sh-Lie structure on the graded space* X_**.*

Proof In the above definition of sh-Lie structures, let $n = 2$ and $x_1 \in X_0, x_2 \in X_1$. It is easy to see then that $l_2 : X_0 \otimes X_0 \to X_0$ satisfies condition (i). Now, in the same definition, let $n = 3$ and $x_1, x_2, x_3 \in X_0$. This will show that $l_2 : X_0 \otimes X_0 \to X_0$ satisfies condition (ii). This establishes the required $\tilde{l}_2$ from a given sh-Lie structure. ∎

In fact the existence of a map $\tilde{l}_2$ satisfying the hypotheses of Theorem 2.1 is equivalent to the existence of a Lie bracket on H_0. We quote the following from [6]:

Lemma 2.3 *The existence of a skew-symmetric linear map* $\tilde{l}_2 : X_0 \otimes X_0 \to X_0$ *that satisfies condition* (i) *is equivalent to the existence of a skew-symmetric linear map* $[\cdot,\cdot] : H_0 \otimes H_0 \to H_0$*.*

In [6] a bracket $[\cdot,\cdot]$ which is induced by $\tilde{l}_2$ satisfies Lemma 2.3. There, it was shown that:

Lemma 2.4 *Assume that $\tilde{l}_2 : X_0 \otimes X_0 \to X_0$ satisfies condition (i). Then the induced bracket on H_0 is a Lie bracket if and only if $\tilde{l}_2$ satisfies condition (ii).*

Remark Although the way the two lemmas above are formulated does not completely show the aforementioned equivalence, the existence of a Lie bracket on H_0 induces a map $\tilde{l}_2$ which satisfies conditions (i) and (ii) of Theorem 2.1 in the strong sense of having 0 on the right-hand side (for both conditions). However, this is a case we are not interested in. The two preceding lemmas will be needed in later chapters.

2.2 The simplified structure

While Theorem 2.1 guarantees the existence of an sh-Lie structure on the graded vector space X_*, we *show* that, under the same conditions, one can always choose an sh-Lie structure such that

1. $l_2 = 0$ in degree > 1.
2. $l_3 = 0$ in degree > 0.
3. $l_n \equiv 0, n > 3$.

Remark Markl has observed (see [6]) that such an sh-Lie structure exists in the case that $l_2(c, b) = 0$ for each cycle c and boundary b. This observation of Markl was proved by Barnich in [5]. However the case we are considering here is more general.

We find it convenient in what follows to refer to the image of a combination of maps by the combination itself: for example $l_2 l_1$ would stand for the image of the map l_2l_1 acting on some element in the appropriate space as the context implies (i.e. l_2l_1 stands for $l_2l_1(x_p \otimes x_q) = l_2(l_1x_p \otimes x_q + (-1)^p x_p \otimes l_1x_q) = l_2(l_1x_p \otimes x_q) + (-1)^p l_2(x_p \otimes l_1x_q)$ for $x_p \in X_p$ and $x_q \in X_q$). l_2l_2 would stand for the three unshuffles of the composite l_2l_2, again acting on an element in the approprite space (where we skip writing the element it's acting on, e.g., $l_2l_2 = l_2l_2(x_0 \otimes x_0' \otimes x_0'') = l_2(l_2(x_0, x_0'), x_0'') - l_2(l_2(x_0, x_0''), x_0') + l_2(l_2(x_0', x_0''), x_0)$ for $x_0 \otimes x_0' \otimes x_0'' \in X_0 \otimes X_0 \otimes X_0$) ...etc. Let's also quote the following from [6] as it is needed in our proof:

Lemma 2.5 *(i) l_2l_1 is a boundary.*
(ii) $l_2l_2 + l_3l_1$ is a boundary.
(iii) More generally $(\sum_{i+j=n+1, j>1} (-1)^{i(j-1)} l_jl_i)$ is a boundary.

To begin with the proof we define l_2 inductively by

$$l_2 = -s \circ l_2l_1$$

(this is just $-sl_2l_1$, i.e., s acts on the image of l_2l_1) where we begin with $l_2 = \tilde{l}_2$ in degree 0, and recall that s satisfies

$$\lambda \circ \eta - 1_{X_*} = l_1 \circ s$$

in degree 0, and

$$-1_{X_*} = l_1 \circ s + s \circ l_1$$

in degree > 0 (see equation (2.1)). One checks that if l_2l_1 is in degree 0 then

$$l_1(-sl_2l_1) = l_2l_1 - \lambda \circ \eta(l_2l_1) = l_2l_1,$$

where $(\lambda \circ \eta)(l_2l_1) = 0$ since l_2l_1 is a boundary. While if l_2l_1 is in degree > 0 then

$$l_1(-sl_2l_1) = l_2l_1 + sl_1(l_2l_1) = l_2l_1,$$

where $l_1(l_2l_1) = 0$ since l_2l_1 is a boundary. So we have a well-defined structure map l_2 satisfying $l_1l_2 = l_2l_1$ (notice that l_2 is also a chain map!). Now we show that l_2 as defined above is *zero* in degree > 1.

First consider l_2 on $\underline{X_1 \otimes X_1}$: take an element $x_1 \otimes x_1' \in X_1 \otimes X_1$. We have $l_2(x_1 \otimes x_1') = -s\{l_2(l_1x_1 \otimes x_1' - x_1 \otimes l_1x_1')\}$. But $l_2(l_1x_1 \otimes x_1') = l_2(x_1 \otimes l_1x_1')$, since by definition

$$l_2(l_1x_1 \otimes x_1') = -sl_2l_1(l_1x_1 \otimes x_1') = -sl_2(l_1x_1 \otimes l_1x_1'),$$

and

$$l_2(x_1 \otimes l_1x_1') = -sl_2l_1(x_1 \otimes l_1x_1') = -sl_2(l_1x_1 \otimes l_1x_1').$$

So $l_2 = 0$ on $X_1 \otimes X_1$. Now consider l_2 on $\underline{X_2 \otimes X_0}$: take $x_2 \otimes x_0 \in X_2 \otimes X_0$. Then $l_2(x_2 \otimes x_0) = -sl_2(l_1x_2 \otimes x_0)$, but $l_2(l_1x_2 \otimes x_0) = -sl_2l_1(l_1x_2 \otimes x_0) = -sl_2(0) = 0$. So $l_2 = 0$ on $X_2 \otimes X_0$.

Proceeding by induction one then shows that $l_2 = 0$ on $X_n \otimes X_0$ for $n \geq 3$: $l_2(x_n \otimes x_0) = -sl_2l_1(x_n \otimes x_0) = -sl_2(l_1x_n \otimes x_0) = -s(0) = 0$. On the other hand consider l_2 on $X_n \otimes X_m, n > 1, m \geq 1$: $l_2(x_n \otimes x_m) = -sl_2l_1(x_n \otimes x_m) = -sl_2(l_1x_n \otimes x_m + (-1)^n x_n \otimes l_1x_m) = -s(0) = 0$. This way one has $l_2 \equiv 0$ in degree > 1. Now consider the map l_3. Define

$$l_3 = s \circ l_2l_2,$$

in degree 0 and then inductively by

$$l_3 = s \circ (l_2l_2 + l_3l_1),$$

in degree > 0. One checks that in degree 0

$$-l_1l_3 = -l_1s(l_2l_2) = l_2l_2 - (\lambda \circ \eta)(l_2l_2) = l_2l_2,$$

where $(\lambda \circ \eta)(l_2 l_2) = 0$ since $l_2 l_2$ is a boundary. While in degree > 0 we have

$$-l_1 l_3 = -l_1(s(l_2 l_2 + l_3 l_1) = l_2 l_2 + l_3 l_1 + s\{l_1(l_2 l_2 + l_3 l_1)\} = l_2 l_2 + l_3 l_1,$$

where $l_1(l_2 l_2 + l_3 l_1) = 0$ since $l_2 l_2 + l_3 l_1$ is a boundary. So we have a well-defined structure map l_3 satisfying $l_1 l_3 + l_3 l_1 + l_2 l_2 = 0$.

Now consider l_3 on $X_1 \otimes X_0 \otimes X_0$: take an element $x_1 \otimes x_0 \otimes x_0' \in X_1 \otimes X_0 \otimes X_0$. By definition we have:
$(l_2 l_2 + l_3 l_1)(x_1 \otimes x_0 \otimes x_0') =$
$-sl_2 l_1(l_2(x_1, x_0), x_0') + sl_2 l_1(l_2(l_2(x_1, x_0'), x_0) - sl_2 l_1(l_2(l_2(x_0, x_0'), x_1)$
$+s\{l_2(l_2(l_1 x_1, x_0), x_0') - l_2(l_2(l_1 x_1, x_0'), x_0) + l_2(l_2((x_0, x_0'), l_1 x_1)\} =$
$-sl_2(l_2(l_1 x_1, x_0), x_0') + sl_2(l_2(l_2(l_1 x_1, x_0'), x_0) - sl_2(l_2(l_2(x_0, x_0'), l_1 x_1)$
$+sl_2(l_2(l_1 x_1, x_0), x_0') - sl_2(l_2(l_1 x_1, x_0'), x_0) + sl_2(l_2((x_0, x_0'), l_1 x_1) = 0.$
So $l_3 = s\{(l_2 l_2 + l_3 l_1)\} = s(0) = 0$ on $X_1 \otimes X_0 \otimes X_0$.

One can then proceed by induction to find that $l_3 = 0$ in degree > 1. Take for example $x_1 \otimes x_1' \otimes x_0$ in $X_1 \otimes X_1 \otimes X_0$, then $l_3(x_1 \otimes x_1' \otimes x_0) = s(l_2 l_2 + l_3 l_1)(x_1 \otimes x_1' \otimes x_0) = s\{l_2(l_2(x_1, x_1'), x_0) - l_2(l_2(x_1, x_0), x_1') + l_2(l_2(x_1', x_0), x_1) + l_3(l_1 x_1 \otimes x_1' \otimes x_0) + l_3(x_1 \otimes l_1 x_1' \otimes x_0)\} = 0$, since l_3 and l_2 are *zero* in degrees 1 and 2 respectively. Now consider the map l_4. Define

$$l_4 = s \circ (l_3 l_2 - l_2 l_3),$$

in degree 0, and then inductively by

$$l_4 = s \circ (l_3 l_2 - l_2 l_3 - l_4 l_1),$$

in degree > 0. As before one can easily check that l_4 is a well-defined map that satisfies the corresponding sh-Lie relation at this step of the construction (i.e. $l_1 l_4 - l_4 l_1 + l_3 l_2 - l_2 l_3 = 0$).

Consider the value of l_4 on $x_0 \otimes x_0' \otimes x_0'' \otimes x_0''' \in X_0 \otimes X_0 \otimes X_0 \otimes X_0$. We find it convenient in the following calculation to use the identity $l_3(l_2(y_0, y_0'), y_0'', y_0''') = (sl_2l_2)(l_2(y_0, y_0'), y_0'', y_0''') = s\{l_2(l_2(l_2(y_0, y_0'), y_0''), y_0''') - l_2(l_2(l_2(y_0, y_0'), y_0'''), y_0'') + l_2(l_2(y_0'', y_0'''), l_2(y_0, y_0'))\}$, for $y_0, y_0', y_0'', y_0''' \in X_0$. By definition l_4 is the value of s on:

$l_3(l_2(x_0, x_0'), x_0'', x_0''') - l_3(l_2(x_0, x_0''), x_0', x_0''') + l_3(l_2(x_0, x_0'''), x_0', x_0'') +$
$l_3(l_2(x_0', x_0''), x_0, x_0''') - l_3(l_2(x_0', x_0'''), x_0, x_0'') + l_3(l_2(x_0'', x_0'''), x_0, x_0') -$
$l_2(l_3(x_0, x_0', x_0''), x_0''') + l_2(l_3(x_0, x_0', x_0'''), x_0'') - l_2(l_3(x_0, x_0'', x_0'''), x_0') +$
$l_2(l_3(x_0', x_0'', x_0'''), x_0) =$
$s\{l_2(l_2(l_2(x_0, x_0'), x_0''), x_0''') - l_2(l_2(l_2(x_0, x_0'), x_0'''), x_0'') + l_2(l_2(x_0'', x_0'''), l_2(x_0, x_0'))\} -$
$s\{l_2(l_2(l_2(x_0, x_0''), x_0'), x_0''') - l_2(l_2(l_2(x_0, x_0''), x_0'''), x_0') + l_2(l_2(x_0', x_0'''), l_2(x_0, x_0''))\} +$
$s\{l_2(l_2(l_2(x_0, x_0'''), x_0'), x_0'') - l_2(l_2(l_2(x_0, x_0'''), x_0''), x_0') + l_2(l_2(x_0', x_0''), l_2(x_0, x_0'''))\} +$
$s\{l_2(l_2(l_2(x_0', x_0''), x_0), x_0''') - l_2(l_2(l_2(x_0', x_0''), x_0'''), x_0) + l_2(l_2(x_0, x_0'''), l_2(x_0', x_0''))\} -$
$s\{l_2(l_2(l_2(x_0', x_0'''), x_0), x_0'') - l_2(l_2(l_2(x_0', x_0'''), x_0''), x_0) + l_2(l_2(x_0, x_0''), l_2(x_0', x_0'''))\} +$
$s\{l_2(l_2(l_2(x_0'', x_0'''), x_0), x_0') - l_2(l_2(l_2(x_0'', x_0'''), x_0'), x_0) + l_2(l_2(x_0, x_0'), l_2(x_0'', x_0'''))\} +$
$s\{l_2(l_1l_3(x_0, x_0', x_0''), x_0''') - l_2(l_1l_3(x_0, x_0', x_0'''), x_0'') + l_2(l_1l_3(x_0, x_0'', x_0'''), x_0') -$
$l_2(l_1l_3(x_0', x_0'', x_0'''), x_0)\} =$
$sl_2\{(l_2l_2 + l_1l_3)(x_0, x_0', x_0''), x_0'''\} - sl_2\{(l_2l_2 + l_1l_3)(x_0, x_0', x_0'''), x_0''\} +$
$sl_2\{(l_2l_2 + l_1l_3)(x_0, x_0'', x_0'''), x_0'\} - sl_2\{(l_2l_2 + l_1l_3)(x_0', x_0'', x_0'''), x_0\} +$
$l_2(l_2(x_0'', x_0'''), l_2(x_0, x_0')) + l_2(l_2(x_0, x_0'), l_2(x_0'', x_0''')) - l_2(l_2(x_0', x_0'''), l_2(x_0, x_0'')) -$
$l_2(l_2(x_0, x_0''), l_2(x_0', x_0''')) + l_2(l_2(x_0', x_0''), l_2(x_0, x_0''')) + l_2(l_2(x_0, x_0'''), l_2(x_0', x_0'')) =$
0 since $l_2l_2 + l_1l_3 = 0$ in degree 0 and l_2 is skew-symmetric. So we have $l_4 = 0$ in degree 0.

Further l_4 is inductively found to be *zero* in higher degrees since $l_2 = 0$ and $l_3 = 0$ in degrees > 1 and > 0 respectively. Next we inductively define for $n > 4$,

$$l_n = s \circ (\sum_{i,j>1} (-1)^{i(j-1)} l_j l_i),$$

in degree 0, and

$$l_n = s \circ (\sum_{i+j=n+1, j>1} (-1)^{i(j-1)} l_j l_i),$$

in degree > 0. (Again these are well-defined maps for the sh-Lie structure.) The combination of maps (the l_k's within the s) in degree 0, and then inductively in degree > 0, leads to 0 so that one has $l_n \equiv 0$ for $n > 4$ (Notice that for l_5 in degree 0 one encounters $l_3 l_3$, the inside l_3 raises the degree from 0 to 1 so that the combination is 0). Summarizing:

Theorem 2.6 *Given a graded space X_* and a skew-symmetric linear map $\tilde{l}_2 : X_0 \otimes X_0 \to X_0$ that satisfies conditions (i) and (ii), there exists an sh-Lie structure on X_* such that*

1. $l_2 = 0$ in degree > 1.

2. $l_3 = 0$ in degree > 0.

3. $l_n \equiv 0, n > 3$.

Corollary 2.7 *Under the same hypotheses as in Theorem* 2.6 *there exists an sh-Lie structure on the graded space*

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow X_2 \xrightarrow{i} X_1 \xrightarrow{l_1} X_0$$

where X_1, X_0, $l_1 : X_1 \to X_0$, and $l_k, k > 1$ are as above, but with $X_2 = ker l_1$ and the inclusion $i : X_2 \to X_1$.

2.3 Examples

We will primarily consider three examples. The first of these fits perfectly into our discussion. It first appeared in [21] where the authors determined an sh-Lie structure on a Courant algebroid. For convenience we recall the definition of a Courant algebroid.

Definition A Courant algebroid is a vector bundle $E \to M$ equipped with a nondegenerate symmetric bilinear form $< \cdot, \cdot >$ on the bundle, a skew-symmetric bracket $[\cdot, \cdot]$ on $\Gamma(E)$, and a bundle map $\rho : E \to TM$ such that the following properties are satisfied:

1. For any $e_1, e_2, e_3 \in \Gamma(E), J(e_1, e_2, e_3) = \mathcal{D}T(e_1, e_2, e_3)$;
2. for any $e_1, e_2 \in \Gamma(E), \rho[e_1, e_2] = [\rho e_1, \rho e_2]$;
3. for any $e_1, e_2 \in \Gamma(E)$ and $f \in C^\infty(M), [e_1, fe_2] = f[e_1, e_2] + (\rho(e_1)f)e_2 - < e_1, e_2 > \mathcal{D}f$;
4. $\rho \circ \mathcal{D} = 0$;
5. for any $e, h_1, h_2 \in \Gamma(E), \rho(e) < h_1, h_2 > = < [e, h_1] + \mathcal{D} < e, h_1 >, h_2 > + < h_1, [e, h_2] + \mathcal{D} < e, h_2 >>$;

where

$$J(e_1, e_2, e_3) = [[e_1, e_2], e_3] + [[e_2, e_3], e_1] + [[e_3, e_1], e_2],$$

and $T(e_1, e_2, e_3)$ is the function on the base space M defined by

$$T(e_1, e_2, e_3) = \frac{1}{3} < [e_1, e_2], e_3 > + c.p.$$

($c.p.$ here denotes the cyclic permutations of the e_i's) and $\mathcal{D} : C^\infty(M) \to \Gamma(E)$ is defined such that the following identity holds

$$< \mathcal{D}f, e > = \frac{1}{2}\rho(e)f.$$

For more details on Courant algebroids we refer the reader to [21] and the references therein.

