

SOME TOPICS IN THE THEORY OF STOCHASTIC PROCESSES

by

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And each man hears as the twilight nears  
to the beat of his dying heart,  
The Devil drum on the darkened pane:  
"You did it, but was it Art?"

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## INTRODUCTION

Kolmogorov [9] was the first to make use of linear methods in Probability theory, in the study of stationary sequences of random variables. Later, Karhunen [8], introduced these methods in a systematic manner, and employed them to give an elegant definition of the stochastic integral,

$$\int_S x(t) dm ,$$

where  $x(t)$  is a second order process, and  $m$  an ordinary  $\sigma$ -finite measure on the real line. (It has been known since, that this is the same as the integral due to Pettis, [13].) In Chapter I of this work, we have utilized these concepts to give a definition of the Stieltjes stochastic integral, that is, an integral of the form,

$$\int_S f(a) dZ ,$$

where  $Z$  is a random set function defined in a certain manner. This definition is seen to be at least as general as the one given by the usual method of approximating sums. Finally, a theorem is proved concerning the representability of a random set function as a Karhunen stochastic integral.

In Chapter II, an integral representation is obtained for a random process of the second order whose covariance function is

a Carleman kernel. This problem has been considered recently by Dolph and Woodbury, [3] who have obtained a series representation which makes use of Hellinger integrals.

The third chapter is devoted to additive functionals of a purely discontinuous Markov process. This study was motivated by a recent, (as yet unpublished) paper of R. Fortet, [5], in which he has considered the continuous case and has obtained many interesting and remarkable results. The principal result of Chapter III is that the conditional characteristic function of such a functional is the solution of a certain integro-differential equation. One of the applications of this theory yields, as a consequence, Feller's equation for the transition probabilities of a discontinuous Markov process.

CHAPTER I.

ON A STOCHASTIC STIELTJES INTEGRAL AND THE REPRESENTABILITY OF A  
RANDOM SET FUNCTION AS A KARHUNEN INTEGRAL.

1. Preliminaries.

In this section we shall give one of the many definitions of a stochastic process, (or random function),  $X(t)$  where we shall assume that  $t$  ranges over the real line,  $R$ . In most applications,  $t$  plays the role of time.

Definition (1.1): Let  $(\Omega)$  be an abstract space whose points or elements are denoted by  $\omega$  and let  $\mathcal{P}$  be a probability measure defined over a  $\sigma$ -algebra  $\Sigma$  of subsets of  $(\Omega)$ . Consider a real or complex-valued function  $X(t, \omega)$ , such that, for every fixed  $t$  in  $R$ ,  $X(t, \omega)$  is a  $\mathcal{P}$ -measurable function of  $\omega$ . Then  $X(t, \omega)$  defines a random variable  $X(t)$ , and as  $t$  ranges over  $R$ , we thus obtain a family of random variables depending on the parameter  $t$ . This family of random variables will be called a stochastic process  $X(t)$ .

We shall write

$$EX(t) = \int_{(\Omega)} X(t, \omega) d\mathcal{P}(\omega); \quad E[\overline{X(t)}X(u)] = \int_{(\Omega)} X(t, \omega) \cdot \overline{X(u, \omega)} d\mathcal{P}(\omega),$$

where  $\overline{X(u, \omega)}$  denotes the complex conjugate of  $X(u, \omega)$ .  $X(t)$  is said to be of the second order if

$$\rho(t, u) = E[\overline{X(t)}X(u)] < \infty, \text{ for all } t \text{ and } u.$$



It will be assumed throughout this chapter that  $X(t)$  is of the second order and that  $EX(t) = 0$  for all  $t$ .  $\rho(t,u)$  is the covariance function of the process. In section (2) we shall give Cramér's definition [2]<sup>1</sup> of an additive random set function, (abbreviated as r.s.f.) and state some of its properties. We shall then use them to define the stochastic Stieltjes integral of a function  $f(a)$  with respect to such a r.s.f. This will be done in section (3) by applying the "linear" methods, first used by Kolmogorov [9] in the study of stationary processes, and later developed systematically by Karhunen, who applied them to define the stochastic integral of the type,

$$\int_S X(t) dm,$$

$m$  being an ordinary  $\sigma$ -finite measure on  $R$ , and  $S$  an  $m$ -measurable set (see [8]). This definition is also known to be the same as that given by Pettis [13], in a different connection. In section 4, our definition of

$$\int_S f(a) dZ$$

is compared with the definitions of Cramér and Karhunen ([2], [8]). Section 5 is devoted to the proof of a theorem on the representability of a r.s.f. as a Karhunen stochastic integral.

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1. Numbers in square brackets refer to bibliography.

The following theorem, well-known in the study of Hilbert space theory is quoted below for convenience, as it is the main tool employed in obtaining the results of this chapter. For the terminology and notation used in this theorem, we refer to Nagy [12].

Theorem 1.2.

If  $L(f)$  is a bounded linear operation on the Hilbert space  $\mathcal{R}$ , there exists a unique element  $g^*$  in  $\mathcal{R}$  such that,

$$L(f) = (f, g^*),$$

for every  $f \in \mathcal{R}$ . The results contained in sections 3 and 5 have been announced in a recent note by the author [7].

2. Additive Random Set Functions.

Starting from a stochastic process  $X(t)$ , define,

$$Z(I) = X(t_2) - X(t_1),$$

where  $I$  is the half-open interval  $(t_1, t_2]$ . If  $I = I_1 + \dots + I_n$ , where the  $I_k$  are finite disjoint half-open intervals, we may extend this definition by writing,

$$Z(I) = \sum_{k=1}^n Z(I_k).$$

If the covariance function of  $X(t)$  has a certain property, to be

stated shortly, this definition can be extended, so that we can define the random variable  $Z(S)$ , where  $S$  is a bounded Borel set. Strictly speaking, we should write  $Z(S, \omega)$  to bring out the dependence on  $\omega$  explicitly. Henceforth, we shall suppress the letter  $\omega$  and simply write  $Z(S)$ .  $Z$  is now a stochastic process whose argument is a bounded Borel set. Similarly, by  $EZ(S)$  we shall mean the integral over  $\bar{\Omega}$  with respect to the probability measure  $\bar{\mathbb{P}}$ . With these brief explanatory remarks we can now (see [2]) characterize an additive random set function as follows: -- An additive random set function is a family of random variables such that,

- (1) For every bounded Borel set  $S$  of real numbers,  $Z(S)$  is a uniquely defined random variable.
- (2) If  $S_1, S_2, \dots$  are disjoint Borel sets, such that

$$S = \sum_{i=1}^{\infty} S_i,$$

is bounded, then

$$Z(S) = \sum_{i=1}^{\infty} Z(S_i),$$

where the series on the right converges in quadratic mean. One could conceive of such a process  $Z(S)$  as the "impulse" received during an arbitrary Borel set  $S$  of time points. We shall assume that  $Z(S)$  is real valued, with  $EZ(S) = 0$ , and  $E[Z(S)]^2 < \infty$  (for every bounded Borel set  $S$ ). Cramér [2] has proved the following theorem:

If the covariance function  $\rho(t, u)$  of the stochastic process

$X(t)$  is of bounded variation in every finite domain (in the sense defined in [2]), there exists an additive r.s.f  $Z(S)$ , uniquely defined for all bounded Borel sets  $S$  and such that,

$$Z(S) = X(t_2) - X(t_1) ,$$

when  $S$  is a half-open interval,

$$(t_1, t_2] .$$

The covariance function of  $Z$  is given by

$$E [Z(S_1) Z(S_2)] = \int_{S_1} \int_{S_2} d^2 \rho (t,u) .$$

In particular, if

$$\rho (t,u) = F [ \inf (t,u) ] ,$$

where  $F(t)$  is a non-negative, never decreasing function, we obtain the spectral random set function of Kerhunen [8] , and we have,

$$E [Z(S_1) \cdot Z(S_2)] = \int_{S_1 \cdot S_2} dF(t) .$$

3. Definition of the Integral  $\int_S f(a) dZ .$

In what follows it will be assumed that we have a real-valued additive r.s.f  $Z(S)$  which is such that,

$$(3.1) \quad E Z(S) = 0,$$

and

$$E [Z(S_1) \cdot Z(S_2)] = \int_{S_1} \int_{S_2} d^2 \rho(t, u),$$

where  $\rho$  is of bounded variation in every finite domain.