Example Let E be a Courant algebroid over a manifold M, and consider the sequence

$$\cdots \to 0 \to \ker\mathcal{D} \xrightarrow{i} C^\infty(M) \xrightarrow{\mathcal{D}} \Gamma(E)$$

where $i : \ker\mathcal{D} \to C^\infty(M)$ is the inclusion, and one assumes that $X_0 = \Gamma(E)$, $X_1 = C^\infty(M)$, and $X_2 = \ker\mathcal{D}$. Define $\tilde{l}_2(e_1, e_2) = [e_1, e_2]$ (this is just l_2 in degree 0). It was shown in [21] that $\tilde{l}_2$ satisfies condition (i) as in Theorem 2.1, whereas condition (ii) from the same Theorem follows from the the first axiom in the definition above, yielding an sh-Lie structure. The authors [21] have also shown that the sh-Lie structure has the explicit formulas

$$\begin{array}{lcll} l_2(e_1, e_2) & = & [e_1, e_2] & \text{in degree } 0 \\ l_2(e, f) & = & < e, \mathcal{D}f > & \text{in degree } 1 \\ l_2 & = & 0 & \text{in degree} > 1 \\ l_3(e_1, e_2, e_3) & = & -T(e_1, e_2, e_3) & \text{in degree } 0 \\ l_3 & = & 0 & \text{in degree} > 0 \\ l_n & = & 0 & \text{for } n > 3. \end{array}$$

What is interesting about this example is that the only nonzero maps are the same ones as in Theorem 2.6 and Corollary 2.7. In addition notice that the structure of this complex is similar to the “simplified” complex that appears in Corollary 2.7. ∎

Remark In this example it is not generally true that $\tilde{l}_2(c, b) = 0$ for an arbitrary cycle c and arbitrary boundary b. Thus it is *not* a consequence of the Markl-Barnich Theorem. On the other hand $\tilde{l}_2(c,b)$ is a boundary and so falls within the scope of our Theorem.

Our next example comes from Lagrangian field theory. In particular, it relates to Poisson brackets of local functionals where the sh-Lie structure exists on a "de Rham complex" as in [6]. We refer the reader to [6] and the references therein for more details regarding this subject. In fact the next chapters will shed some light on this subject along with canonical transformations of local functionals.

Example let $E \to M$ be a vector bundle where the base space M is an n-dimensional manifold and let $J^\infty E$ be the infinite jet bundle of E. Consider the complex

$$\Omega^{0,0}(J^\infty E) \to \Omega^{1,0}(J^\infty E) \to \cdots \to \Omega^{n-1,0}(J^\infty E) \to \Omega^{n,0}(J^\infty E)$$

with a differential d_H which in local coordinates takes the form $d_H = dx^i D_i$, i.e. if $\alpha = \alpha_I dx^I$ then $d_H\alpha = D_i\alpha_I dx^i \wedge dx^I$. Here D_i is the total derivative defined on the algebra of local functions on $J^\infty E$. It is defined by $D_i = \dfrac{\partial}{\partial x^i} + u^a_{iJ}\dfrac{\partial}{\partial u^a_J}$ (we assume the summation convention, i.e., the sum is over all a and multi-index J). In this case $\tilde{l}_2$ was defined in [6] by

$$\tilde{l}_2(P\nu, Q\nu) = \omega(\mathbf{E}(P), \mathbf{E}(Q))\nu = \omega^{ab}(\mathbf{E}_b(Q))\mathbf{E}_a(P)\nu \tag{2.2}$$

where $P\nu, Q\nu \in \Omega^{n,0}(J^\infty E)$ and $\mathbf{E}$ is the Euler-Lagrange operator with components

$$\mathbf{E}_a(P) = (-D)_I(\frac{\partial P}{\partial u^a_I}).$$

The bilinear mapping ω is a skew-symmetric total differential operator with the ω^{ab}'s as its components (see [6] for more details). It was shown in [6] that $\tilde{l}_2$ satisfies conditions (i) and (ii) as in Theorem 2.1. In fact $\tilde{l}_2$ satisfies condition (i) in a strong sense (with 0 on the right-hand side of the equation). Markl noted in [6] and Barnich proved in [5] that with this strong condition the higher order maps can be chosen to be zero (the result in Theorem 2.6 is yet stronger since it does not require that the right-hand side of (i) be zero, only that it be a boundary). Here is a summary of the structure

$$\begin{array}{lccl} l_2(P\nu, Q\nu) & = & \omega(\mathbf{E}(P), \mathbf{E}(Q))\nu & \text{in degree } 0 \\ l_2 & = & 0 & \text{in degree} > 0 \\ l_3(P\nu, Q\nu, R\nu) & is & nonzero & \text{in degree } 0 \\ l_3 & = & 0 & \text{in degree} > 0 \\ l_n & = & 0 & \text{for } n > 3. \quad \blacksquare \end{array}$$

Our last example is within the context of symplectic manifolds, where we refer the reader to [1]. We include details in this example on how the sh-Lie structure maps are obtained. Notice that in this example the strong version of condition (i) of Theorem 2.1 does not hold in general, but our weaker hypothesis does hold.

Example Consider the following sequence

$$0 \to \mathbf{R} \xrightarrow{i} \Omega^0(M) \xrightarrow{d} \Omega^1_C(M) \to 0,$$

where $\Omega^0(M)$ is the set of smooth real-valued functions on the symplectic manifold (M, ω) and $\Omega^1_C(M)$ is the set of closed one forms on M. We take $X_0 = \Omega^1_C(M), X_1 = \Omega^0(M)$ and $X_2 = \mathbf{R}$. The differential for this complex is $l_1 = i : \mathbf{R} \to \Omega^0(M)$, and $l_1 = d : \Omega^0(M) \to \Omega^1_C(M)$, where i is the inclusion mapping and d is the differential operator. We then define a linear skew-symmetric map $\tilde{l}_2$ on $X_0 \otimes X_0$ by

$$\tilde{l}_2(\alpha \otimes \beta) = \{\alpha, \beta\},$$

where $\{.,.\}$ is a Poisson bracket on $\Omega^1(M)$ (e.g. see definition 3.3.7 in [1]). Notice that $\tilde{l}_2$ satisfies the two conditions needed to guarantee the existence of an sh-Lie algebra (Theorem 2.1) and that it does not satisfy the strong condition of Barnich and Markl.

Now to extend $\tilde{l}_2$, first take an element in $X_1 \otimes X_0$ say $f \otimes \beta$, then $\tilde{l}_2 l_1(f \otimes \beta) = \tilde{l}_2(df \otimes \beta) = \{df, \beta\}$. Now notice that $h = L_{\beta^\#} f + c = -i_{X_f}\beta + L_{\beta^\#} f + i_{X_f} i_{\beta^\#}\omega + c \in X_1$, where $c \in \mathbf{R}$ is a constant, satisfies

$$\begin{aligned} l_1(h) &= d(-i_{X_f}\beta + L_{\beta^\#} f + i_{X_f} i_{\beta^\#}\omega + c) \\ &= -d i_{X_f}\beta - i_{X_f} d\beta + dL_{\beta^\#} f + d(i_{X_f} i_{\beta^\#}\omega) \\ &= -L_{X_f}\beta + L_{\beta^\#} df + d(i_{X_f} i_{\beta^\#}\omega) \\ &= \{df, \beta\}, \end{aligned}$$

where ω is the *symplectic* 2-form on M. So if we take $c = 0$ we get $l_2(f \otimes \beta) = L_{\beta^\#} f$. Then l_2 would be defined on $X_0 \otimes X_1$ by skew-symmetry. To proceed take an element in $X_1 \otimes X_1$ say $f \otimes g$, and notice that $l_2 l_1(f \otimes g) = l_2(df \otimes g - f \otimes dg) = L_{df^\#} g - (-L_{dg^\#} f) = L_{X_f} g + L_{X_g} f = 0$. Hence l_2 on $X_1 \otimes X_1$ is *zero*. Now take an element in $X_2 \otimes X_0$ say $k \otimes \beta$, then $l_2 l_1(k \otimes \beta) = l_2(k \otimes \beta + 0) = L_{\beta^\#} k = 0$ since k is a constant function. Therefore l_2 is *zero* on $X_2 \otimes X_0$. By skew-symmetry l_2 will also be *zero* on $X_0 \otimes X_2$.

Next we turn to l_3. Take an element in $X_0 \otimes X_0 \otimes X_0$, say $\alpha \otimes \beta \otimes \gamma$. Then $l_2 l_2$ maps it into $\{\{\alpha, \beta\}, \gamma\} - \{\{\alpha, \gamma\}, \beta\} + \{\{\beta, \gamma\}, \alpha\}$ which is 0 (the

Jacobi identity), so we have $l_3 = 0$ on $X_0 \otimes X_0 \otimes X_0$. Now take $f \otimes \beta \otimes \gamma \in X_1 \otimes X_0 \otimes X_0$. Under $l_2 l_2 + l_3 l_1$ it is mapped to

$$l_2(L_{\beta^\#} f, \gamma) - l_2(L_{\gamma^\#} f, \beta) + l_2(\{\beta, \gamma\}, f) + 0 =$$

$$L_{\gamma^\#} L_{\beta^\#} f - L_{\beta^\#} L_{\gamma^\#} f - L_{\{\beta,\gamma\}^\#} f =$$

$$L_{\{\beta,\gamma\}^\#} f - L_{\{\beta,\gamma\}^\#} f = 0.$$

So l_3 is *zero* on $X_1 \otimes X_0 \otimes X_0$. Utilizing skew-symmetry we have $l_3 = 0$ in degree 1. On higher degrees l_3 is trivially *zero* since $X_n = 0$ for $n > 2$, and furthermore $l_n = 0$ for $n > 3$. To summarize

$$\begin{array}{llll} l_2(\alpha, \beta) & = & \{\alpha, \beta\} & \text{in degree } 0 \\ l_2(f, \beta) & = & L_{\beta^\#} f & \text{in degree } 1 \\ l_2 & = & 0 & \text{in degree} > 1 \\ l_n & = & 0 & \text{for } n > 2. \quad \blacksquare \end{array}$$

2.4 Chain maps and induced sh-Lie structures

Suppose that $f : X_* \to Y_*$ is a *0*-degree chain map between two exact chain complexes (X_*, d_X) and (Y_*, d_Y). Specifically, we assume that $H_k(X) = H_k(Y) = 0$ for $k > 0$, $H_0(X)$ and $H_0(Y)$ are generally non-trivial, and $X_n = 0$ for $n < 0$. Notice that since f is a chain map we have $d_Y \circ f = f \circ d_X$. Also suppose that f is *onto*, and that there exists an sh-Lie structure on X_*. Observe that, by Corollary 2.2, this sh-Lie structure yields a mapping $\tilde{l}_2 : X_0 \otimes X_0 \to X_0$ that satisfies the same hypotheses as in Theorem 2.1. We assume that the following mapping, $\hat{l}_2$, is well-defined on $Y_0 \otimes Y_0$:

$$\hat{l}_2(y_0, y_1) = \hat{l}_2(f(x_0), f(x_1)) := f(\tilde{l}_2(x_0, x_1))$$

where $y_0, y_1 \in Y_0$ and $x_0, x_1 \in X_1$ with $y_0 = f(x_0), y_1 = f(x_1)$. Notice that $\hat{l}_2$ is skew-symmetric and linear since $\tilde{l}_2$ is. Now let $y_0 \in Y_0$ and $y_1 \in Y_1$ be arbitrary with $y_0 = f(x_0), y_1 = f(x_1)$ for some $x_0 \in X_0$ and $x_1 \in X_1$, then

$$\begin{aligned}
\hat{l}_2(y_0, d_Y y_1) &= \hat{l}_2(f(x_0), d_Y f(x_1)) \\
&= \hat{l}_2(f(x_0), f(d_X x_1)) \\
&= f(\tilde{l}_2(x_0, d_X x_1)) \\
&= f(d_X x_2) \\
&= d_Y f(x_2)
\end{aligned}$$

since $\tilde{l}_2$ satisfies condition (i) of Theorem 2.1 and for some $x_2 \in X_1$. This shows that $\hat{l}_2$ satisfies the same condition (i) of Theorem 2.1. On the other hand if $y_0, y_1, y_2 \in Y_0$ are arbitrary with $y_0 = f(x_0), y_1 = f(x_1), y_2 = f(x_2)$ for some $x_0, x_1, x_2 \in X_0$, then

$$\begin{aligned}
\sum_\sigma \hat{l}_2(\hat{l}_2(y_{\sigma(0)}, y_{\sigma(1)}), y_{\sigma(2)}) &= \sum_\sigma \hat{l}_2(\hat{l}_2(f(x_{\sigma(0)}), f(x_{\sigma(1)})), f(x_{\sigma(2)})) \\
&= f(\sum_\sigma \tilde{l}_2(\tilde{l}_2(x_{\sigma(0)}, x_{\sigma(1)}), x_{\sigma(2)})) \\
&= f(d_X x_3) = d_Y f(x_3)
\end{aligned}$$

(where the sum is over the unshuffles (2,1) with the corresponding permutation sign $(-1)^\sigma$) since $\tilde{l}_2$ satisfies condition (ii) of Theorem 2.1 and for some $x_3 \in X_1$. This shows that $\hat{l}_2$ satisfies the same condition. Therefore we have:

Theorem 2.8 *Suppose that $f : X_* \to Y_*$ is a 0-degree chain map between exact chain complexes, and that there exists an sh-Lie structure on X_*. If there exists a well-defined mapping $\hat{l}_2$ as above, then this mapping $\hat{l}_2$ extends to an sh-Lie structure on Y_*.*

This concludes our discussion of abstract sh-Lie structures. Our focus throughout the remainder of the dissertation is on applications to field theories.

Chapter 3

Canonical transformations of local functionals

In this chapter we consider canonical transformations of local functionals and their relation to existing sh-Lie structures. This will set the stage for applying ideas of reduction which we leave for the next two chapters. Recall that a transformation on the space of local functionals is *canonical* if and only if it preserves the Poisson bracket.

We further say that an automorphism of the bundle E is *canonical* if and only if its induced transformation on the space of local functionals $\mathcal{F}$ is canonical. (Observe that the space of local functionals is not a Poisson algebra and so there is no underlying Poisson manifold.)

After presenting background material in section 3.1, we find conditions for the induced transformations on the space of local functionals to be canonical. Eventually, in the last section of this chapter, we show how these induced canonical transformations relate to the existing sh-Lie structure maps.

3.1 Background material

We introduce some of the terminology and concepts that are used in this chapter as well as in later chapters, in addition to some of the simpler results that will be needed. Our exposition and notation closely follows that in [6]. First let $E \to M$ be a vector bundle where the base space M is an n-dimensional manifold and let $J^\infty E$ be the infinite jet bundle of E. The restriction of the infinite jet bundle over an appropriate open set $U \subset M$ is trivial with fiber an infinite dimensional vector space V^∞. The bundle

$$\pi^\infty : J^\infty E_U = U \times V^\infty \to U$$

then has induced coordinates given by

$$(x^i, u^a, u^a_i, u^a_{i_1 i_2}, \ldots,).$$

We use multi-index notation and the summation convention throughout the dissertation. If $j^\infty\phi$ is the section of $J^\infty E$ induced by a section ϕ of the bundle E, then $u^a \circ j^\infty\phi = u^a \circ \phi$ and

$$u^a_I \circ j^\infty\phi = (\partial_{i_1}\partial_{i_2}...\partial_{i_r})(u^a \circ j^\infty\phi)$$

where r is the order of the symmetric multi-index $I = \{i_1, i_2, ..., i_r\}$, with the convention that, for $r = 0$, there are no derivatives. For more details see [2] and [18].

Let Loc_E denote the algebra of local functions where a local function on $J^\infty E$ is defined to be the pull-back of a smooth function on some finite jet bundle J^pE via the projection from $J^\infty E$ to J^pE. Let Loc^0_E denote the subalgebra of Loc_E such that $P \in Loc^0_E$ if, and only if, $(j^\infty\phi)^*P$ has compact support for all $\phi \in \Gamma E$ with compact support, where ΓE denotes the set of sections of the bundle $E \to M$. The de Rham complex of differential forms $\Omega^*(J^\infty E, d)$ on $J^\infty E$ possesses a differential ideal, the ideal C of contact forms θ which satisfy $(j^\infty\phi)^*\theta = 0$ for all sections ϕ with compact support. This ideal is generated by the contact one-forms, which in local coordinates assume the form $\theta^a_J = du^a_J - u^a_{iJ}dx^i$. Contact one-forms of order 0 satisfy $(j^1\phi)^*(\theta) = 0$. In local coordinates, contact forms of order zero assume the form $\theta^a = du^a - u^a_i dx^i$.

Using the contact forms, we see that the complex $\Omega^*(J^\infty E, d)$ splits as a bicomplex $\Omega^{r,s}(J^\infty E)$ (though the finite level complexes $\Omega^*(J^p E)$ do not), where $\Omega^{r,s}(J^\infty E)$ denotes the space of differential forms on $J^\infty E$ with r horizontal components and s vertical components. The bigrading is described by writing a differential p-form $\alpha = \alpha^{\mathbf{J}}_{IA}(\theta^A_{\mathbf{J}} \wedge dx^I)$ as an element of $\Omega^{r,s}(J^\infty E)$, with $p = r + s$, and

$$dx^I = dx^{i_1} \wedge ... \wedge dx^{i_r} \qquad \theta^A_{\mathbf{J}} = \theta^{a_1}_{J_1} \wedge ... \wedge \theta^{a_s}_{J_s}.$$

Now let C_0 denote the set of contact one-forms of order zero. In local coordinates contact one-forms of order zero can be written as $\theta^a = du^a - u^a_\mu dx^\mu$. Notice that both C_0 and $\Omega^{n,1} = \Omega^{n,1}(J^\infty E)$ are modules over Loc_E. Let $\Omega^{n,1}_0$ denote the subspace of $\Omega^{n,1}$ which is locally generated by the forms $\{(\theta^a \wedge d^n x)\}$ over Loc_E. We assume the existence of a mapping, ω, from $\Omega^{n,1}_0 \times \Omega^{n,1}_0$ to Loc_E, such that ω is a skew-symmetric module homomorphism in each variable separately. In local coordinates let $\omega^{ab} = \omega(\theta^a \wedge \nu, \theta^b \wedge \nu)$, where ν is a volume form on M (notice that in local coordinates ν takes the form $\nu = f d^n x = f dx^1 \wedge dx^2 \wedge ... \wedge dx^n$ for some function $f : U \to \mathbf{R}$ and U is a subset of M on which the x^i's are defined).