(a) The Linear Space of the Process  $Z$ :

Let  $L(Z)$  be the set of all random variables of the form,

$$(3.2) \quad c_1 Z(S_1) + \dots + c_n Z(S_n),$$

where the  $S_k$  are bounded Borel sets, and the  $c_k$  are real constants. Closing the set  $L(Z)$  with respect to convergence in the mean, we obtain an extended set  $L_2(Z)$ . The elements of  $L_2(Z)$  are either of the form (3.2) or are limits in the mean of random variables of the form (3.2). Define the inner product of two arbitrary elements  $z_1$  and  $z_2$  of  $L_2$  (we shall write  $L_2$  for  $L_2(Z)$  from now on), by the relation,

$$(z_1, z_2) = E(z_1 \cdot z_2).$$

It is well known, (see [2]) that the set  $L_2$  is a Hilbert space with the norm,

$$\|z\| = \sqrt{E(z^2)}.$$

$L_2$  is referred to as the linear space of  $Z$  .

(b) Denote by  $z$  the elements of  $L_2$  . For any  $z$  write,

$$(3.3) \quad E \int z \cdot Z(S) \_ ] = P_z (S) .$$

Let

$$S = \sum_{i=1}^{\infty} S_1$$

be a bounded Borel set, the  $S_1$  being disjoint. Then since,

$$Z(S) = \sum_{i=1}^{\infty} Z(S_1) ,$$

(the series converging in quadratic mean), we have

$$\left\| Z(S) - \sum_{i=1}^n Z(S_1) \right\|^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty .$$

Now,

$$\begin{aligned} & \left( E \int z \cdot Z(S) \_ ] - \sum_{i=1}^n E \int z \cdot Z(S_1) \_ ] \right)^2 \\ &= \left( E \int z \cdot \left\{ Z(S) - \sum_{i=1}^n Z(S_1) \right\} \_ ] \right)^2 \\ &\leq \| z \|^2 \cdot \left\| Z(S) - \sum_{i=1}^n Z(S_1) \right\|^2 , \text{ by the Schwarz} \\ & \hspace{15em} \text{inequality.} \end{aligned}$$

$\rightarrow 0$  as  $n \rightarrow \infty$  for every  $z$ .

Hence,

$$E \int z \cdot Z(S) \_ ] = \sum_{i=1}^{\infty} E \int z \cdot Z(S_1) \_ ] .$$

Using the notation of (3.3) we have

$$(3.4) \quad P_z(S) = \sum_{i=1}^{\infty} P_z(S_i),$$

for every  $z$  in  $L_2$ . Also, from (3.1), taking  $S = 0$ , the empty set, we obtain,

$$\|z(0)\|^2 = E \int z(0)^2 = 0.$$

Again by Schwarz's inequality,

$$(3.5) \quad P_z(0) = 0,$$

for every  $z$  in  $L_2$ . Thus from relations (3.4) and (3.5) we find that, for every  $z$  in  $L_2$ ,  $P_z$  is a signed measure. Therefore we may write (using the notation of Halmos [6\_7]),

$$P_z(S) = P_z^+(S) - P_z^-(S),$$

where both  $P_z^+$  and  $P_z^-$  are measures which are finite for bounded sets  $S$ . Actually, as is done in the theory of signed measures, we might include sets  $S$  which make one of (but not both)  $P_z^+$  or  $P_z^-$  infinite.

Write,

$$|P_z|(S) = P_z^+(S) + P_z^-(S).$$

### Theorem 3.1

Let  $f(a)$  be a real valued function such that,

(i) the integral

$$\int_S f(a) d |P_z|$$

exists and is finite for every  $z$ .

$$(ii) \quad \frac{1}{\|z\|} \left| \int_S f(a) dP_z \right|$$

is bounded. Then there exists a unique element,  $I(S)$  in  $L_2$ , such that

$$E [z \cdot I(S)] = \int_S f(a) dP_z ,$$

for every  $z$ .

Proof:

Let

$$J_S(z) = \int_S f(a) dP_z .$$

For any two real numbers  $\alpha$  and  $\beta$ , we have,

$$\begin{aligned} J_S(\alpha z_1 + \beta z_2) &= \alpha \int_S f(a) dP_{z_1} + \beta \int_S f(a) dP_{z_2} \\ &= \alpha J_S(z_1) + \beta J_S(z_2) . \end{aligned}$$



From (11) ,

$$\left| \mathcal{J}_S(z) \right| \leq m \cdot \|z\| ,$$

for all  $z$  in  $L_2$ . Therefore,  $\mathcal{J}_S(z)$  is a bounded linear operation on the space  $L_2$ . Hence from theorem (2.1), there exists a unique element in  $L_2$ , say  $I(S)$ , such that for every  $z$  in  $L_2$  ,

$$(3.6) \quad (z, I(S)) = E[z \cdot I(S)] = \mathcal{J}_S(z) = \int_S f(a) dP_z .$$

We shall define this element  $I(S)$  occurring in (3.6) as the stochastic Stieltjes integral of  $f(a)$  with respect to the process  $Z(S)$  and write,

$$(3.7) \quad I(S) = \int_S f(a) dZ.$$

Let  $S_1$  and  $S_2$  be two disjoint bounded Borel sets. The integral  $I(S_1 + S_2)$  is defined by,

$$\begin{aligned} E[z \cdot I(S_1 + S_2)] &= \int_{S_1 + S_2} f(a) dP_z \\ &= \int_{S_1} f(a) dP_z + \int_{S_2} f(a) dP_z \\ &= E[z \cdot I(S_1)] + E[z \cdot I(S_2)] \end{aligned}$$

$$= \mathbb{E} \int z \cdot \{I(S_1) + I(S_2)\} dz .$$

Hence

$$\mathbb{E} \int z \cdot \{I(S_1 + S_2) - I(S_1) - I(S_2)\} dz = 0,$$

for all  $z$  in  $L_2$ . Therefore,

$$(3.8) \quad I(S_1 + S_2) = I(S_1) + I(S_2) \quad (\text{in quadratic mean}) .$$

Similarly one can show that,

$$\int_S \int [f(a) + g(a)] dz = \int_S f(a) dz + \int_S g(a) dz ,$$

and

$$\int_S c \cdot f(a) dz = c \int_S f(a) dz , \quad c \text{ being a constant.}$$

#### 4. Comparison with Older Definitions : --

The stochastic Stieltjes integral with respect to  $Z$  has been defined by Cramér [2], and, in the case of the spectral function, by Karhunen [8] by the method of approximating sums. We shall here consider only the case when  $Z$  is Karhunen's spectral function. Denote by

$$(D_1) \quad \int_S f(a) dz ,$$

the definition by means of approximating sums and by

$$(D_2) \int_S f(a) dZ ,$$

the integral defined in the preceding section. We prove the following:

Theorem 4.1 :

If  $f(a)$  is such that

$$(D_1) \int_S f(a) dZ$$

exists, then

$$(D_2) \int_S f(a) dZ$$

exists and the two are equal.

Proof:

Since  $Z$  is a spectral r . s . f,

$$E \int Z(S) \bar{Z}^2 = m(S) ,$$

where  $m$  is a measure.

Case (i):  $f(a)$  is a simple function and  $m(R) < \infty$ . Here  $R$  denotes the real line. Clearly, it suffices to consider integrals over  $R$ .