We will assume throughout the dissertation that ω satisfies the conditions that make our Poisson bracket, which will be defined soon, satisfy the Jacobi identity.

Define the operator D_i (total derivative) by $D_i = \frac{\partial}{\partial x^i} + u^a_{iJ}\frac{\partial}{\partial u^a_J}$ (recall we assume the summation convention, i.e., the sum is over all a and multi-index J), and recall that the Euler-Lagrange operator maps $\Omega^{n,0}(J^\infty E)$ into $\Omega^{n,1}(J^\infty E)$ and is defined by

$$\mathbf{E}(P\nu) = \mathbf{E}_a(P)(\theta^a \wedge \nu)$$

where $P \in Loc_E, \nu$ is a volume form on the base manifold M, and the components $\mathbf{E}_a(P)$ are given by

$$\mathbf{E}_a(P) = (-D)_I(\frac{\partial P}{\partial u^a_I}).$$

For simplicity of notation we may use $\mathbf{E}(P)$ for $\mathbf{E}(P\nu)$. We will also use $\tilde{D}_i$ for $\frac{\partial}{\partial \tilde{x}^i} + \tilde{u}^a_{iJ}\frac{\partial}{\partial \tilde{u}^a_J}$ and $\tilde{\mathbf{E}}_a(P)$ for $(-\tilde{D})_I(\frac{\partial P}{\partial \tilde{u}^a_I})$ so that $\mathbf{E}(P) = \tilde{\mathbf{E}}_a(P)(\tilde{\theta}^a \wedge \nu)$ in the $(\tilde{x}^\mu, \tilde{u}^a)$ coordinate system.

Let $\Omega^{k,l}_c(J^\infty E)$ be the subspace of $\Omega^{k,l}(J^\infty E)$, for $(k,l) \neq (n,0)$, such that $\alpha \in \Omega^{k,l}_c(J^\infty E)$ if, and only if, $(j^\infty\phi)^*\alpha$ has compact support for all $\phi \in \Gamma E$ with compact support, and let $\Omega^{n,0}_c(J^\infty E)$ be the subspace of $\Omega^{n,0}(J^\infty E)$ such that $P\nu \in \Omega^{n,0}_c(J^\infty E)$ if, and only if, $(j^\infty\phi)^*(P\nu)$ and $(j^\infty\phi)^*\mathbf{E}_a(P)$ have compact support for all $\phi \in \Gamma E$ with compact support and for all a. We are interested in the complex

$$0 \to \Omega^{0,0}_c(J^\infty E) \to \Omega^{1,0}_c(J^\infty E) \to \cdots \to \Omega^{n-1,0}_c(J^\infty E) \to \Omega^{n,0}_c(J^\infty E)$$

with the differential d_H defined by $d_H = dx^i D_i$, i.e., if $\alpha = \alpha_I dx^I$ then $d_H\alpha = D_i\alpha_I dx^i \wedge dx^I$. Notice that this complex is exact when the vector bundle $E \to M$ is trivial (e.g. see [9]).

Now let $\mathcal{F}$ be the space of local functionals where $\mathcal{P} \in \mathcal{F}$ if, and only if, $\mathcal{P} = \int_M P\nu$ for some $P \in Loc_E^0$, and define a Poisson bracket on $\mathcal{F}$ by

$$\{\mathcal{P}, \mathcal{Q}\}(\phi) = \int_M [\omega(\mathbf{E}(P), \mathbf{E}(Q)) \circ j\phi]\nu,$$

where $\phi \in \Gamma E$, ν is a volume form on $M, \mathcal{P} = \int_M P\nu, \mathcal{Q} = \int_M Q\nu$, and $P, Q \in Loc_E^0$. Using local coordinates (x^μ, u_I^a) on $J^\infty E$, observe that for $\phi \in \Gamma E$ such that the support of ϕ lies in the domain Ω of some chart x of M, one has

$$\{\mathcal{P}, \mathcal{Q}\}(\phi) = \int_{x(\Omega)} ([\omega^{ab}\mathbf{E}_a(P)\mathbf{E}_b(Q)] \circ j\phi \circ x^{-1})(x^{-1})^*(\nu)$$

where x^{-1} is the inverse of $x = (x^\mu)$.

We assume that ω satisfies the necessary conditions for the above bracket to satisfy the Jacobi identity, e.g. see [20]. Notice that it follows from the identity (7.11) of [20] that the bracket satisfies the Jacobi identity if the skew-symmetric matrix $\{\omega^{ab}\}$ is a Poisson tensor in the sense that:

$$\omega^{cd}\frac{\partial \omega^{ab}}{\partial u^d} + \omega^{ad}\frac{\partial \omega^{bc}}{\partial u^d} + \omega^{bd}\frac{\partial \omega^{ca}}{\partial u^d} = 0 \tag{3.1}$$

where $\{u^a\}$ are coordinates on the fiber of the trivial bundle E. This condition is met in the case of the Poisson sigma model, which we include later in Chapter 4*, and more generally for any trivial vector bundle with a Poisson structure on its fibers.*

The functions P and Q in our definition of the Poisson bracket of local functionals are representatives of $\mathcal{P}$ and $\mathcal{Q}$ respectively, since generally these are not unique. In fact $\mathcal{F} \simeq H_c^n(J^\infty E)$, where $H_c^n(J^\infty E) = \Omega_c^{n,0}(J^\infty E)/(\mathrm{im} d_H \bigcap \Omega_c^{n,0}(J^\infty E))$ and $\mathrm{im} d_H$ is the image of the differential d_H.

Let $\psi : E \to E$ be an automorphism, sending fibers to fibers, and let $\psi_M : M \to M$ be the induced diffeomorphism of M. Notice that ψ induces an automorphism $j^\infty\psi : J^\infty E \to J^\infty E$ where

$$(j^\infty\psi)((j^\infty\phi)(p)) = j^\infty(\psi \circ \phi \circ \psi_M^{-1})(\psi_M(p)),$$

for all $\phi \in \Gamma E$ and all p in the domain of ϕ. In these coordinates the independent variables transform via $\tilde{x}^\mu = \psi_M^\mu(x^\nu)$. Local coordinate representatives of ψ_M and $j^\infty\psi$ may be described in terms of charts (Ω, x) and $(\tilde{\Omega}, \tilde{x})$ of M, and induced charts $((\pi^\infty)^{-1}(\Omega), (x^\mu, u_I^a))$ and $((\pi^\infty)^{-1}(\tilde{\Omega}), (\tilde{x}^\mu, \tilde{u}_I^a))$ of $J^\infty E$.

Remark We will also use $j\psi$ for $j^\infty\psi$.

Remark In the chapters on reduction we will consider (left) Lie group actions on E and their induced (left) actions on $J^\infty E$. Such actions are defined by group homomorphisms from the group into the group of automorphisms of E.

Definition $\omega : \Omega_0^{n,1} \times \Omega_0^{n,1} \to Loc_E$ is *covariant* with respect to an automorphism $\psi : E \to E$ of the above form if, and only if

$$\omega((j^\infty\psi)^*\theta, (j^\infty\psi)^*\theta') = (\det\psi_M)(j^\infty\psi)^*(\omega(\theta, \theta')),$$

for all $\theta, \theta' \in \Omega_0^{n,1}(J^\infty E)$.

Observe that

$$
\begin{aligned}
(j^\infty\psi)^*\tilde{\theta}^a &= (j^\infty\psi)^*(d\tilde{u}^a - \tilde{u}^a_\mu d\tilde{x}^\mu) \\
&= d(\tilde{u}^a \circ j^\infty\psi) - (\tilde{u}^a_\mu \circ j^\infty\psi)d(\tilde{x}^\mu \circ j^\infty\psi) \\
&= \frac{\partial\psi^a_E}{\partial x^\nu}dx^\nu + \frac{\partial\psi^a_E}{\partial u^b}du^b - (\frac{\partial\psi^a_E}{\partial x^\nu} + \frac{\partial\psi^a_E}{\partial u^b}u^b_\nu)(J^{-1})^\nu_\mu \frac{\partial\tilde{x}^\mu \circ j^\infty\psi}{\partial x^\lambda}dx^\lambda \\
&= \frac{\partial\psi^a_E}{\partial x^\nu}dx^\nu + \frac{\partial\psi^a_E}{\partial u^b}du^b - \frac{\partial\psi^a_E}{\partial x^\nu}dx^\nu - \frac{\partial\psi^a_E}{\partial u^b}u^b_\nu dx^\nu \\
&= \frac{\partial\psi^a_E}{\partial u^b}(du^b - u^b_\nu dx^\nu) \\
&= \frac{\partial\psi^a_E}{\partial u^b}\theta^b
\end{aligned}
$$

where we have assumed that $\psi^a_E = \tilde{u}^a \circ \psi$ and J is the Jacobian matrix of the transformation $\tilde{x}^\nu = \psi^\nu_M(x^\mu)$.

Lemma 3.1 *The following are equivalent*
(i) $\omega((j\psi)^*\theta, (j\psi)^*\theta') = (\det\psi_M)(j\psi)^*(\omega(\theta,\theta'))$, *for all* $\theta, \theta' \in \Omega^{n,1}_0(J^\infty E)$.
(ii) $\tilde{\omega}^{ab} \circ j\psi = (\det\psi_M)\omega^{cd}\frac{\partial\psi^a_E}{\partial u^c}\frac{\partial\psi^b_E}{\partial u^d}$.

Proof Notice that

$$
\begin{aligned}
\det\psi_M(j\psi)^*(\omega(\mathbf{E}(P), \mathbf{E}(Q)) &= (\det\psi_M)\omega(\mathbf{E}(P), \mathbf{E}(Q)) \circ j\psi \\
&= \det\psi_M[\omega(\tilde{\theta}^a \wedge \nu, \tilde{\theta}^b \wedge \nu)\tilde{\mathbf{E}}_a(P)\tilde{\mathbf{E}}_b(Q)] \circ j\psi \\
&= \det\psi_M(\tilde{\omega}^{ab} \circ j\psi)(\tilde{\mathbf{E}}_a(P) \circ j\psi)(\tilde{\mathbf{E}}_b(Q) \circ j\psi)
\end{aligned}
$$

and that

$$
\begin{aligned}
(j\psi)^*(\mathbf{E}(P)) &= (j\psi)^*(\tilde{\mathbf{E}}_a(P)(\tilde{\theta}^a \wedge \nu)) \\
&= (\tilde{\mathbf{E}}_a(P) \circ j\psi)\frac{\partial\psi^a_E}{\partial u^c}(\det\psi_M)(\theta^c \wedge \nu).
\end{aligned}
$$

Now

$$\begin{aligned}\omega((j\psi)^*\mathbf{E}(P),(j\psi)^*\mathbf{E}(Q)) &= (\det\psi_M)^2\frac{\partial\psi_E^a}{\partial u^c}\frac{\partial\psi_E^b}{\partial u^d}(\tilde{\mathbf{E}}_a(P)\circ j\psi)(\tilde{\mathbf{E}}_b(Q)\circ j\psi)\\ &\quad\ \omega(\theta^c\wedge\nu,\theta^d\wedge\nu)\\ &= (\det\psi_M)^2\frac{\partial\psi_E^a}{\partial u^c}\frac{\partial\psi_E^b}{\partial u^d}\omega^{cd}(\tilde{\mathbf{E}}_a(P)\circ j\psi)\\ &\quad\ (\tilde{\mathbf{E}}_b(Q)\circ j\psi).\end{aligned}$$

Hence $\omega((j\psi)^*\mathbf{E}(P),(j\psi)^*\mathbf{E}(Q)) = (\det\psi_M)(j\psi)^*(\omega(\mathbf{E}(P),\mathbf{E}(Q))$ for all P, Q in Loc_E if, and only if, $(\tilde{\omega}^{ab}\circ j\psi) = (\det\psi_M)\dfrac{\partial\psi_E^a}{\partial u^c}\dfrac{\partial\psi_E^b}{\partial u^d}\omega^{cd}$. ∎

3.2 Automorphisms preserving the Poisson structure

Let $L : J^\infty E \to \mathbf{R}$ be a Lagrangian in Loc_E (generally we will assume that any element of Loc_E is a Lagrangian). Let $\hat{L} = L\circ(x^\mu, u_I^a)^{-1}$ and let $\tilde{L} = L\circ(\tilde{x}^\mu,\tilde{u}_I^a)^{-1}$. Then, in local coordinates, $\tilde{L}$ is related to $\hat{L}$ by the equation

$$(\tilde{L}\circ j\bar{\psi})\det(J) = \hat{L},$$

where $j\bar{\psi} = (\tilde{x}^\nu,\tilde{u}_K^b)\circ(x^\mu,u_I^a)^{-1}$ and J is the Jacobian matrix of the transformation $\psi_M = \tilde{x}^\nu\circ(x^\mu)^{-1}$. With abuse of notation we may assume coordinates and charts are the same and write $\tilde{x}^\nu = \psi_M(x^\mu)$. For simplicity, we have also assumed that ψ_M is orientation-preserving. In this case the functional

$$\tilde{\mathcal{L}} = \int_{\tilde{\Omega}}\tilde{L}d^n\tilde{x}$$

is the transformed form of the functional

$$\hat{\mathcal{L}} = \int_{\Omega}\hat{L}d^n x$$

where $\hat{L}$ and $\tilde{L}$ are related as above, Ω is the domain of integration and $\tilde{\Omega}$ is the transformed domain under $j\bar{\psi}$ (see [20] pp.249-250). Notice that both of these are local coordinate expressions of the equation $\mathcal{L} = \int_M L\nu$, for appropriately restricted charts. Now suppose that ψ is an automorphism of E, $j\psi$ its induced automorphism on $J^\infty E$, and ψ_M its induced (orientation-preserving) diffeomorphism on M. Also suppose that $\hat{L}$ and $\tilde{L}$ are two Lagrangians related by the equation $(\tilde{L} \circ j\psi)\det(\psi_M) = \hat{L}$. We have:

Lemma 3.2 *Let P be a Lagrangian as above, then*

$$\mathbf{E}_a((P \circ j\psi)\det(\psi_M)) = \det(\psi_M)\frac{\partial \psi_E^c}{\partial u^a}(\tilde{\mathbf{E}}_c(P) \circ j\psi). \tag{3.2}$$

Proof First notice that $\mathbf{E}_{u^a}(\hat{L}) = \det(\psi_M)\frac{\partial \psi_E^c}{\partial u^a}(\mathbf{E}_{\tilde{u}^c}(\tilde{L})\circ j\psi)$ (see [20] pp.250). But $(\tilde{L} \circ j\psi)\det(\psi_M) = \hat{L}$. The identity 3.2 follows by letting $P = \tilde{L}$. *Notice that this is justified since $\tilde{L}$ is arbitrary in the sense that given any L' there exists an $\hat{L}$ derived from a Lagrangian L as above such that $(L' \circ j\psi)\det(\psi_M) = \hat{L}$ since $j\psi$ is an automorphism.* ∎

Let $\hat{\psi}$ denote the mapping representing the induced action of the automorphism on sections of E, i.e., $\hat{\psi} : \Gamma E \to \Gamma E$ where $\hat{\psi}(\phi) = \psi \circ \phi \circ \psi_M^{-1}$ and ϕ is a section of E. This induces a mapping on the space of local functionals given by

$$\begin{aligned}
(\mathcal{P} \circ \hat{\psi})(\phi) &= \mathcal{P}(\psi \circ \phi \circ \psi_M^{-1}) \\
&= \int_M [P \circ j(\psi \circ \phi \circ \psi_M^{-1})]\nu \\
&= \int_M [P \circ j\psi \circ j\phi \circ \psi_M^{-1})]\nu \\
&= \int_M [P \circ j\psi \circ j\phi](\det\psi_M)\nu,
\end{aligned}$$

where

$$\mathcal{P}(\phi) = \int_M (P \circ j\phi)\nu,$$

and ϕ is a section of E.

We find conditions on those automorphisms of the space of local functionals under which the Poisson structure is preserved.

Recall that $\{\mathcal{P}, \mathcal{Q}\} = \int_M \omega(\mathbf{E}(P), \mathbf{E}(Q))\nu$, and hence

$$\{\mathcal{P}, \mathcal{Q}\}(\phi) = \int_M [\omega(\mathbf{E}(P), \mathbf{E}(Q)) \circ j\phi]\nu. \tag{3.3}$$

Now

$$\{\mathcal{P} \circ \hat{\psi}, \mathcal{Q} \circ \hat{\psi}\}(\phi) = (\{\mathcal{P}, \mathcal{Q}\} \circ \hat{\psi})(\phi) \tag{3.4}$$

is equivalent to

$$\int_M [\omega(\mathbf{E}((P \circ j\psi)\mathrm{det}\psi_M), \mathbf{E}((Q \circ j\psi)\mathrm{det}\psi_M)) \circ j\phi]\nu =$$

$$\int_M [(\omega(\mathbf{E}(P), \mathbf{E}(Q)) \circ j\psi \circ j\phi)\mathrm{det}\psi_M]\nu,$$

but since this latter equation holds for all sections ϕ of E it is equivalent to

$$\omega(\mathbf{E}((P \circ j\psi)\mathrm{det}\psi_M), \mathbf{E}((Q \circ j\psi)\mathrm{det}\psi_M)) = (\omega(\mathbf{E}(P), \mathbf{E}(Q)) \circ j\psi)\mathrm{det}\psi_M$$

up to a *divergence*. The last equation is equivalent to

$$\omega^{ab}\mathbf{E}_b((Q \circ j\psi)\mathrm{det}\psi_M)\mathbf{E}_a((P \circ j\psi)\mathrm{det}\psi_M) = ([\tilde{\omega}^{ab}\tilde{\mathbf{E}}_b(Q)\tilde{\mathbf{E}}_a(P)] \circ j\psi)\mathrm{det}\psi_M$$

up to a divergence, or (Lemma 3.2)

$$\omega^{ab}(\mathrm{det}\psi_M)^2 \frac{\partial \psi_E^d}{\partial u^b} \frac{\partial \psi_E^c}{\partial u^a} (\tilde{\mathbf{E}}_d(Q) \circ j\psi)(\tilde{\mathbf{E}}_c(P) \circ j\psi) =$$

$$(\tilde{\omega}^{ab} \circ j\psi)(\tilde{\mathbf{E}}_b(Q) \circ j\psi)(\tilde{\mathbf{E}}_a(P) \circ j\psi)(\mathrm{det}\psi_M)$$

up to a divergence. Finally, since the last equation is true for all P and Q it is equivalent to

$$\tilde{\omega}^{ab} \circ j\psi = (\det\psi_M)\omega^{cd}\frac{\partial\psi_E^a}{\partial u^c}\frac{\partial\psi_E^b}{\partial u^d}$$

which is equivalent to the covariance of ω. (Notice that if the last equality does not hold then by some choice of P and Q the equations above will not hold up to a divergence.) We have established the following:

Theorem 3.3 *Let $\psi : E \to E$ be an automorphism of E sending fibers to fibers, and let $\Psi : \mathcal{F} \to \mathcal{F}$ be the induced mapping defined by $\Psi(\mathcal{P}) = \mathcal{P} \circ \hat{\psi}$ (noting that $\mathcal{P} \circ \hat{\psi}$ is defined as above) where $\hat{\psi} : \Gamma E \to \Gamma E$ is given by $\hat{\psi}(\phi) = \psi \circ \phi \circ \psi_M^{-1}$. Then Ψ is canonical in the sense that*

$$\{\Psi(\mathcal{P}), \Psi(\mathcal{Q})\} = \Psi(\{\mathcal{P}, \mathcal{Q}\})$$

for all $\mathcal{P}, \mathcal{Q} \in \mathcal{F}$ iff ω is covariant with respect to ψ.

Definition An automorphism ψ of E is *canonical* provided the induced mapping $\Psi : \mathcal{F} \longrightarrow \mathcal{F}$ is canonical (in the sense of the preceding Theorem).

Example Consider $M = \mathbf{R}, E = \mathbf{R} \times \mathbf{R}^2$, and let

$$\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Consider the action $\psi_g : E \to E$ defined by $\psi_g(x, u^1, u^2) = (x, g \cdot (u^1, u^2))$ for some $g \in SO(2)$. It can easily be shown that ω is covariant with respect to ψ_g and hence the induced action is canonical. As an illustration let g be the counter-clockwise rotation by 90^o (so that $gu^1 = u^2$ and $gu^2 = -u^1$), and let $\mathcal{P} = \int_{\mathbf{R}} P(u^1)dx$ and $\mathcal{Q} = \int_{\mathbf{R}} Q(u^2)dx$ for real-valued differentiable functions P and Q. Then $\Psi(\mathcal{P}) = \int_{\mathbf{R}} P(u^2)dx, \Psi(\mathcal{Q}) = \int_{\mathbf{R}} Q(-u^1)dx$ and $\{\Psi(\mathcal{P}), \Psi(\mathcal{Q})\} = -\int_{\mathbf{R}} P'(u^2)Q'(-u^1)dx$.

On the other hand $\{\mathcal{P}, \mathcal{Q}\} = -\int_{\mathbf{R}} P'(u^1)Q'(u^2)dx$ so that $\Psi(\{\mathcal{P}, \mathcal{Q}\}) = -\int_{\mathbf{R}} P'(u^2)Q'(-u^1)dx$. ∎

The following will be needed in our subsequent work.

Proposition 3.4 *$d_H((j\psi)^*\eta) = (j\psi)^*(d_H\eta)$ for $\eta \in \Omega^{m,0}$, m arbitrary.*

Proof Let $\eta = \alpha_I d\tilde{x}^I$ where each $\alpha_I : J^\infty E \to \mathbf{R}$. We assume that the α_I's depend on the transformed variables $(\tilde{x}^j, \tilde{u}^b_K)$. Then

$$\begin{aligned}
d_H((j\psi)^*(\alpha)) &= d_H((\alpha_I \circ j\psi)d(\tilde{x}^I \circ \psi)) \\
&= \{D_i(\alpha_I \circ j\psi)\}dx^i \wedge d(\tilde{x}^I \circ \psi) \\
&= \{(\frac{\partial \alpha_I}{\partial \tilde{x}^j} \circ j\psi)\frac{\partial(\tilde{x}^j \circ \psi)}{\partial x^i} + (\frac{\partial \alpha_I}{\partial \tilde{u}^b_K} \circ j\psi)\frac{\partial j\psi^b_K}{\partial x^i} + \\
&\quad u^a_{iI}(\frac{\partial \alpha_I}{\partial \tilde{u}^b_K} \circ j\psi)\frac{\partial j\psi^b_K}{\partial u^a_I}\}dx^i \wedge d(\tilde{x}^I \circ \psi) \\
&= \{(\frac{\partial \alpha_I}{\partial \tilde{x}^j} \circ j\psi)d(\tilde{x}^j \circ \psi) + (\frac{\partial \alpha_I}{\partial \tilde{u}^b_K} \circ j\psi) \cdot (\frac{\partial j\psi^b_K}{\partial x^i} + u^a_{iI}\frac{\partial j\psi^b_K}{\partial u^a_I}) \\
&\quad dx^i\} \wedge d(\tilde{x}^I \circ \psi) \\
&= \{(\frac{\partial \alpha_I}{\partial \tilde{x}^j} \circ j\psi) + (\frac{\partial \alpha_I}{\partial \tilde{u}^b_K} \circ j\psi) \cdot (\frac{\partial j\psi^b_K}{\partial x^i} + u^a_{iI}\frac{\partial j\psi^b_K}{\partial u^a_I})(J^{-1})^i_j\} \\
&\quad d(\tilde{x}^j \circ \psi) \wedge d(\tilde{x}^I \circ \psi) \\
&= \{(\frac{\partial \alpha_I}{\partial \tilde{x}^j} \circ j\psi) + (\frac{\partial \alpha_I}{\partial \tilde{u}^b_K} \circ j\psi) \cdot (\tilde{u}^b_{Kj} \circ j\psi)\} \\
&\quad d(\tilde{x}^j \circ \psi) \wedge d(\tilde{x}^I \circ \psi) \\
&= \{(\tilde{D}_j\alpha_I) \circ j\psi\}(d(\tilde{x}^j \circ \psi) \wedge d(\tilde{x}^I \circ \psi)) = (j\psi)^*(d_H(\alpha)),
\end{aligned}$$

where we have assumed that $j\psi^b_K = \tilde{u}^b_K \circ j\psi$ and J is the Jacobian matrix of the transformation $\tilde{x}^\nu = \psi^\nu_M(x^\mu)$ as before. ∎

3.3 The effect of a canonical automorphism on the structure maps of the sh-Lie algebra

In this section we utilize the complex

$$0 \rightarrow \Omega_c^{0,0}(J^\infty E) \rightarrow \Omega_c^{1,0}(J^\infty E) \rightarrow \cdots \rightarrow \Omega_c^{n,0}(J^\infty E)$$

which we require to be exact. It is known that this is true when the bundle $E \rightarrow M$ is trivial. One can define an sh-Lie algebra on this "de Rham complex" by first defining a skew-symmetric linear mapping $\tilde{l}_2$ on $\Omega^{n,0}$ (recall Theorem 2.1). We may define such a mapping on $\Omega^{n,0}$ by

$$\tilde{l}_2(P\nu, Q\nu) = \omega^{ab}\mathbf{E}_a(P)\mathbf{E}_b(Q)\nu = \omega(\mathbf{E}(P), \mathbf{E}(Q))\nu. \tag{3.5}$$

Recall that for each automorphism ψ we have $(j\psi)^*(P\nu) = (P \circ j\psi)(\det\psi_M)\nu$. Therefore

$$\tilde{l}_2((j\psi)^*(P\nu), (j\psi)^*(Q\nu)) = (j\psi)^*(\tilde{l}_2(P\nu, Q\nu))$$

for all $P, Q \in Loc_E$ if and only if

$$\omega^{ab}\mathbf{E}_b((Q \circ j\psi)\det\psi_M)\mathbf{E}_a((P \circ j\psi)\det\psi_M) = [(\tilde{\omega}^{ab}\tilde{\mathbf{E}}_b(Q)\tilde{\mathbf{E}}_a(P)) \circ j\psi](\det\psi_M)$$

for all $P, Q \in Loc_E$, which by Lemma 3.2 is equivalent to

$$\omega^{ab}(\det\psi_M)^2 \frac{\partial\psi^d}{\partial u^b}\frac{\partial\psi^c}{\partial u^a}(\tilde{\mathbf{E}}_d(Q) \circ j\psi)(\tilde{\mathbf{E}}_c(P) \circ j\psi) =$$

$$(\tilde{\omega}^{ab} \circ j\psi)(\tilde{\mathbf{E}}_b(Q) \circ j\psi)(\tilde{\mathbf{E}}_a(P) \circ j\psi)(\det\psi_M).$$

The last equation is true for all $P, Q \in Loc_E$, so it is equivalent to

$$\tilde{\omega}^{ab} \circ j\psi = (\det\psi_M)\omega^{cd}\frac{\partial\psi_E^a}{\partial u^c}\frac{\partial\psi_E^b}{\partial u^d}$$

which in turn is equivalent to the covariance of ω. Now consider l_3 in degree 0. We have

$$\begin{aligned}
l_3((j\psi)^*(P\nu),(j\psi)^*(Q\nu),(j\psi)^*(R\nu)) &= s[\tilde{l}_2(\tilde{l}_2((j\psi)^*(P\nu),(j\psi)^*(Q\nu)),(j\psi)^*\\
&\quad (R\nu))]\\
&= s[\tilde{l}_2((j\psi)^*(\tilde{l}_2(P\nu,Q\nu)),(j\psi)^*(R\nu))]\\
&= s[(j\psi)^*(\tilde{l}_2(\tilde{l}_2(P\nu,Q\nu),R\nu))]\\
&= s[-(j\psi)^*(l_1 l_3(P\nu,Q\nu,R\nu))]\\
&= s[-l_1((j\psi)^* l_3(P\nu,Q\nu,R\nu))]
\end{aligned}$$

since in this case $l_1 = d_H$ so it commutes with the pull-back (Proposition 3.4). Proceeding using the identity

$$-s \circ l_1 = 1 + l_1 \circ s$$

the above becomes $(1 + l_1 \circ s)[(j\psi)^*(l_3(P\nu,Q\nu,R\nu))] =$

$$(j\psi)^*(l_3(P\nu,Q\nu,R\nu)) + l_1 \circ s[(j\psi)^*(l_3(P\nu,Q\nu,R\nu))].$$

So $l_3((j\psi)^*(P\nu),(j\psi)^*(Q\nu),(j\psi)^*(R\nu)) = (j\psi)^*(l_3(P\nu,Q\nu,R\nu))$ up to an exact form. We have shown:

Theorem 3.5 *Let $\psi : E \to E$ be an automorphism of E sending fibers to fibers, and let $j\psi : J^\infty E \to J^\infty E$ be its induced automorphism on $J^\infty E$. Then*

$$\tilde{l}_2((j\psi)^*\alpha,(j\psi)^*\beta) = (j\psi)^*(\tilde{l}_2(\alpha,\beta))$$

for all $\alpha, \beta \in \Omega^{n,0}(J^\infty E)$ *iff* ω *is covariant with respect to* ψ*. Moreover we then have*

$$l_3((j\psi)^*\alpha, (j\psi)^*\beta, (j\psi)^*\gamma) = (j\psi)^* l_3(\alpha, \beta, \gamma) + l_1(\delta),$$

for all $\alpha, \beta, \gamma \in \Omega^{n,0}(J^\infty E)$*, and for some* $\delta \in \Omega^{n-2,0}(J^\infty E)$*.*

Recall that l_2 and l_3 are *zero* in higher degrees, and so are the higher order maps for this case of field theory as in the second example of Chapter 2.

Chapter 4

Reduction I : The base manifold preserved

In this chapter we consider the action of a Lie group G on the bundle E such that the induced action on the base manifold M is the identity. We assume that the induced transformations on the space of local functionals are canonical for all $g \in G$. We then show how one obtains an (induced) sh-Lie structure on a corresponding reduced space when $\Omega_c^{*,0}$ is exact (recall that $\Omega_c^{*,0}$ is exact if the bundle $E \to M$ is trivial). A brief discussion of functional invariance is also included.

We apply the ideas of canonical transformations and reduction to a Poisson sigma model which appears at the end of this chapter. Poisson sigma models have proven to be of interest in many areas of physics. In particular they have been used to describe certain two-dimensional theories of gravity by Ikeda [14], topological field theories by Schaller and Strobl [23], and to obtain a path integral proof of Kontsevich's theorem on deformation quantization by Catanneo and Felder [10]. These are but a sample of the many authors who have made important contributions relating to these model theories.

4.1 Reduction and the existence of an sh-Lie structure on a reduced graded space

Let M be a manifold, $E \to M$ a vector bundle, and $J^\infty E$ the infinite jet bundle of E. Let G be a Lie group acting on E via automorphisms (as in Chapter 3) and hence inducing an action of G on $J^\infty E$. We assume the induced action $\hat{\psi}_g$ on ΓE is canonical with respect to the Poisson bracket of local functionals for all $g \in G$. Notice that G acts via canonical tranformations on the space of local functionals if and only if for every $j\psi_g$

$$\tilde{l}_2((j\psi_g)^* f_1, (j\psi_g)^* f_2) = (j\psi_g)^*(\tilde{l}_2(f_1, f_2)),$$

where $\tilde{l}_2$ is defined on the vector space $\Omega^{n,0}(J^\infty E)$ as in the Chapter 3 (in fact $\tilde{l}_2(f_1, f_2) = \dfrac{1}{2}[\omega(\mathbf{E}(f_1), \mathbf{E}(f_2)) - \omega(\mathbf{E}(f_2), \mathbf{E}(f_1))]$ by the skew-symmetry of ω, see also equation 3.5).

Definition Given an automorphism ψ of the bundle E, a differential form $\alpha \in \Omega^{k,l}(J^\infty E)$ is *ψ-invariant* if, and only if, $(j\psi)^*\alpha = \alpha$. If G acts on E via automorphisms $\psi_g : E \to E, g \in G$, then α is *G-invariant* if, and only if, it is ψ_g-invariant for all $g \in G$.

Let $\Omega^{k,l}_\psi(J^\infty E)$ denote the space of all ψ-invariant forms on $J^\infty E$ which are in $\Omega^{k,l}_c(J^\infty E)$, and let $\Omega^{k,l}_G(J^\infty E)$ denote the space of all G-invariant forms in $\Omega^{k,l}_c(J^\infty E)$.

One also needs the following:

Definition Assume that G acts on E such that $J^\infty E/G$ has a manifold structure and the projection map $\pi : J^\infty E \to J^\infty E/G$ is smooth. Then $\Omega_c^{k,0}(J^\infty E/G)$ is the subspace of k-forms $\alpha \in \Omega_c^k(J^\infty E/G)$ such that $\pi^*\alpha \in \Omega_G^{k,0}(J^\infty E)$, and $\Omega_c^{*,0}(J^\infty E/G)$ is the *reduced graded space* of the graded space $\Omega_c^{*,0}(J^\infty E)$ with respect to G.

Remark In this chapter we use π for the canonical projection map $\pi : J^\infty E \to J^\infty E/G$, *not* as the generic mapping π we use in other chapters as the bundle mapping from E to M.

Recall that in this chapter we assume that the map ψ_M representing the transformation of the independent variables ($\tilde{x}^\nu = \psi_M^\nu(x^\mu)$) is the identity for all $g \in G$. So acting on an element of $\Omega_c^{k,0}(J^\infty E)$ gives an element of the same space and the reduction to $\Omega_c^{*,0}(J^\infty E/G)$ makes sense.

Our assumption will also enable us to define a differential on the reduced graded space, and it will insure that the space $\Omega_c^{n,0}(J^\infty E)$ does not collapse to zero upon reduction (due to a reduction in the number of independent variables so that any n-form in $\Omega_c^{n,0}(J^\infty E/G)$ would be trivial) which is desired so that the induced sh-Lie structure would not necessarily be trivial.

Proposition 4.1 *If ψ_M is the identity map, then $\pi^* : \Omega_c^{k,0}(J^\infty E/G) \to \Omega_G^{k,0}(J^\infty E)$ is onto.*

Proof Notice that if ψ_M is the identity map one can choose coordinates $\{x^i\}$ on $J^\infty E/G$ such that $\pi^* x^i = x^i, i = 1, 2, \cdots, n$ (where by an abuse of notation we denote by x^i both coordinates on M and $J^\infty E/G$) are the coordinates on M. Now let $f_I dx^I \in \Omega_G^{k,0}(J^\infty E)$ be arbitrary where I is a multi-index with $|I| = k$. Then for every I, f_I is a smooth G-invariant function on $J^\infty E$. Thus there exist smooth functions α_I on $J^\infty E/G$ such that $f_I = \alpha_I \circ \pi$ for all I. Now $\pi^*(\alpha_I dx^I) = (\alpha_I \circ \pi)dx^I = f_I dx^I$. The result follows. ∎

Observe that if $\alpha \in \Omega_c^{k,0}(J^\infty E/G)$ and $\pi^*\alpha = \alpha \circ d\pi = 0$ then $\alpha = 0$ since $d\pi$ is onto. So we have:

Corollary 4.2 *If ψ_M is the identity map, then we have an isomorphism $\pi^* : \Omega_c^{k,0}(J^\infty E/G) \longrightarrow \Omega_G^{k,0}(J^\infty E)$.*

In this setting it can be shown that $\Omega_c^{*,0}(J^\infty E/G)$ is a complex with a differential $\hat{d}_H : \Omega_c^{m,0}(J^\infty E/G) \longrightarrow \Omega_c^{m+1,0}(J^\infty E/G)$ defined by

$$\hat{d}_H h = (\pi^*)^{-1}(d_H(\pi^* h)).$$

This is well-defined since $d_H(\pi^* h)$ is invariant under the group action which follows from the fact that $\pi^* h$ is invariant under the group action so that $(j\psi_g)^*(d_H(\pi^* h)) = d_H((j\psi_g)^*(\pi^* h)) = d_H(\pi^* h)$. Also notice that $\hat{d}_H \circ \hat{d}_H = 0$ easily follows from $d_H \circ d_H = 0$. So $\hat{d}_H$ is a well-defined differential.

Reduction hypothesis: *Assume that every invariant d_H-exact form is the horizontal differential of an invariant form. This hypothesis will guarantee that the reduced graded space with the differential $\hat{d}_H$ is exact. Subsequently we will determine sufficient conditions which will insure that this is true.*

This assumption will also yield the two conditions, (i) and (ii) below, that are needed to obtain the sh-Lie structure on the reduced graded space.