Let

$$f(a) = v_k \quad \text{if} \quad a \in S_k ,$$

where the  $S_k$  are disjoint and

$$\sum_{k=1}^n S_k = R .$$

Then, by definition

$$I = (D_1) \int_R f(a) dZ = \sum_{k=1}^n v_k Z(S_k) .$$

$$(4.1) \quad E(zI) = \sum_{k=1}^n v_k E [z \cdot Z(S_k)]$$

$$= \sum_{k=1}^n v_k P_Z(S_k) = \int_R f(a) dP_Z ,$$

by the definition of the ordinary Lebesgue-Stieltjes integral of a simple function. Relation (4.1) show that

$$I = (D_2) \int_R f(a) dZ .$$

Case (ii):  $f(a)$  is bounded and  $m(R) < \infty$ . Then,  $f(a)$  is approximated uniformly by a sequence of simple functions  $f_n(a) \cdot (n = 1, 2, \dots)$  (see [8], p. 37). Let  $I(f)$  be the

integral defined according to  $(D_1)$  . By definition

$$I(f) = \lim_{n \rightarrow \infty} I(f_n) .$$

Now from case (1),  $I(f_n)$  coincides with the  $(D_2)$  integral of  $f_n$ .

Hence,

$$(4.2) \quad \mathbb{E} \int z I(f_n) = \int_R f_n(a) dP_z .$$

Since

$$\lim_{n \rightarrow \infty} f_n(a) = f(a)$$

uniformly,

$$(4.3) \quad \lim_{n \rightarrow \infty} \int_R f_n(a) dP_z = \int_R f(a) dP_z .$$

Further, since

$$\lim_{n \rightarrow \infty} I(f_n) = I(f),$$

weak convergence holds, i. e.,

$$(4.4) \quad \lim_{n \rightarrow \infty} \mathbb{E} \int z \cdot I(f_n) = \mathbb{E} \int z \cdot I(f) .$$

From (4.2), (4.3), and (4.4) we obtain

$$\mathbb{E} \int z \cdot I(f) = \int_R f(a) dP_z$$

(for all  $z$  in  $L_2$ ), which proves that,

$$I(f) = (D_2) \int_R f(a) dZ .$$

Case (iii):  $f(a)$  bounded,  $m(R) = \infty$ , but  $m$  is a  $\sigma$ -finite measure. That is, we can write

$$R = \sum_{n=1}^{\infty} R_n ,$$

where the  $R_n$  are disjoint and  $m(R_n) < \infty$ , for every  $n$ . Definition  $(D_1)$  gives

$$I = \int_R f(a) dZ = \sum_{n=1}^{\infty} \int_{R_n} f(a) dZ,$$

if the series converges in quadratic mean. From case (ii),

$$I_n = \int_{R_n} f(a) dZ$$

satisfies the relation,

$$E(z I_n) = \int_{R_n} f(a) dP_z .$$

Let

$$J_n = \sum_{k=1}^n I_k$$

and

$$W_n = \sum_{k=1}^n R_k .$$

Then,

$$(4.5) \quad E (z J_n) = \int_{W_n} f(a) dP_z .$$

Since

$$\lim_{n \rightarrow \infty} J_n = I ,$$

$$(4.6) \quad \lim_{n \rightarrow \infty} E(z \cdot J_n) = E(z I),$$

for every  $z$  in  $L_2$  . Also,

$$\int_{W_n} f(a) dP_z \longrightarrow \int_R f(a) dP_z , \quad \text{as } n \rightarrow \infty .$$

Hence,

$$E ( z I ) = \int_R f(a) dP_z ,$$

which proves again that

$$I = (D_2) \int_R f(a) dZ .$$

The case when  $f(a)$  is unbounded can be treated similarly. Thus we have shown that the integral defined in section 3 is at least as general as the integral given by the older definition.