Lemma 4.3 *Suppose that $\Omega_c^{*,0}(J^\infty E)$ is exact. If for every d_H-exact form $\alpha \in \Omega_G^{k,0}(J^\infty E)$ there exists $\gamma \in \Omega_G^{k-1,0}(J^\infty E)$ such that $\alpha = d_H\gamma$, then the reduced graded space is exact.*

Proof Suppose that $\hat{d}_H\beta = 0$, then $\pi^*(\hat{d}_H\beta) = 0$ and by the definition of $\hat{d}_H$ this implies that $d_H(\pi^*\beta) = 0$. Now exactness of $\Omega_c^{*,0}$ implies that $\pi^*\beta = d_H\gamma$ for some γ, and γ can be chosen so that it is invariant by assumption since $d_H\gamma$ is, so $\gamma = \pi^*\tau$ for some τ. By the definition of $\hat{d}_H$ then $d_H(\pi^*\tau) = \pi^*(\hat{d}_H\tau)$ so that $\pi^*\beta = \pi^*(\hat{d}_H\tau)$ or

$$\pi^*(\beta - \hat{d}_H\tau) = 0$$

from which $\beta - \hat{d}_H\tau = 0$, and therefore $\beta = \hat{d}_H\tau$. (*Observe that* $\pi^*\delta = \delta \circ d\pi = 0$ *implies that* $\delta = 0$ *for* $\delta \in \Omega_c^{k,0}(J^\infty E/G)$ *since* $d\pi$ *is onto.*) ∎

Remark We have used the simplified notation $\Omega_c^{*,0}$ for $\Omega_c^{*,0}(J^\infty E)$.

Corollary 4.4 *Under the same hypotheses as in the preceding lemma, the subcomplex of G-invariant forms,* $\Omega_G^{*,0}(J^\infty E)$, *is exact.*

Now we proceed to find a mapping on the reduced space $\Omega_c^{n,0}(J^\infty E/G)$ analogous to and induced by $\tilde{l}_2$ on the space $\Omega_c^{n,0}(J^\infty E)$. Define $\hat{l}_2$ by

$$\hat{l}_2(f_1, f_2) = (\pi^*)^{-1}(\tilde{l}_2(\pi^* f_1, \pi^* f_2))$$

where $f_1, f_2 \in \Omega_c^{n,0}(J^\infty E/G)$. Notice that this is well-defined since $\tilde{l}_2(\pi^* f, \pi^* h)$ is invariant under the group action by the following calculation

$$(j\psi_g)^*\tilde{l}_2(\pi^* f, \pi^* h) = \tilde{l}_2((j\psi_g)^*(\pi^* f), (j\psi_g)^*(\pi^* h)) = \tilde{l}_2(\pi^* f, \pi^* h),$$

and the map $(\pi^*)^{-1}$ exists by Corollary 4.2.

Skew-symmetry and linearity of $\hat{l}_2$ follow from the skew-symmetry and linearity of $\tilde{l}_2$. Furthermore $\hat{l}_2$ satisfies

$$
\begin{array}{ll}
(i) & \hat{l}_2(\hat{d}_H k_1, h) = \hat{d}_H k_2, \\
(ii) & \displaystyle\sum_{\sigma \in unsh(2,1)} (-1)^\sigma \hat{l}_2(\hat{l}_2(f_{\sigma(1)}, f_{\sigma(2)}), f_{\sigma(3)}) = \hat{d}_H k_3,
\end{array}
$$

where $k_1 \in \Omega_c^{n-1,0}(J^\infty E/G)$ while $h, f_1, f_2, f_3 \in \Omega_c^{n,0}(J^\infty E/G)$, and for some $k_2, k_3 \in \Omega_c^{n-1,0}(J^\infty E/G)$. *Subsequently we will suppress some of the notation and assume the summands are over the appropriate unshuffles with their corresponding signs.*

To verify (i) notice that

$$
\begin{aligned}
\pi^*(\hat{l}_2(\hat{d}_H k_1, h)) &= \tilde{l}_2(\pi^*(\hat{d}_H k_1), \pi^* h) \\
&= \tilde{l}_2(d_H(\pi^* k_1), \pi^* h) \\
&= d_H K_2,
\end{aligned}
$$

for some $K_2 \in \Omega_c^{n-1,0}(J^\infty E)$. But by our assumption K_2 can be chosen to be invariant under the group action since $d_H K_2$ is, i.e., $K_2 = \pi^* k_2$ for some $k_2 \in \Omega_c^{n-1,0}(J^\infty E/G)$, and then $d_H K_2 = d_H(\pi^* k_2) = \pi^*(\hat{d}_H k_2)$ by the definition of $\hat{d}_H$. This implies that $\hat{l}_2(\hat{d}_H k_1, h) = \hat{d}_H k_2$. (*Recall that* $\pi^*\alpha = \alpha \circ d\pi = 0$ *implies that* $\alpha = 0$ *for* $\alpha \in \Omega_c^{n,0}(J^\infty E/G)$ *since* $d\pi$ *is onto.*)

While to verify (ii), notice that

$$
\begin{aligned}
\pi^*(\sum_\sigma \hat{l}_2(\hat{l}_2(f_{\sigma(1)}, f_{\sigma(2)}), f_{\sigma(3)})) &= \sum_\sigma \tilde{l}_2(\tilde{l}_2(\pi^* f_{\sigma(1)}, \pi^* f_{\sigma(2)}), \pi^* f_{\sigma(3)}) \\
&= d_H K_3,
\end{aligned}
$$

where the sum is over the unshuffles (2,1) with the corresponding permutation sign, and for some $K_3 \in \Omega_c^{n-1,0}(J^\infty E)$ and all $f_1, f_2, f_3 \in \Omega_c^{n,0}(J^\infty E/G)$. Again K_3 can be chosen to be invariant under the group action since $d_H K_3$ is, i.e., $K_3 = \pi^* k_3$ for some $k_3 \in \Omega_c^{n-1,0}(J^\infty E/G)$, and then $d_H K_3 = d_H(\pi^* k_3) = \pi^*(\hat{d}_H k_3)$ by the definition of $\hat{d}_H$. This verifies (ii).

We have shown (see Lemmas 2.3, 2.4 and Theorem 2.1):

Theorem 4.5 *Suppose that* $\Omega_c^{*,0}$ *is exact. Then there exists a skew-symmetric bilinear bracket on* $H_0(\hat{d}_H) \times H_0(\hat{d}_H)$ *that satisfies the Jacobi identity, where we use* $H_0(\hat{d}_H)$ *for* $H^n(\Omega_c^{*,0}(J^\infty E/G), \hat{d}_H)$. *This bracket is induced by the map* $\hat{l}_2$.

Theorem 4.6 *If* $\Omega_c^{*,0}$ *is exact then the skew-symmetric linear map* $\hat{l}_2$ *as defined above on the space* $\Omega_c^{n,0}(J^\infty E/G)$ *extends to an sh Lie structure on the graded space* $\Omega_c^{*,0}(J^\infty E/G)$.

4.2 Exactness of the reduced graded vector space

In this section we find sufficient conditions under which our reduction hypothesis in the last section holds. Thus we consider the question: If α is in the reduced space $\Omega_c^{k,0}(J^\infty E/G)$ and $\hat{d}_H\alpha = 0$, then is $\alpha = \hat{d}_H\beta$ for some β? Suppose $\hat{d}_H\alpha = 0$ for $\alpha \in \Omega_c^{k,0}(J^\infty E/G)$, then $d_H(\pi^*\alpha) = 0$ so that $\pi^*\alpha = d_H\gamma$ for some $\gamma \in \Omega_c^{k-1,0}(J^\infty E)$ since $\Omega_c^{*,0}(J^\infty E)$ is exact. Notice that $d_H\gamma$ is invariant under the group action (since $d_H\gamma = \pi^*\alpha$) so $d_H\gamma = j\psi_g^*(d_H\gamma)$ for all $g \in G$. But since d_H commutes with $j\psi_g^*$ by Proposition 3.4, we have $d_H\gamma = d_H(j\psi_g^*\gamma)$ for all $g \in G$. So

$$\gamma = j\psi_g^*\gamma + d_H\tau_g,$$

where $\tau_g \in \Omega_c^{k-2,0}(J^\infty E)$ depends on g. Consider $\gamma' = \gamma + d_H\Delta$ for some fixed $\Delta \in \Omega_c^{k-2,0}(J^\infty E)$ and notice that $d_H\gamma' = d_H\gamma = \pi^*\alpha$. Now $\gamma' =$

$\gamma + d_H\Delta = j\psi_g^*\gamma + d_H\tau_g + d_H\Delta$ so that $j\psi_g^*\gamma = \gamma' - d_H\tau_g - d_H\Delta$, and hence $j\psi_g^*\gamma' = j\psi_g^*\gamma + j\psi_g^*(d_H\Delta) = \gamma' - d_H\tau_g - d_H\Delta + j\psi_g^*(d_H\Delta)$. But if γ' is invariant under the group action then we must have $-d_H\Delta - d_H\tau_g + j\psi_g^*(d_H\Delta) = 0$, or

$$d_H(j\psi_g^*\Delta - \Delta - \tau_g) = 0,$$

(recall that d_H commutes with $j\psi_g^*$ by Proposition 3.4) so that $(j\psi_g^*\Delta - \Delta - \tau_g)$ must be *exact*. In this case, let β be such that $\gamma' = \pi^*\beta$, and notice that $\pi^*(\hat{d}_H\beta) = d_H\gamma' = \pi^*\alpha$ so that $\hat{d}_H\beta = \alpha$. Observe that τ_g depends on g and γ whereas Δ depends on γ.

Thus the existence of a $\Delta \in \Omega_c^{k-2,0}(J^\infty E)$ for every $\alpha \in \Omega_c^{k,0}(J^\infty E/G)$ (as above) such that $(j\psi_g^*\Delta - \Delta - \tau_g)$ is *exact* for all $g \in G$, is a necessary and sufficient condition for the exactness of the reduced graded space.

We find the above criterion too general and rather complicated, and find it useful to consider a special case. Suppose that G is *compact* and let $\alpha \in \Omega_c^{k,0}(J^\infty E)$ be a closed form that is invariant under the group action. By exactness of $\Omega_c^{*,0}$ there exists a β such that $d_H\beta = \alpha$. Observe that $d_H(j\psi_g^*\beta) = j\psi_g^*(d_H\beta) = j\psi_g^*\alpha = \alpha$ for all $g \in G$. So

$$\int_G d_H(j\psi_g^*\beta)dg = \int_G \alpha dg = \alpha \int_G dg = \alpha \cdot vol(G) = \alpha$$

assuming that $vol(G) = 1$. Now let

$$\hat{\beta} = \int_G (j\psi_g^*\beta)dg$$

and notice that $d_H\hat{\beta} = \displaystyle\int_G d_H(j\psi_g^*\beta)dg = \alpha$, and $j\psi_h^*(\hat{\beta}) = \displaystyle\int_G j\psi_h^*(j\psi_g^*\beta)dg = \displaystyle\int_G (j\psi_{gh}^*\beta)d(gh) = \hat{\beta}$. So we have:

Proposition 4.7 *If the group G acting (canonically) on E is compact and $\Omega_c^{*,0}(J^\infty E)$ is exact, then every d_H-closed form that is G-invariant is the horizontal differential of a G-invariant form. Consequently $\Omega_c^{*,0}(J^\infty E/G)$ is exact and admits an (induced) sh-Lie structure.*

4.3 The existence of an sh-Lie structure on the subcomplex of G-invariant forms

In this section we consider the subcomplex of G-invariant forms

$$\cdots \to \Omega_G^{n-1,0}(J^\infty E) \xrightarrow{d_H} \Omega_G^{n,0}(J^\infty E) \tag{4.1}$$

Working with the subcomplex of G-invariant forms is rather interesting. We shall maintain the same assumptions made earlier in this chapter, in particular the hypotheses of Lemma 4.3. Recall that throughout this chapter we require that the mapping ψ_M representing the tranformation of the independent variables be the identity.

In fact, if the base manifold M is one-dimensional these assumptions are not needed for the subcomplex (4.1) to be exact. However the absence of these assumptions does not guarantee the existence of an sh-Lie structure.

Recall that by Corollary 4.4 the subcomplex of G-invariant forms (4.1) is exact, and observe that $\tilde{l}_2(\alpha,\beta) = \tilde{l}_2((j\psi)^*\alpha,(j\psi)^*\beta) = (j\psi)^*\tilde{l}_2(\alpha,\beta)$ for all $\alpha,\beta \in \Omega_G^{n,0}(J^\infty E)$. So $\tilde{l}_2$ can be restricted to the subspace $\Omega_G^{n,0}(J^\infty E)$. Now notice that conditions (i) and (ii) that guarantee the existence of an sh-Lie structure, as stated earlier in this chapter, are readily established (this sh-Lie structure is just the restriction of the original one to the subcomplex of G-invariant forms). So we have:

Theorem 4.8 *Under the same hypotheses as in Lemma* 4.3*, there exists an sh-Lie structure on the subcomplex of G-invariant forms $\Omega_G^{*,0}(J^\infty E)$.*

Example Consider $M = \mathbf{R}, E = \mathbf{R} \times \mathbf{R}^2$, and let

$$\omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Consider the action of $G = SO(2)$ on E defined by $\psi_g(x, u^1, u^2) = (x, g \cdot (u^1, u^2)), g \in SO(2)$. As we noticed in the example in Chapter 3, ω is covariant with respect to ψ_g and hence the induced action is canonical for all $g \in G$. Consider a subset of $J^\infty E$ defined by $(J^\infty E)' = J^\infty E - \{u \in J^\infty E | u = (u^1_I, u^2_I) = (0,0), I = 0,1,2,3,\cdots\}$, where $u^1_0 = u^1, u^1_1 = u^1_x, u^1_2 = u^1_{xx}$, ...etc.

Notice that at a point in $(J^\infty E)'$ u^1 and u^2 cannot be zero at the same time, u^1_x and u^2_x cannot be zero at the same time, and so on. We note that $(J^\infty E)'/G = \mathbf{R} \times (\mathbf{R}^+) \times (\mathbf{R}^+) \times S^1 \times (\mathbf{R}^+) \times S^1 \times \cdots$, where $\mathbf{R}^+$ is the set of all positive real numbers (without the 0). As an illustration let $\alpha = u^1 dx$ and notice that $(j\psi_g)^*\alpha = (\cos\theta)u^1 + (\sin\theta)u^2$, whereas $\beta = (1/2)[(u^1)^2 + (u^2)^2]dx$ is invariant under the induced action of G and so is $\gamma = (1/2)[(u^1_x)^2 + (u^2_x)^2]dx$. As for the sh-Lie structure, first notice that the resolution space is rather simple:

$$0 \to Loc^0_E \to \Omega^{1,0}_c(J^\infty E).$$

One finds that

$$\tilde{l}_2(Pdx, Qdx) = (\mathbf{E}_1(P)\mathbf{E}_2(Q) - \mathbf{E}_2(P)\mathbf{E}_1(Q))dx,$$

while $l_2(Pdx, f) = 0$ for $f \in Loc_E^0$ since $l_2 l_1(Pdx, f) = l_2(l_1(Pdx), f) + l_2(Pdx, l_1 f) = 0 + l_2(Pdx, d_H f) = 0$, which follows from $\mathbf{E}_i(d_H f) = 0, i = 1, 2$. Further $l_2 = 0$ in higher degrees. We note that l_3 is non-zero in degree 0, but is zero in higher degrees. Let for example $P_1 = u^1 u_x^2, P_2 = u^1 u^2$, and $P_3 = (u^1)^2$, then $\tilde{l}_2 \tilde{l}_2(P_1 dx, P_2 dx, P_3 dx) = 4u^1 u_x^1 dx = d_H(2(u^1)^2)$, so that we can choose $l_3(P_1 dx, P_2 dx, P_3 dx) = -2(u^1)^2$.

The subcomplex of G-invariant forms is exact and so the sh-Lie structure can be restricted to it. Observe that, for example, $\tilde{l}_2(\beta, \gamma) = \tilde{l}_2((1/2)[(u^1)^2 + (u^2)^2]dx, (1/2)[(u_x^1)^2 + (u_x^2)^2]dx) = (-u^1 u_{xx}^2 + u^2 u_{xx}^1)dx$ is invariant under the induced group action of $G = SO(2)$. ∎

Remark In [17] Kogan and Olver provide a definitive account of invariant Euler-Lagrange equations using moving frames and other tools from differential geometry. We note that their invariantization map $\imath$ in our case will just map horizontal differentials to themselves (i.e. in local coordinates $\imath(dx^i) = dx^i$). Consequently the invariant derivatives are the same as the ordinary derivatives $\mathcal{D}_i = D_i$ (here we are borrowing some of the notation from [17]) whereas the twisted invariant adjoints of $\mathcal{D}_i$ turn out to be $\mathcal{D}_i^\dagger = -\mathcal{D}_i$. Now suppose that $I^1, \cdots, I^m$ is a fundamental set of differential invariants (on $J^\infty E$). Let $\hat{I}^1, \cdots, \hat{I}^m$ be coordinates on $J^\infty E/G$ such that $\hat{I}^k \circ \pi = I^k, k = 1, \cdots, m$ where π is the canonical projection map $\pi : J^\infty E \to J^\infty E/G$, and let $\hat{L}$ be the corresponding Lagrangian defined on $J^\infty E/G$ for some (invariant) Lagrangian $L = \hat{L} \circ \pi$ defined on $J^\infty E$. The *invariant Eulerian* (in [17]) is defined by

$$\mathcal{E}_\alpha(\tilde{L}) = \sum_K \mathcal{D}_K^\dagger \frac{\partial \tilde{L}}{\partial I_{,K}^\alpha}$$

where $\tilde{L}$ indicates the Lagrangian is written in terms of the $I^\alpha_{,K}$'s. But this operator $\mathcal{E}_\alpha$, in our case, reduces to

$$\mathcal{E}_\alpha(\tilde{L}) = \sum_K (-D)_K \frac{\partial \tilde{L}}{\partial I^\alpha_{,K}}.$$

A *corresponding invariant Eulerian* can be defined on $J^\infty E/G$. To accomplish this, first define the "total derivative" $\hat{D}_k$ on $J^\infty E/G$ by $(\hat{D}_k \hat{P}) \circ \pi = D_k(\hat{P} \circ \pi)$ where $\hat{P}$ is a smooth function on $J^\infty E/G$. Now let $\hat{I}^\alpha_{,K} = \hat{D}_K \hat{I}^\alpha$ and notice that $\hat{I}^\alpha_{,K} \circ \pi = I^\alpha_{,K} = D_K I^\alpha$. Finally, define

$$\hat{\mathcal{E}}_\alpha(\hat{L}) = \sum_K (-\hat{D})_K \frac{\partial \hat{L}}{\partial \hat{I}^\alpha_{,K}}$$

and observe that $\hat{\mathcal{E}}_\alpha(\hat{L}) \circ \pi = \mathcal{E}_\alpha(\tilde{L})$. The reader should consult [17] for more details.