5. Representability of a Random Set Function as a Stochastic Integral.

As a motivation for this section, consider a stochastic process  $x(t)$  of the second order with covariance function  $r(t,s)$ . Then we easily see that the Karhunen integral,

$$X(S) = \int_S x(t) \, d\mu ,$$

(when it exists) is an example of an additive  $r \cdot s \cdot f \cdot$  defined in (2). The question naturally arises: under what conditions can a  $r \cdot s \cdot f \cdot Z(S)$  be expressed in the form of such an integral? We are able to prove the following result:

Theorem 5.1:

Suppose that the following conditions hold:

- (5.2) The linear space  $L_2(z)$  is separable.



(5.3)<sup>2</sup> Define the set function,

$$F(S) = \sup_z \frac{1}{\|z\|} \left| \mathbb{E} \int z \cdot Z(S) \right|, \quad z \neq 0.$$

The total variation (see [10 a], p. 278),  $\nu(S)$  of  $F(S)$  is finite for every bounded  $S$ . Then, there exists an element  $w(t)$  in  $L_2$ , for almost all  $t$ , such that,

$$Z(S) = \int_S w(t) \, d\nu,$$

the integral being taken in the sense of Karhunen.

Proof:

First of all, we need the result that  $\nu$  is a non-negative additive set function (in fact, a measure). This is proved separately as a lemma and is not new.

Now by definition,

---

2. This condition, as well as the lemma that follows, has been used by Dunford and Pettis in a similar situation (Transactions of the American Mathematical Society 47, 1946). An earlier "proof" of the author without conditions of this sort, but merely with the condition (i)  $\mathbb{E} \int Z(S)^2 = 0$ , whenever  $\mu(S) = 0$ ,  $\mu$  being a given measure, was found to be incorrect. Consequently, the theorem in its present form owes much to this condition of Dunford and Pettis. In [7], the result is stated with both (i) and (5.3), but (i) is unnecessary.

$$\begin{aligned}
|P_z|(S) &= \text{Total variation of } P_z \text{ over } S \\
&= \sup \sum_{i=1}^n |P_z(S_i)| \text{ over all finite} \\
&\quad \text{sequences of bounded sets } S_i \text{ contained} \\
&\quad \text{in } S. \\
&\leq \|z\| \cdot \sup \sum_{i=1}^n F(S_i) \\
&\leq \|z\| \cdot \text{Total variation of } F \text{ over } S \\
&= \|z\| \cdot \nu(S).
\end{aligned}$$

Therefore,  $|P_z| \ll \nu$ , for every  $z$ , and hence  $P_z \ll \nu$  for every  $z$ . Here the symbol  $\ll$  is used to mean "absolutely continuous with respect to".

Applying the Radon-Nikodym theorem to the family of signed measures  $P_z$ , we obtain, for every bounded Borel set  $S$ ,

$$(5.4) \quad P_z(S) = \int_S G_z(t) d\nu.$$

Further,  $|P_z|(S)$  which is the total variation of  $P_z$  over  $S$  is given by,

$$(5.5) \quad |P_z|(S) = \int_S |G_z(t)| d\nu.$$

Thus we have,

$$\int_S |G_z(t)| d\nu \leq \|z\| \cdot \nu(S),$$

for all  $z$  in  $L_2$ . And this holds for every bounded  $S$ . Therefore

$$(5.6) \quad |G_z(t)| \leq \|z\|,$$

for almost all  $t$ .

If  $a, b$  are real numbers

$$\begin{aligned} \int_S G_{az_1+bz_2}(t) d\nu &= P_{az_1+bz_2}(S) \\ &= aP_{z_1}(S) + bP_{z_2}(S) \\ &= a \int_S G_{z_1}(t) d\nu + b \int_S G_{z_2}(t) d\nu. \end{aligned}$$

Thus we get,

$$\int_S [G_{az_1+bz_2}(t) - aG_{z_1}(t) - bG_{z_2}(t)] d\nu = 0,$$

and this holds for all bounded Borel sets  $S$ . Hence,

$$(5.7) \quad G_{az_1+bz_2}(t) = a G_{z_1}(t) + b G_{z_2}(t),$$

for almost all  $t$ . Now, the exceptional set of  $t$ -points in (5.6) depends, in general on  $z$ , and in (5.7) the exceptional set may depend on the numbers  $a$ ,  $b$  as well as on  $z_1$  and  $z_2$ . It is this fact which presents the greatest difficulty in the proof and necessitates the imposing of the restriction (5.2).