4.4 Functional invariance

In this section we consider the implications of invariance on the space of local functionals. Assume that ψ is canonical, i.e.,

$$\{\mathcal{P} \circ \hat{\psi}, \mathcal{Q} \circ \hat{\psi}\} = \{\mathcal{P}, \mathcal{Q}\} \circ \hat{\psi}$$

for all functionals $\mathcal{P}, \mathcal{Q}$.

Definition We say that $\mathcal{P}$ is *invariant* under ψ if, and only if, $\mathcal{P} \circ \hat{\psi} = \mathcal{P}$.

Notice that this holds if, and only if

$$\int_M (P \circ j\psi \circ j\phi) \det \psi_M \nu = \int_M (P \circ j\phi)\nu$$

for all $\phi \in \Gamma E$, which in turn holds if $(P \circ j\psi)\det\psi_M = P$. Similarly one says that $P \in Loc_E^0$ is *invariant* under ψ if, and only if, $(P \circ j\psi)\det\psi_M = P$. Let $\mathcal{F}_\psi$ denote the set of all functionals $\mathcal{P}$ such that $\mathcal{P} \circ \hat{\psi} = \mathcal{P}$. Observe that

$$\mathcal{P}, Q \in \mathcal{F}_\psi \Rightarrow \{\mathcal{P}, Q\} \in \mathcal{F}_\psi$$

so $\mathcal{F}_\psi$ is a Lie subalgebra of $\mathcal{F}$. Let $Loc_E^0(\psi)$ denote the subset of Loc_E^0 consisting of $P \in Loc_E^0$ such that

$$P = (P \circ j\psi)\det\psi_M.$$

We note that $Loc_E^0(\psi)$ is a subspace of Loc_E^0, while for automorphisms ψ such that $\det\psi_M = 1, Loc_E^0(\psi)$ is a subalgebra of Loc_E^0.

Proposition 4.9 *If $P\nu \in \Omega^{n,0}(J^\infty E)$ is ψ-invariant for an automorphism ψ, then so is $\mathbf{E}(P\nu)$.*

Proof In local coordinates $\mathbf{E}(P\nu) = \mathbf{E}_a(P)(\theta^a \wedge \nu)$. So

$$\begin{aligned}
(j\psi)^*(\mathbf{E}(P\nu)) &= (j\psi)^*(\mathbf{E}_a(P))(j\psi)^*(\theta^a \wedge \nu) \\
&= (\mathbf{E}_a(P) \circ j\psi)(\frac{\partial \psi_E^a}{\partial u^b}\theta^b \wedge (\det\psi_M)\nu) \\
&= (\mathbf{E}_a(P) \circ j\psi)\frac{\partial \psi_E^a}{\partial u^b}(\det\psi_M)(\theta^b \wedge \nu).
\end{aligned}$$

Now, since $P\nu$ is ψ-invariant we have

$$\begin{aligned}
P\nu &= (j\psi)^*(P\nu) \\
&= (j\psi)^*(P)(j\psi)^*\nu \\
&= (P \circ j\psi)(\det\psi_M)\nu,
\end{aligned}$$

and therefore $(P \circ j\psi)\det\psi_M = P$. Finally,

$$\begin{aligned}
\mathbf{E}(P\nu) &= \mathbf{E}_a(P)(\theta^a \wedge \nu) \\
&= \mathbf{E}_a((P \circ j\psi)\det\psi_M)(\theta^a \wedge \nu) \\
&= (\det\psi_M)\frac{\partial \psi_E^b}{\partial u^a}(\mathbf{E}_b(P) \circ j\psi)(\theta^a \wedge \nu) \\
&= (j\psi)^*(\mathbf{E}(P\nu)),
\end{aligned}$$

where we have used Lemma 3.2 in the last calculation. ∎

If G is a Lie group which acts on E via canonical automorphisms ψ_g, for all $g \in G$, then we write

$$Loc_E^0(G) = \bigcap_{g \in G} Loc_E^0(\psi_g) \quad , \quad \mathcal{F}_G = \bigcap_{g \in G} \mathcal{F}_{\psi_g}.$$

Clearly $\mathcal{F}_G$ is a Lie sub-algebra of $\mathcal{F}$, and if $P \in Loc_E^0(G)$, then $\mathbf{E}(P\nu)$ is G-invariant.

Notice that if ϕ is a section of the bundle $E \to M$ then $j^\infty\phi$ is a section of $\pi^\infty : J^\infty E \to M$. Sections of this type are said to be *holonomic* as they are induced by sections of $E \to M$. It is easily shown that not all sections of π^∞ are holonomic. Observe that $\pi \circ j^\infty\phi$ is a section of the bundle $\tau : J^\infty E/G \to M$ since $\pi^\infty = \tau \circ \pi$. Similarly we say that a section η of τ is *holonomic* if it has the form $\eta = \pi \circ j^\infty\phi$ for some section $j^\infty\phi$ of π^∞. Let Γ denote the set of all holonomic sections of τ. Note that Γ is not a linear space over $\mathbf{R}$ since π is not linear. Indeed $J^\infty E/G$ is generally not a vector bundle.

Definition We say that $\tilde{\mathcal{P}}$ is a *reduced local functional* if it is a mapping from the set Γ of holonomic sections of the bundle $\tau : J^\infty E/G \to M$ into $\mathbf{R}$

such that

$$\tilde{\mathcal{P}}(\eta) = \int_M \eta^*(\tilde{P})\nu$$

for some smooth mapping $\tilde{P} : J^\infty E/G \to \mathbf{R}$ and for every $\eta \in \Gamma$. We denote the set of all reduced local functionals by $\tilde{\mathcal{F}}$.

In this definition, when we say that $\tilde{P} : J^\infty E/G \to \mathbf{R}$ is smooth we mean that $\tilde{P} \circ \pi$ is in Loc_E^0.

Proposition 4.10 *There is a bijection Ξ from $\tilde{\mathcal{F}}$ onto $\mathcal{F}_G$. The mapping Ξ is defined as follows: if $\tilde{\mathcal{P}}$ is defined by*

$$\tilde{\mathcal{P}}(\eta) = \int_M \eta^*(\tilde{P})\nu$$

for some smooth mapping $\tilde{P}$, then $\Xi(\tilde{\mathcal{P}}) = \mathcal{P}$ is defined by

$$\mathcal{P}(\phi) = \int_M (j^\infty\phi)^*(P)\nu$$

where $P = \tilde{P} \circ \pi$.

The proof of the proposition is straightforward and is omitted.

Remark It follows from the proposition that the set $\tilde{\mathcal{F}}$ of reduced local functionals inherits a Lie-structure from that on $\mathcal{F}_G$. In the sequel it is identified with $\mathcal{F}_G$.

Notice that by Proposition 4.9 the complex

$$\Omega_c^{0,0} \xrightarrow{d_H} \Omega_c^{1,0} \xrightarrow{d_H} \cdots \xrightarrow{d_H} \Omega_c^{n-1,0} \xrightarrow{d_H} \Omega_c^{n,0} \xrightarrow{\mathbf{E}} \Omega_c^{n,1} \to \cdots$$

induces a subcomplex

$$\Omega_\psi^{0,0} \xrightarrow{d_H} \Omega_\psi^{1,0} \xrightarrow{d_H} \cdots \xrightarrow{d_H} \Omega_\psi^{n-1,0} \xrightarrow{d_H} \Omega_\psi^{n,0} \xrightarrow{\mathbf{E}} \Omega_\psi^{n,1} \to \cdots$$

This subcomplex is exact up to the term $\Omega_{\psi}^{n-1,0}$ (and including it, i.e., $H_{\psi}^{n-1} = 0$), if we assume that $\Omega_{c}^{*,0}$ is itself exact and that every exact ψ-invariant form is the horizontal differential of some ψ-invariant form. Similarly, the subcomplex

$$\Omega_G^{0,0} \xrightarrow{d_H} \Omega_G^{1,0} \xrightarrow{d_H} \cdots \xrightarrow{d_H} \Omega_G^{n-1,0} \xrightarrow{d_H} \Omega_G^{n,0} \xrightarrow{\mathbf{E}} \Omega_G^{n,1} \rightarrow \cdots$$

is exact up to the term $\Omega_G^{n-1,0}$ (under the same assumptions).

4.5 An example: A Poisson sigma model

A number of authors [10, 14, 23] have investigated a class of physical theories called Poisson sigma models. These models focus on fields which are defined on a 2-dimensional manifold Σ with range in a Poisson manifold M. These models seem to have first arisen in various theories of 2-dimensional gravity but have been applied to areas such as topological field theory and in the reformulation of Kontsevich's work on deformation quantization [10]. We consider an application of our results to the version of the Poisson sigma model presented in the work of Ikeda [14] but we utilize the notation of [6].

Assume that V is a finite-dimensional vector space, say of dimension N and with basis $\{T_A\}$, and let $\{T^A\}$ denote the basis of the space V^* dual to V. We assume the existence of a Poisson tensor W on V^*. Thus W is a bivector field

$$W = W_{AB}(\frac{\partial}{\partial T_A} \wedge \frac{\partial}{\partial T_B})$$

where, for each A, T_A is identified as a coordinate mapping $T_A : V^* \rightarrow \mathbf{R}$ and V is identified with V^{**}. The fact that W is Poisson means that it is a tensor and the $\{W_{AB}\}$ are functions on V^* (assumed to be polynomials in the coordinates $\{T_A\}$ in the present model) such that

$$W_{AD}\frac{\partial W_{BC}}{\partial T_D} + W_{BD}\frac{\partial W_{CA}}{\partial T_D} + W_{CD}\frac{\partial W_{AB}}{\partial T_D} = 0$$

and $W_{AB} = -W_{BA}$. Now V^* is a Poisson manifold with

$$\{f, g\} = W_{AB}\frac{\partial f}{\partial T_A}\frac{\partial g}{\partial T_B}$$

for smooth functions f, g defined on V^*.

Observe that the Poisson field W is not dependent on the basis used to represent it. If the components W_{AB} of W relative to a basis $\{T_A\}$ of V satisfy the Poisson conditions given above then the components $\overline{W}_{AB}$ of W relative to any other basis $\{\overline{T}_B\}$ of V will also satisfy these same conditions. Notice that $W_{AB} = \{T_A, T_B\}$ and that if $\{\overline{T}_A\}$ is a different basis and $\{\overline{W}_{AB}\}$ are the components of W relative to it then $\overline{W}_{AB} = \{\overline{T}_A, \overline{T}_B\}$ as well.

The fields of Ikeda's model are ordered pairs (ψ, h) where ψ is a mapping from the 2-dimensional manifold Σ into V^* and h is a mapping from Σ into $T^*\Sigma \otimes V$. In components

$$\psi(x) = \psi_A(x)T^A \quad , \quad h(x) = h^A_\mu(x)(dx^\mu \otimes T_A)$$

where $\{x^\mu\}$ are coordinates on Σ. One form of Ikeda's Lagrangian for 2-dimensional gravity is

$$L = \epsilon^{\mu\nu}[h^A_\mu D_\nu \psi_A - \frac{1}{2}W_{AB}h^A_\mu h^B_\nu],$$

where ϵ is the skew-symmetric Levi-Civita tensor on Σ such that $\epsilon^{01} = 1$ and

$$D_\nu \psi_A = \partial_\nu \psi_A + W_{AB}h^B_\nu.$$

It is our intent to show how some of our work relates to Ikeda's model. To cast this model in our formalism let E denote the vector bundle over Σ with total space $E = V^* \oplus [T^*\Sigma \otimes V]$ and with the obvious projection of E onto Σ. The fields (ψ, h) are sections of this bundle.

First we show how to define a Poisson bracket on the relevant space of local functionals. To accomplish this, we want to construct a mapping ω as in Chapter 3 in such a manner that the Jacobi condition is satisfied.

For this purpose we find it convenient to introduce a positive definite metric μ on V with its induced metric μ^* on V^*. Moreover we choose $\{T_A\}$ to be an orthonormal basis relative to μ and we define $\{T^B\}$ by $T^B(v) = \mu(v, T_B)$ for each B and for all $v \in V$. It follows that $\{T^A\}$ is a μ^*-orthonormal basis of V^* and that the basis $\{T^A\}$ is dual to $\{T_B\}$. Define a tensor $\tilde{W}$ on V by

$$\tilde{W} = W^{AB}\left(\frac{\partial}{\partial T^A} \wedge \frac{\partial}{\partial T^B}\right)$$

where $W^{AB} = \mu^{AC}\mu^{BD}W_{CD}$, and $\mu^{PQ} = \mu^*(T^P, T^Q)$ for $1 \leq P, Q \leq N$. Thus $\tilde{W}$ is the tensor on V induced by W and the metric μ.

We reformulate this data in terms of the jet bundle of E. In particular the tensors $W, \tilde{W}$ induce a bilinear mapping ω on the jet bundle which is used to define a Lie structure on the space of local functionals.

Local coordinates on E may be denoted (x^μ, u_A, w_μ^B) and those on the jet bundle $J^\infty E$ by $(x^\mu, u_{A_I}, w^B_{\mu,J})$. Thus if (ψ, h) is a section of E we have

$$x^\mu((\psi, h)(p)) = x^\mu(p) \quad , \quad u_A((\psi, h)(p)) = \psi_A(p)$$

and

$$w_\mu^B((\psi, h)(p)) = h_\mu^B.$$

Clearly, there is a corresponding splitting of the jet coordinates. It follows that in local coordinates each local function P on $J^\infty E$ is a function of $(x^\mu, u_{A,I}, w^B_{\nu,J})$. Now the $\{W_{AB}\}$ are functions of the coordinates $\{T_A\}$ on V and these coordinates are denoted $\{u_A\}$ on the bundle E. Consequently we can regard the $\{W_{AB}\}$ as being functions on the jet bundle $J^\infty E$ which depend polynomially on the coordinates $\{u_A\}$ and are in fact independent of the coordinates $x^\mu, u_{A,I}, w^B_{\mu,J}$ for $|I| \geq 1$. The function ω is required to be a mapping from $\Omega^{n,1}_0 \times \Omega^{n,1}_0$ into Loc_E. Observe that there are two types of contact forms θ_A, θ^B_μ on $J^\infty E$, those which arise from the coordinates $\{u_A\}$ and those which arise from $\{w^B_\mu\}$. Since each fiber of E is a direct sum of two vector spaces the matrix of components of ω is a block diagonal matrix with two blocks defined by

$$\omega_{AB} = W_{AB} \quad \text{and} \quad \omega^{A,B}_{\mu,\nu} = \delta_{\mu\nu} W^{AB}.$$

Here $\delta_{\mu\nu}$ is the usual Kronecker delta symbol. In the first block we have written the indices of ω^{ab} as lower indices as they represent components relative to a basis of the dual of V. In the second block it is appropriate to write the components $\omega^{(\mu,A),(\nu,B)}$ as $\omega^{A,B}_{\mu,\nu}$ for similar reasons. Notice that the matrix of ω is skew-symmetric and

$$\omega_{AD}\frac{\partial \omega_{BC}}{\partial u_D} + \omega_{BD}\frac{\partial \omega_{CA}}{\partial u_D} + \omega_{CD}\frac{\partial \omega_{AB}}{\partial u_D} = 0.$$

The other components of ω satisfy a similar conditon as they too are determined by the $\{W_{AB}\}$. It follows from this fact and equation (7.11) of [20] that the bracket of local functionals defined on sections of E by

$$\{\mathcal{P}, \mathcal{Q}\}(\phi) = \int_\Sigma [\omega^{ab}\mathbf{E}_a(P)\mathbf{E}_b(Q)] \circ (j^\infty \phi)\nu$$

satifies the Jacobi identity (also see equation (3.1) and the discussion in section 3.1).

Ikeda shows that the Euler operators for the Lagrangian L in this model are given by

$$\mathbf{E}^A(L) = \epsilon^{\mu\nu} R^A_{\mu\nu} \quad , \quad \mathbf{E}^\mu_A(L) = \epsilon^{\mu\nu} D_\nu \psi_A$$

where

$$R^A_{\mu\nu} = \partial_\mu h^A_\nu - \partial_\nu h^A_\mu + \frac{\partial W_{BC}}{\partial T_A} h^B_\mu h^C_\nu .$$

This suggests that for every local function P we should define

$$\mathbf{E}^A(P) = (-D)_I(\frac{\partial P}{\partial u_{A,I}}) \quad , \quad \mathbf{E}^\mu_B(P) = (-D)_J(\frac{\partial P}{\partial w^B_{\mu,J}}).$$

Consequently, the Poisson bracket assumes the following form:

$\{\mathcal{P}, \mathcal{Q}\}(\psi, h) =$

$$\int_\Sigma [\omega_{AB} \mathbf{E}^A(P) \mathbf{E}^B(Q)] \circ j^\infty(\psi, h)\nu + \int_\Sigma [\omega^{A,B}_{\mu,\nu} \mathbf{E}^\mu_A(P) \mathbf{E}^\nu_B(Q)] \circ j^\infty(\psi, h)\nu.$$

Now we characterize the automorphisms of E that induce canonical transformations, relative to the Poisson bracket, on the space of local functionals $\mathcal{F}$. Recall that in section 3.2 we have referred to such automorphisms as canonical automorphisms. Suppose that Ψ is a linear automorphism of E, i.e., assume that there are matrices R, S which are functions on Σ such that

$$\Psi([T^A \oplus (dx^\mu \otimes T_B)]) = (R^A_C T^C) \oplus (dx^\mu \otimes (S^D_B T_D)).$$

We determine conditions which insure that Ψ is a canonical automorphism of E. According to Lemma 3.1 such an automorphism will be canonical if, and only if, its components satisfy condition (ii) of the Lemma.