Since the space  $L_2$  is separable, there exists a (denumerable) fundamental set,  $\Gamma$  in  $L_2$ . Let  $L_0$  be the linear set which spans  $\Gamma$ . That is,  $L_0$  consists of all finite linear combinations with rational coefficients of elements of  $\Gamma$ , and  $L_0$  is dense in  $L_2$ .

If  $z \in L_0$  only, one can find a single exceptional set  $S_0$  for both (5.6) and (5.7). ( $S_0 =$  union of a denumerable number of sets of  $\nu$ -measure zero.) Thus (5.6) and (5.7) hold for every  $z$  in  $L_0$  and for  $t \in S - S_0$ . We shall now make use of (5.2) and (5.3) to show that the same set  $S_0$  serves for every  $z \in L_2$ .

Let  $z$  be any element of  $L_2$ . Then there exists a sequence  $\{z_n\}$  of elements in  $L_0$ , such that,

$$E \int [z_n - z]^2 \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Hence,

$$\|z_n - z_m\| \rightarrow 0, \quad \text{as } m, n \rightarrow \infty. (z_n, z_m \in L_0).$$

From (5.6) and (5.7) we obtain,

$$\left| G_{z_n}(t) - G_{z_m}(t) \right| = \left| G_{z_n - z_m}(t) \right| \leq \|z_n - z_m\|,$$

for

$$t \in S - S_0 .$$

$$\rightarrow 0 \quad \text{as } m, n \rightarrow \infty .$$

Hence

$$(5.8) \quad \lim_{n \rightarrow \infty} G_{z_n}(t) \text{ exists} = G_0(t),$$

say, for  $t \in S - S_0$  . Since,

$$\left| G_{z_n}(t) \right| \leq \|z_n\| \quad \text{and} \quad \lim_{n \rightarrow \infty} \|z_n\| = \|z\| ,$$

it follows that

$$(5.9) \quad \left| G_0(t) \right| \leq \|z\| \quad (t \in S - S_0) .$$

Also, for every  $n$ ,

$$\left| G_{z_n}(t) \right| \leq C < \infty .$$

( $C$  is some constant, independent of  $n$ .) Hence by the Lebesgue convergence theorem ,

$$(5.10) \quad \lim_{n \rightarrow \infty} \int_S G_{z_n}(t) \, d\nu = \int_S G_0(t) \, d\nu .$$

Also, since

$$\lim_{n \rightarrow \infty} P_{z_n}(S) = P_z(S),$$

we obtain,

$$\begin{aligned} P_z(S) &= \lim_{n \rightarrow \infty} P_{z_n}(S) = \lim_{n \rightarrow \infty} \int_S G_{z_n}(t) d\gamma \\ &= \int_S G_0(t) d\gamma . \end{aligned}$$

Thus,

$$(5.11) \quad \int_S G_z(t) d\gamma = \int_S G_0(t) d\gamma .$$

On account of (5.9) and (5.11) we may, for our purposes replace  $G_z(t)$  by  $G_0(t)$ . Now let  $z_1, z_2$  be elements of  $L_2$ , and  $a, b$  real numbers.  $\{a_n\}, \{b_n\}$  are rational number sequences  $\rightarrow a$  and  $b$  respectively, as  $n \rightarrow \infty$ .  $\{z_{1n}\}, \{z_{2n}\}$  are sequences in  $L_0$  which  $\rightarrow z_1$  and  $z_2$  respectively. Writing,