Observe that $\Psi_A = u_A \circ \Psi = R_A^C u_C$, $\Psi_\mu^A = w_\mu^A \circ \Psi = S_D^A w_\mu^D$ and

$$\frac{\partial \Psi_A}{\partial u_B} = R_A^B \quad , \quad \frac{\partial \Psi_\mu^A}{\partial w_\nu^B} = S_B^A \delta_\mu^\nu .$$

Consequently, if the matrices satisfy the conditions

$$\overline{W}_{AB} = W_{CD} R_A^C R_B^D \quad , \quad \overline{W}^{AB} = W^{CD} S_C^A S_D^B$$

where $\overline{W}_{AB}$ are the components of the tensor W relative to a new basis $\overline{T}_A = M_A^C T_C$ and $\overline{W}^{AB}$ are the components of the tensor $\tilde{W}$ relative to the dual basis $\overline{T}^A = (M^{-1})_D^A T^D$, then

$$\tilde{\omega}_{AB} = \omega(\tilde{\theta}_A, \tilde{\theta}_B) = \overline{W}_{AB} = W_{CD} R_A^C R_B^D = \omega(\theta_C, \theta_D) \frac{\partial \Psi_A}{\partial u_C} \frac{\partial \Psi_B}{\partial u_D} .$$

(In the equation above we have dropped the volume form ν from the definition of the components of ω for simplicity, see section 3.1.) Similarly, we must have

$$\tilde{\omega}_{\mu,\nu}^{A,B} = \omega(\tilde{\theta}_\mu^A, \tilde{\theta}_\nu^B) = \delta_{\mu\nu} \overline{W}^{AB} = W^{CD} \frac{\partial \Psi_\mu^A}{\partial w_\lambda^C} \frac{\partial \Psi_\nu^B}{\partial w_\rho^D} \delta_{\lambda\rho}$$

which is the same as

$$\tilde{\omega}_{\mu,\nu}^{A,B} = \omega_{\lambda,\rho}^{C,D} \frac{\partial \Psi_\mu^A}{\partial w_\lambda^C} \frac{\partial \Psi_\nu^B}{\partial w_\rho^D} .$$

Notice that these computations will be consistent if we require that $R = M$ and $S = M^{-1}$ since M is the matrix transforming the basis $\{T_A\}$ to $\{\overline{T}_A\}$, and since we require that W and $\tilde{W}$ be tensors. Moreover we must also have that M be orthogonal if we want the transformed basis to remain μ-orthonormal. These remarks give us the conditions required in order that a linear automorphism be canonical.

If G is a Lie group and $M : G \to O(n)$ is a *representation* of G by orthogonal matrices then there is a representation Φ of G via canonical automorphisms of $E = V^* \oplus (T^*\Sigma \otimes V)$ defined by

$$\Phi(g)([T^A \oplus (dx^\mu \otimes T_B)]) = (M(g)^A_C T^C) \oplus (dx^\mu \otimes ([M(g)^{-1}]^D_B T_D)).$$

The fact that $\Phi : G \to Aut(E)$ is a *group homomorphism* is a consequence of the fact that M defines a linear left action of G on V via

$$g \cdot T_A = [M(g)^{-1}]^B_A T_B$$

with a corresponding linear left action of G on V^* defined by

$$g \cdot T^A = M(g)^A_B T^B.$$

The following theorem is a consequence of these remarks.

Theorem 4.11 *For each orthogonal $n \times n$ matrix M there is a canonical gauge automorphism Ψ_M of the bundle $V^* \oplus (T^*\Sigma \otimes V) \longrightarrow \Sigma$ which is linear on fibers of the bundle and which transforms the basis $\{T^A \oplus (dx^\mu \otimes T_B)\}$ via*

$$\Psi_M([T^A \oplus (dx^\mu \otimes T_B)]) = (M^A_C T^C) \oplus (dx^\mu \otimes ((M^{-1})^D_B T_D)).$$

Moreover, if $M : G \to O(n)$ is a representation of a Lie group G by orthogonal $n \times n$ matrices, then the mapping $\Phi : G \to Aut(V^* \oplus (T^*\Sigma \otimes V))$ defined by $\Phi(g) = \Psi_{M(g)}$ for $g \in G$, is a representation of G by canonical automorphisms of $V^* \oplus (T^*\Sigma \otimes V)$. It follows that the space of local functionals defined on sections of the bundle $V^* \oplus (T^*\Sigma \otimes V) \to \Sigma$ admits a reduction as does the complex $\{\Omega_c^{k,0}[J^\infty(V^* \oplus (T^*\Sigma \otimes V))], k = 0, 1, 2, \cdots, n\}$.

Remark It is not difficult to show that Ikeda's Lagrangian given above is invariant under the action of the Lie group G defined in Theorem 4.11. This action is not physically significant as one can "freeze out" the gauge freedom by choosing a basis of V once for all. It could be interesting to elevate this global (rigid) symmetry to a local symmetry, thereby introducing compensating fields. We hope to determine the significance of such fields in future work.

Chapter 5

Reduction II : Symmetry reduction

In this chapter we use the ideas of symmetry reduction as in [3] where the action of a connected Lie group G on the bundle $\pi : E \to M$ is transversal to the fibers of E. The essence of symmetry reduction, in our context, is finding a cochain map between two cochain complexes. Specifically, we find a cochain map between the variational bicomplexes of $J^\infty E$ and $J^\infty \overline{E}$ where $\overline{E} = E/G$. In this case the reduced graded vector space is $\Omega_c^{*,0}(J^\infty \overline{E})$.

As before, we assume that G acts on E via automorphisms and that it induces an action of G on $J^\infty E$. We also assume the induced action $\hat{\psi}_g$ on ΓE is canonical with respect to the Poisson bracket of local functionals for all $g \in G$. Recall that G acts via canonical tranformations on the space of local functionals if, and only if, for every $j^\infty \psi_g$

$$\tilde{l}_2((j^\infty \psi_g)^* f_1, (j^\infty \psi_g)^* f_2) = (j^\infty \psi_g)^* (\tilde{l}_2(f_1, f_2)),$$

where $\tilde{l}_2$ is defined on the vector space $\Omega^{n,0}(J^\infty E)$ as in equation 3.5, i.e., $\tilde{l}_2(P\nu, Q\nu) = \omega^{ab}\mathbf{E}_a(P)\mathbf{E}_b(Q)\nu = \omega(\mathbf{E}(P), \mathbf{E}(Q))\nu$.

We begin with an introduction of symmetry reduction. Then we show how one gets an sh-Lie structure on the reduced complex. Finally, in the last section of this chapter, we discuss how local functionals are reduced in this context.

5.1 Symmetry reduction

We begin with an introduction of the basic ideas of symmetry reduction and refer the reader to [3] for more details. Suppose that G is a p-dimensional Lie group acting on E and inducing an action on M. Also suppose that the dimension of M is n and that of E is $n+m$. We assume that G acts transversally to the fibers of E, and that it acts projectably on $\pi : E \to M$, and regularly and effectively on both E and M with orbits of dimension $q < n$. It follows that the quotient spaces $\overline{E} = E/G$ and $\overline{M} = M/G$ are smooth manifolds of dimensions $n+m-q$ and $n-q$ respectively, and that the following diagram commutes with all smooth maps.

$$\begin{array}{ccccc} & E & \xrightarrow{\pi_{\overline{E}}} & \overline{E} & \\ \pi & \downarrow & & \downarrow & \overline{\pi} \\ & M & \xrightarrow{\pi_{\overline{M}}} & \overline{M} & \end{array}$$

On E one uses local coordinates (x^i, u^a) and local coordinates $(\hat{x}^i, y^r, v^\alpha)$ adapted to G such that the locally G-invariant sections (as defined below) of $\pi : E \to M$ are given by $v^\alpha = f(y^r)$. So, if for example, $G = SO(3)$ acts on $M = \mathbf{R}^3 - \{0\}$ and $E = M \times \mathbf{R}^2$ then (x, y, z, u^1, u^2) are coordinates on E

while

$$\hat{x} = x \quad \hat{y} = y \quad r = \sqrt{x^2 + y^2 + z^2} \quad v^1 = u^1 \quad v^2 = u^2$$

are coordinates on an open subset of E adapted to G where the last three, namely (r, v^1, v^2), can serve as coordinates on $\overline{E}$.

Let $J^\infty \overline{E}$ denote the infinite jet bundle of $\overline{E}$, and let $\Omega^{r,s}_{\mathrm{pr}G}(J^\infty E)$ denote the subspace of $\Omega^{r,s}(J^\infty E)$ that consists of the forms that are invariant under the prolonged action of G.

Let $\mathfrak{g}$ denote the Lie algebra of G, and let Γ denote the vector space of infinitesimal generators of the action of G on E. A differential form γ on $J^\infty E$ is an invariant of the action if $\mathcal{L}_{\mathrm{pr}X}\gamma = 0$ for all $X \in \Gamma$, and where $\mathrm{pr}X$ is the prolongation of X to $J^\infty E$.

A vector X_σ at $\sigma = j^\infty(s)(p) \in J^\infty E$, where s is a local section of the bundle $\pi : E \to M$, is called a *total vector* at σ if

$$X_\sigma(f) = [(\pi^\infty_M)_*(X_\sigma)](f \circ j^\infty(s))$$

for all $f : U \to \mathbf{R}$ where U is any subset of $J^\infty E$ containing σ, and where $\pi^\infty_M : J^\infty E \to M$ is the usual projection. We denote the space of all total vector fields on $J^\infty E$ by $\mathrm{Tot}(J^\infty E)$. If X is a vector field on M then $\mathrm{tot}X$ denotes the lifting of X to a total vector field on $J^\infty E$. In local coordinates total vectors are spanned by the total derivatives $D_i = \dfrac{\partial}{\partial x^i} + u^a_{iJ}\dfrac{\partial}{\partial u^a_J}, i = 1, 2, \cdots, n$.

A section s of $\pi : E \to M$ (here we use s for sections rather than ϕ which was used in previous chapters) is locally G-invariant if for all $g \in G$ sufficiently close to the identity $g \cdot [s(g^{-1} \cdot p)] = s(p)$. The jet space of

G-invariant local sections of E is the bundle $\mathrm{Inv}_G^\infty(E) \to M$ defined by

$$\mathrm{Inv}_G^\infty(E) = \{\sigma \in J^\infty E | \sigma = j^\infty(s)(p), \quad s \text{ is a locally } \\ G\text{-invariant section of } E\}.$$

In local coordinates $(\hat{x}^i, y^r, v^\alpha)$ of E adapted to G the locally G-invariant sections of $\pi : E \to M$ are given by $v^\alpha = f(y^r)$ and therefore

$$\mathrm{Inv}_G^\infty(E) = \{\sigma = (\hat{x}^i, y^r, v^\alpha, v_i^\alpha, v_r^\alpha, v_{ij}^\alpha, v_{ir}^\alpha, v_{rs}^\alpha, \cdots) | v_i^\alpha = 0, v_{ij}^\alpha = 0, v_{ir}^\alpha = 0, \cdots\}.$$

If $s : M \to E$ is a G-invariant local section then there exists a unique local section of $\overline{\pi}$ such that

$$\overline{s}(\pi_{\overline{M}}(p)) = \pi_{\overline{E}}(s(p)). \tag{5.1}$$

This correspondence between G-invariant local sections of π and local sections of $\overline{\pi}$ induces a projection map $\Pi : \mathrm{Inv}_G^\infty(E) \to J^\infty\overline{E}$ defined by $\Pi(j^\infty(s)(p)) = j^\infty(\overline{s})(\pi_{\overline{M}}(p))$, see [3].

One can describe the correspondence between G-invariant objects on $\pi : E \to M$ and the associated objects on $\overline{\pi} : \overline{E} \to \overline{M}$. If $s : M \to E$ is G-invariant then we define $\overline{s} = \varrho(s)$ to be the unique section of $\overline{\pi}$ satisfying 5.1.

A G-invariant form α on E satisfying $X \lrcorner\, \alpha = 0$ for all X in Γ is said to be G-*basic*. If α is G-basic then there exists a unique form $\overline{\alpha} = \varrho(\alpha)$ on $\overline{E}$ such that

$$\alpha = (\pi_{\overline{E}})^*(\overline{\alpha}).$$

If $f : J^\infty E \to \mathbf{R}$ is a G-invariant function then there is a unique function $\overline{f} : J^\infty\overline{E} \to \mathbf{R}$ satisfying $\overline{f}(\overline{\sigma}) = f(\sigma)$ where $\overline{\sigma} \in J^\infty\overline{E}, \sigma \in \mathrm{Inv}_G^\infty(E)$ and $\Pi(\sigma) =$

$\overline{\sigma}$. We let $\varrho(f) = \overline{f}$. More generally if $\alpha \in \Omega^{r,s}_{\mathrm{pr}G}(J^\infty E)$ and $\mathrm{tot}X \lrcorner\ \alpha = 0$ for all X in Γ, then there is a unique $\overline{\alpha} \in \Omega^{r,s}(J^\infty \overline{E})$ such that

$$\imath^*(\alpha) = \Pi^*(\overline{\alpha})$$

where $\imath : \mathrm{Inv}^\infty_G(E) \to J^\infty E$ is the canonical inclusion (see [3]). We let $\varrho(\alpha) = \overline{\alpha}$.

Example Suppose that $f : J^\infty E \to \mathbf{R}$ is a G-invariant function expressed in local coordinates adapted to G by $f(\hat{x}^i, y^r, v^\alpha, v^\alpha_i, v^\alpha_r, v^\alpha_{ij}, v^\alpha_{ir}, v^\alpha_{rs}, \cdots)$ then

$$\varrho(f)(y^r, v^\alpha, v^\alpha_r, v^\alpha_{rs}, \cdots) = f(\hat{x}^i, y^r, v^\alpha, 0, v^\alpha_r, 0, 0, v^\alpha_{rs}, \cdots).$$

As an illustration suppose that $G = SO(3)$ acts on $M = \mathbf{R}^3 - \{0\}$ and that $E = M \times \mathbf{R}$. Let $f = u_{xx} + u_{yy} + u_{zz}$. Using local coordinates (x, y, z, u) on E and local coordinates adapted to G

$$\hat{x} = x \quad \hat{y} = y \quad r = \sqrt{x^2 + y^2 + z^2} \quad v = u,$$

one calculates (by the chain rule) $u_x = u_{\hat{x}} + \frac{x}{r} u_r$, $u_y = u_{\hat{y}} + \frac{y}{r} u_r$, and

$$u_{xx} = v_{\hat{x}\hat{x}} + \frac{x}{r} v_{\hat{x}r} + \frac{x}{r}(v_{r\hat{x}} + \frac{x}{r} v_{rr}) + v_r \frac{r^2 - x^2}{r^3},$$

$$u_{yy} = v_{\hat{y}\hat{y}} + \frac{y}{r} v_{\hat{y}r} + \frac{y}{r}(v_{r\hat{y}} + \frac{y}{r} v_{rr}) + v_r \frac{r^2 - y^2}{r^3},$$

$$u_{zz} = \frac{z^2}{r^2} v_{rr} + \frac{r^2 - z^2}{r^3} v_r.$$

Now set $v_{\hat{x}\hat{x}} = v_{\hat{x}r} = v_{r\hat{x}} = 0$ and $v_{\hat{y}\hat{y}} = v_{\hat{y}r} = v_{r\hat{y}} = 0$ to get $\boxed{\frac{x^2}{r^2} v_{rr} + \frac{r^2 - x^2}{r^3} v_r}$ for u_{xx} and $\boxed{\frac{y^2}{r^2} v_{rr} + \frac{r^2 - y^2}{r^3} v_r}$ for u_{yy} so that $\varrho(f) = v_{rr} + \frac{2}{r} v_r$. $\blacksquare$

Example Let $\alpha \in \Omega^{r,0}_{\mathrm{pr}G}(J^\infty E)$ be G-invariant and suppose that $\mathrm{tot}X \lrcorner\ \alpha = 0$ for all X in Γ then α can be expressed in local coordinates adapted to G as $\alpha = A_{i_1 i_2 \cdots i_r}(dy^{i_1} \wedge dy^{i_2} \wedge \cdots \wedge dy^{i_r})$ where the $A_{i_1 i_2 \cdots i_r}$'s are G-invariant, and $\varrho(\alpha) = \varrho(A_{i_1 i_2 \cdots i_r})(dy^{i_1} \wedge dy^{i_2} \wedge \cdots \wedge dy^{i_r})$. ∎

Finally, recall that a q multi-vector on M is an alternating tensor of type $(q, 0)$. Let Γ_M be a Lie algebra of vector fields on M, then a *q-chain* $\mathcal{X}$ on Γ_M is a (non-zero) q multi-vector that can be expressed as $\mathcal{X} = J(X_1 \wedge X_2 \wedge \cdots \wedge X_q)$, where J is a function on M and $X_1, X_2, \cdots, X_q \in \Gamma_M$.

Now we show how a cochain map between the cochain "de Rham complexes" of E and $\overline{E}$ may be defined. Recall that Γ denotes the *vector space of infinitesimal generators* of the action of G on E. One assumes the existence of a G-invariant q-chain on Tot Γ

$$\mathcal{X} = J(\mathrm{tot}X_1 \wedge \mathrm{tot}X_2 \wedge \cdots \wedge \mathrm{tot}X_q),$$

where J is a function on $J^\infty E$ and $X_1, X_2, \cdots, X_q \in \Gamma$, such that the map $\varrho_{\mathcal{X}} : \Omega^{r,s}_{\mathrm{pr}G}(J^\infty E) \to \Omega^{r-q,s}(J^\infty \overline{E})$ defined by

$$\varrho_{\mathcal{X}}(\gamma) = (-1)^{q(r+s)} \varrho(\mathcal{X} \lrcorner\ \gamma)$$

is a d_H cochain map between cochain complexes, and such that

$$\mathbf{E}(\varrho_{\mathcal{X}} \delta) = \varrho_{\mathcal{X}}(\mathbf{E}(\delta))$$

for all $\delta \in \Omega^{n,0}_{\mathrm{pr}G}(J^\infty E)$. In [3] it was proved that if the action of G on E is free and G is unimodular (i.e. $\sum_{a=1}^{q} C^a_{ia} = 0$ where the C^a_{bc}'s are the structure constants of $\mathfrak{g}$) then the existence of a G-invariant q-chain follows. In fact, in [3] the existence of such q-chains is studied in detail. Notice that the map $\varrho_{\mathcal{X}}$ is onto but generally *not* one-to-one.

Remark The above correspondence ϱ can be restricted to compact-support subcomplexes as defined earlier in section 3.1, in particular to the first row of such subcomplexes (the row consisting of purely horizontal forms). If $s : M \to E$ is a G-invariant section of compact support then $\overline{s} = \varrho(s) : \overline{M} \to \overline{E}$ is of compact support. If $\alpha \in \Omega^{r,0}_{\text{pr}G}(J^\infty E)$ satisfies $\text{tot}X \lrcorner\ \alpha = 0$ for all X in Γ, and $(j^\infty s)^*(\alpha)$ is of compact support for all sections $s : M \to E$ of compact support then for (the unique form) $\overline{\alpha} = \varrho(\alpha) \in \Omega^{r,0}(J^\infty \overline{E})$ we have: $(j^\infty \overline{s})^*(\overline{\alpha})$ is of compact support for all sections $\overline{s} : \overline{M} \to \overline{E}$ of compact support, ...etc.

Henceforth we will work with the compact-support subcomplexes that consist of purely horizontal forms, where $\varrho_{\mathcal{X}}$ serves as a d_H cochain map between these subcomplexes (though some of our conclusions apply to the more general complexes).

5.2 An sh-Lie structure on the reduced graded vector space

We assume that $(\Omega^{*,0}_c(J^\infty E), d_H)$ and $(\Omega^{*,0}_c(J^\infty \overline{E}), d_H)$ are exact. The map $\tilde{l}_2$ on $\Omega^{n,0}_c(J^\infty E)$ induces a map $\hat{l}_2$ on $\Omega^{n-q,0}_c(J^\infty \overline{E})$ as follows. Define $\hat{l}_2 : \Omega^{n-q,0}_c(J^\infty \overline{E}) \otimes \Omega^{n-q,0}_c(J^\infty \overline{E}) \to \Omega^{n-q,0}_c(J^\infty \overline{E})$ by

$$\hat{l}_2(\varrho_{\mathcal{X}}\alpha, \varrho_{\mathcal{X}}\beta) = \varrho_{\mathcal{X}}(\tilde{l}_2(\alpha, \beta))$$

for all G-invariant α, β in $\Omega^{n,0}_c(J^\infty E)$. Now the question is: Is the map $\hat{l}_2$ well-defined?

First recall that the cochain map $\varrho_{\mathcal{X}}$ is onto. Now assume that ω is covariant with respect to the group action of G (i.e. covariant with respect to ψ_g for all $g \in G$ as in section 3.1) so that by Theorem 3.5 it follows that if α and β are in $\Omega^{n,0}_{\mathrm{pr}G}(J^\infty E)$ then so is $\tilde{l}_2(\alpha, \beta)$. Finally, we assume that if $\varrho_{\mathcal{X}}\alpha = \varrho_{\mathcal{X}}\alpha'$ for any $\alpha, \alpha' \in \Omega^{n,0}_{\mathrm{pr}G}(J^\infty E)$ then $\hat{l}_2(\varrho_{\mathcal{X}}\alpha, \varrho_{\mathcal{X}}\beta) - \hat{l}_2(\varrho_{\mathcal{X}}\alpha', \varrho_{\mathcal{X}}\beta) = \varrho_{\mathcal{X}}\tilde{l}_2(\alpha - \alpha', \beta) = \varrho_{\mathcal{X}}[\omega(\mathbf{E}(\alpha - \alpha'), \mathbf{E}(\beta))\nu] = 0$, so that $\hat{l}_2$ is *well-defined.*

Observe that if the volume form ν is G-invariant then $\hat{l}_2$ is given by

$$\hat{l}_2(\varrho_{\mathcal{X}}\alpha, \varrho_{\mathcal{X}}\beta) = \overline{\omega}(\mathbf{E}(\varrho_{\mathcal{X}}\alpha), \mathbf{E}(\varrho_{\mathcal{X}}\beta))\overline{\nu}$$

where $\overline{\omega} : \Omega^{n-q,1}_0(J^\infty \overline{E}) \times \Omega^{n-q,1}_0(J^\infty \overline{E}) \to Loc_{\overline{E}}$ is defined by

$$\overline{\omega}(\varrho_{\mathcal{X}}\gamma, \varrho_{\mathcal{X}}\delta) := \varrho(\omega(\gamma, \delta)),$$

for γ, δ that lie in the image of $\mathbf{E}$ in $\Omega^{n,1}_{\mathrm{pr}G}(J^\infty E)$. Thus $\overline{\omega}(\mathbf{E}(\varrho_{\mathcal{X}}\alpha), \mathbf{E}(\varrho_{\mathcal{X}}\beta)) = \overline{\omega}(\varrho_{\mathcal{X}}\mathbf{E}(\alpha), \varrho_{\mathcal{X}}\mathbf{E}(\beta)) = \varrho[\omega(\mathbf{E}(\alpha), \mathbf{E}(\beta))]$. Here $\overline{\nu}$ satisfies $(\overline{\pi}^\infty_M)^*\overline{\nu} = \varrho_{\mathcal{X}}(\pi^\infty_M)^*\nu$ where $\overline{\pi}^\infty_M : J^\infty \overline{E} \to \overline{M}$ and $\pi^\infty_M : J^\infty E \to M$ are the usual projections of the corresponding jet bundles.

Skew-symmetry and linearity of $\hat{l}_2$ follow easily from the skew-symmetry and linearity of $\tilde{l}_2$ in addition to the linearity of $\varrho_{\mathcal{X}}$. Furthermore $\hat{l}_2$ satisfies

(i) $\quad \hat{l}_2(d_H k_1, h) = d_H k_2,$

(ii) $\quad \displaystyle\sum_{\sigma \in unsh(2,1)} (-1)^\sigma \hat{l}_2(\hat{l}_2(f_{\sigma(1)}, f_{\sigma(2)}), f_{\sigma(3)}) = d_H k_3,$

for all $k_1 \in \Omega^{n-q-1,0}_c(J^\infty \overline{E}), h, f_1, f_2, f_3 \in \Omega^{n-q,0}_c(J^\infty \overline{E})$ and for some $k_2, k_3 \in \Omega^{n-q-1,0}_c(J^\infty \overline{E})$. Subsequently we will suppress some of the notation and assume the summands are over the appropriate unshuffles with their corresponding signs.

Notice that (i) follows in the strong sense $\hat{l}_2(d_H k_1, h) = 0$ since $\mathbf{E}(d_H k_1) = 0$. While to verify (ii), let $f_i = \varrho_{\mathcal{X}} F_i$ for $i = 1, 2, 3$, and where $f_1, f_2, f_3 \in$

$\Omega_c^{n-q,0}(J^\infty\overline{E})$ are arbitrary, and notice that

$$\begin{aligned}\sum_\sigma \hat{l}_2(\hat{l}_2(f_{\sigma(1)}, f_{\sigma(2)}), f_{\sigma(3)}) &= \varrho_{\mathcal{X}} \sum_\sigma \tilde{l}_2(\tilde{l}_2(F_{\sigma(1)}, F_{\sigma(2)}), F_{\sigma(3)}) \\ &= \varrho_{\mathcal{X}}(d_H K_2) \\ &= d_H(\varrho_{\mathcal{X}} K_2),\end{aligned}$$

since $\varrho_{\mathcal{X}}$ is a d_H cochain map. Here $K_2 \in \Omega_{\mathrm{pr}G}^{n-1,0}(J^\infty E)$ and the sum is over the unshuffles (2,1) with the corresponding permutation sign. We have shown:

Theorem 5.1 *Assume that $\Omega_c^{*,0}(J^\infty E)$ and $\Omega_c^{*,0}(J^\infty\overline{E})$ are exact. Then the skew-symmetric linear map $\hat{l}_2$ as defined above on the space $\Omega_c^{n-q,0}(J^\infty\overline{E})$ extends to an sh-Lie structure on the exact graded space $(\Omega_c^{*,0}(J^\infty\overline{E}), d_H)$.*

We have also shown (see Lemmas 2.3 and 2.4):

Theorem 5.2 *Assume that $\Omega_c^{*,0}(J^\infty E)$ and $\Omega_c^{*,0}(J^\infty\overline{E})$ are exact. Then there exists a skew-symmetric bilinear bracket on $\overline{H}_0 \times \overline{H}_0$ that satisfies the Jacobi identity. Here $\overline{H}_0 = H^{n-q}(\Omega_c^{*,0}(J^\infty\overline{E}), d_H)$, and the bracket is induced by the map $\hat{l}_2$.*

In fact, if ν is G-invariant with $\varrho_{\mathcal{X}}(\pi_M^*\nu) = \overline{\pi}_M^*\overline{\nu}$ then this bracket can be identified with $\{\mathcal{P}, \mathcal{Q}\} = \int_{\overline{M}} \overline{\omega}(\mathbf{E}(P), \mathbf{E}(Q))\overline{\nu}$, where $\mathcal{P} = \int_{\overline{M}} P\overline{\nu}$ and $\mathcal{Q} = \int_{\overline{M}} Q\overline{\nu}$ represent arbitrary elements of $\overline{H}_0$, and $P, Q \in Loc_{\overline{E}}^0$.

Example Consider $M = \mathbf{R}^3 - \{0\}, E = M \times \mathbf{R}^2$, and let

$$\omega = \begin{pmatrix} 0 & r \\ -r & 0 \end{pmatrix},$$

where $r = \sqrt{x^2+y^2+z^2}$ and x, y, z are cartesian coordinates on M. Recall that ω defines the map $\tilde{l}_2$. Now let $G = SO(3)$ act on E by rotations of the base manifold M. The Lie algebra of the action has generators

$$X_1 = x\partial y - y\partial x, \quad X_2 = y\partial z - z\partial y, \quad X_3 = z\partial x - x\partial z.$$

Observe that ω is covariant with respect to G. In fact the components of ω are G-invariant in this case. Now notice that $\mathcal{X} = \frac{r}{y}(\text{tot}X_1 \wedge \text{tot}X_2)$ is a G-invariant 2-chain, and that $\varrho_{\mathcal{X}}(dx \wedge dy \wedge dz) = r^2 dr$, where $\nu = dx \wedge dy \wedge dz$ is G-invariant and $\overline{\nu} = r^2 dr$. We conclude that there exists a reduced sh-Lie structure. Now suppose that

$$P = u^1(u^2_{xx} + u^2_{yy} + u^2_{zz}) \quad \text{and} \quad Q = u^1 u^2,$$

then

$$\tilde{l}_2(P\nu, Q\nu) = r[(u^2_{xx} + u^2_{yy} + u^2_{zz})u^1 - (u^1_{xx} + u^1_{yy} + u^1_{zz})u^2]\nu,$$

while

$$\varrho_{\mathcal{X}}(P\nu) = u^1(u^2_{rr} + \frac{1}{r}u^2_r)\overline{\nu} \quad , \quad \varrho_{\mathcal{X}}(Q\nu) = u^1 u^2 \overline{\nu},$$

and

$$\begin{aligned}
\hat{l}_2(\varrho_{\mathcal{X}}(P\nu), \varrho_{\mathcal{X}}(Q\nu)) &= r[(u^2_{rr} + \frac{2}{r}u^2_r)u^1 - (u^1_{rr} + \frac{2}{r}u^1_r)u^2]\overline{\nu} \\
&= [(ru^2_{rr} + 2u^2_r)u^1 - (ru^1_{rr} + 2u^1_r)u^2]\overline{\nu}
\end{aligned}$$

where we have used $\hat{l}_2(\varrho_{\mathcal{X}}(P\nu), \varrho_{\mathcal{X}}(Q\nu)) = \varrho_{\mathcal{X}}(\tilde{l}_2(P\nu, Q\nu))$ which was defined earlier.

Now notice that if, for example, one has

$$\int_M \tilde{l}_2(P\nu, Q\nu) = \int_{\mathbf{R^3}} r[(u^2_{xx} + u^2_{yy} + u^2_{zz})u^1 - (u^1_{xx} + u^1_{yy} + u^1_{zz})u^2] dx \wedge dy \wedge dz$$

and one is interested in sections that are G-invariant, i.e., sections that depend only on r, then the above integral reduces to

$$4\pi \int_0^\infty r^3[(u^2_{rr} + \frac{2}{r}u^2_r)u^1 - (u^1_{rr} + \frac{2}{r}u^1_r)u^2]dr = 4\pi \int_{\overline{M}} \varrho_{\mathcal{X}} \tilde{l}_2(P\nu, Q\nu),$$

where notice that the 4π is obtained by integrating out the variables which the fields/sections do not depend on.

More generally, if $P\nu$ is G-invariant and one has $\int_{\mathbf{R^3}} P\nu$ then for sections that are G-invariant this integral reduces to $4\pi \int_0^\infty \varrho(P)\overline{\nu}$. $\blacksquare$

5.3 Reduction of local functionals

As we saw in the previous section the map $\varrho_{\mathcal{X}}$ defines a correspondence between local functionals on $J^\infty E$ and local functionals on $J^\infty \overline{E}$. This correspondence is "natural" when one is interested in G-invariant sections. Given the functional

$$\int_M P\nu$$

on $J^\infty E$, where $P\nu$ is a G-invariant horizontal n-form, we define its *reduced functional* to be the functional on $J^\infty \overline{E}$ given by

$$\int_{\overline{M}} \varrho_{\mathcal{X}}(P\nu).$$

Notice that if the action of G does not have a vertical component along the fibers of $\pi : E \to M$, while ϕ is a G-invariant local section of π with support in Ω, $\overline{\phi}$ is the section of $\overline{\pi} : \overline{E} \to \overline{M}$ corresponding to ϕ, then

$$\int_\Omega (j^\infty \phi)^* P\nu = V \int_{\overline{\Omega}} (j^\infty \overline{\phi})^* \varrho_{\mathcal{X}}(P\nu)$$

where $\overline{\Omega} = \pi_{\overline{M}}(\Omega)$, and V is obtained when integrating out the variables which ϕ does not depend on (from the left-hand side integral).

Example Consider the Euler equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p \quad \text{and} \quad \nabla \cdot \mathbf{u} = 0,$$

where $\mathbf{u} = (u, v, w)$ and p are the dependent variables while $\mathbf{x} = (x, y, z)$ and t are the independent variables. It is known that the $SO(3)$ invariant solutions are given by $\mathbf{u} = (a(t)/r^3)\mathbf{x}$ where $a(t)$ is a function of t, while the energy is given by the functional $\mathcal{E}(t) = \dfrac{1}{2}\displaystyle\int_\Omega |\mathbf{u}|^2 d\mathbf{x}$ where Ω is the region of $\mathbf{R}^3$ over which the solution is defined (we refer the reader to [4] and [20] for more details). For an $SO(3)$ invariant solution, Ω is a spherical region and, the energy reduces to $2\pi \displaystyle\int_{\overline{\Omega}} (a(t)^2/r^2) dr$ where $\overline{\Omega}$ is the subset of $\mathbf{R}$ corresponding to Ω via the action of $SO(3)$.

Remark In Chapter 4 reduction of local functionals and sh-Lie structures was studied for the case when the Lie group G acts only on the fibers, i.e., when the induced action on the base manifold M is the *identity map.* In this Chapter we assumed that the action is *transversal* to the fibers. One may consider a general case where one has a combination of these two kinds of action, involving two different Lie groups. For example, one can consider a reduction from π to $\overline{\pi}$ under a Lie group action as was done in this Chapter. Then another reduction on $\Omega_c^{*,0}(J^\infty \overline{E})$ may be obtained when another Lie group acts on the fibers of $\overline{\pi}$ as in Chapter 4.

Bibliography

[1] R. Abraham and J.E. Marsden, *Foundations of mechanics*, Addison-Wesley, 1994.

[2] I.M. Anderson, *The Variational Bicomplex, Preprint*, Utah State University, 1996.

[3] I.M. Anderson and M. Fels, *Symmetry reduction of variational bicomplexes and the principle of symmetric criticality*, Amer. J. Math. **119** (1997), 609–670.

[4] I.M. Anderson, M. Fels, and C. Torre, *Group invariant solutions without transversality*, Commun. Math. Phys. **212** (2000), 653–686.

[5] G. Barnich, *Brackets in the jet-bundle approach to field theory*, Proceedings of the conference on Secondary Calculus and Cohomological Physics, Moscow, Russia, 24-31 Aug 1997, Contemp. Math. **219** (1998), 17–27.

[6] G. Barnich, R. Fulp, T. Lada, and J. Stasheff, *The sh Lie structure of Poisson brackets in field theory*, Commun. Math. Phys. **191** (1998), 585–601.

[7] G. Barnich and M. Henneaux, *Isomorphisms between the Batalin-Vikovisky antibracket and the Poisson bracket*, J. Math. Phys. **37** (11) (1996), 5273–5296.

[8] G. Barnich and M. Henneaux, *Consistent couplings between fields with a gauge freedom and deformations of the master equation*, Physics Letters B **311** (1993), 123–129.

[9] R. Bott and L.W. Tu, *Differential Forms in Algebraic Topology*, Graduate Texts in Mathematics, vol. 82, Springer-Verlag, 1982.

[10] A. Cattaneo and G. Felder, *A path integral approach to the Kontsevich quantization formula*, `math.QA/9902090`.

[11] B.A. Dubrovin, *Geometry of Hamiltonian Evolutionary Systems*, Monographs and Textbooks in Physical Science, vol. 22, Bibliopolis, 1991.

[12] R. Fulp, T. Lada, and J. Stasheff, *Sh-Lie algebras Induced by Gauge Transformations*, Commun. Math. Phys. **231** (2002), 25–43.

[13] M. Henneaux and C. Teitelboim, *Quantization of gauge systems*, Princeton Univ. Press, 1992.

[14] N. Ikeda, *Two-dimensional gravity and nonlinear gauge theory*, Ann. Phys. **235** (1994), 435–464.

[15] T. Lada and M. Markl, *Strongly homotopy Lie algebras*, Comm. in algebra **23** (1995), 2147–2161.

[16] T. Lada and J.D. Stasheff, *Introduction of sh Lie algebras for physicists*, International J. Theor. Phys. **32** (1993), 1087–1103.

[17] I.A. Kogan and P.J. Olver, *Invariant Euler-Lagrange Equations and the Invariant Variational Bicomplex*, to appear in to Acta Applicandae.

[18] I.S. Krasil'shchik and A.M. Vinogradov (eds.), *Symmetries and Conservaion Laws for differential equations of mathematical physics*, Translations of Mathematical Monographs, vol. 182, American Mathematical Society, Providence, R.I., 1998.

[19] J.E. Marsden and T.S. Ratiu, *Introduction to Mechanics and Symmetry*, Texts in Applied Mathematics, vol. 17, Springer-Verlag, 1994.

[20] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Graduate Texts in Mathematics, vol. 107, Springer-Verlag, 1986.

[21] D. Roytenberg and A. Weinstein, *Courant algebroids and strongly homotopy Lie algebras*, `arXiv:math.QA/9802118 v1`.

[22] D.J. Saunders, *The Geometry of Jet Bundles*, London Mathematical Lecture Notes, vol. 142, Cambridge Univ. Press, 1989.

[23] P. Schaller and T. Strobl, *Poisson structure induced (topological) field theories*, Modern Phys. Lett. A **9** (1994), 3129–3136.

[24] B. Zwiebach, *Closed string field theory: Quantum action and the Batalin-Vilkovisky master equation*, Nucl. Phys. B **390** (1993), 33–152.