$$z = az_1 + bz_2 \quad \text{and} \quad z_n = a_n z_{1n} + b_n z_{2n} ,$$

we see that

$$\|z - z_n\| \rightarrow 0,$$

as  $n \rightarrow \infty$ . Then from the relation,

$$G_{z_n} = a_n G_{z_{1n}} + b_n G_{z_{2n}},$$

we obtain the limiting relation (as  $n \rightarrow \infty$ ),

$$G_0(t) = a G_1(t) + b G_2(t),$$

for  $t \in S-S_0$ , where arguing as above we may identify  $G_0$ ,  $G_1$  and  $G_2$  with  $G_z(t)$ ,  $G_{z_1}(t)$  and  $G_{z_2}(t)$  respectively. In other words, the relation

$$(5.12) \quad G_{az_1+bz_2}(t) = a G_{z_1}(t) + b G_{z_2}(t)$$

holds for any  $z_1, z_2 \in L_2$  and  $t \in S-S_0$ . From (5.9) and (5.12), it is clear that for every  $t \in S-S_0$ ,  $G_z(t)$  is a bounded linear operation defined on the Hilbert space  $L_2$ . Hence there exists an element  $w(t)$  in  $L_2$ , such that,

$$G_z(t) = (z, w(t)) = E \{ z w(t) \},$$

for  $z$  in  $L_2$  and  $t \in S-S_0$ . Finally,

$$(5.13) \quad E \int z \cdot Z(S) = P_z(S) \\ = \int_S G_z(t) d\nu = \int_S E \{ z w(t) \} d\nu,$$

for all  $z$  in  $L_2$ . The relation (5.13) defines  $Z(S)$  as the Karhunen integral,

$$\int_S w(t) d\gamma .$$

Proof of Lemma:

Clearly  $\gamma(0) = 0$ , and  $\gamma$  is non-negative by definition. Now,

$$\gamma(S) = \sup \sum_{i=1}^n F(E_i) ,$$

for all finite sets  $E_1, \dots, E_n$  of disjoint bounded sets contained in  $S$ . Suppose

$$S = \sum_{k=1}^{\infty} S_k ,$$

(disjoint, bounded sets), and let  $E_1, \dots, E_n$  be any finite number of disjoint sets contained in  $S$ . Then,

$$E_i = \sum_k (E_i \cdot S_k) .$$

By the definition of total variation, we have,

$$\gamma(S) \geq \sum_{i=1}^n \sum_{k=1}^N F(E_i \cdot S_k) ,$$

where  $N$  is any positive integer.

$$= \sum_{k=1}^N \sum_{i=1}^n F(E_i \cdot S_k) .$$



Since this is true for any sequence of sets  $E_1, \dots, E_n$ ,

$$\gamma(S) \geq \sum_{k=1}^N \gamma(S_k),$$

for any  $N$ . As  $\gamma(E) < \infty$ , we have,

$$(5.16) \quad \gamma(S) \geq \sum_{k=1}^{\infty} \gamma(S_k).$$

By definition of  $\gamma$ , we can find a finite sequence  $E_1, \dots, E_n$  all contained in  $S$ , such that

$$(a) \quad \gamma(S) - \epsilon < \sum_{i=1}^n F(E_i),$$

where  $\epsilon > 0$  is arbitrary. Also for each  $E_i$  (by definition of  $F$ ) we can find a  $z_i$  in  $L_2$  such that,

$$(b) \quad \frac{1}{\|z_i\|} \left| P_{z_i}(E_i) \right| > F(E_i) - \delta_i.$$

Hence combining (a) and (b), by choosing the  $\delta_i$  sufficiently small, we get,

$$\gamma(S) - \epsilon' < \sum_{i=1}^n \frac{1}{\|z_i\|} \left| P_{z_i}(E_i) \right|.$$

Further,

$$E_1 = \sum_k (E_1 \cdot S_k),$$

and

$$\left| P_{z_1}(E_1) \right| \leq \sum_k \left| P_{z_1}(E_1 \cdot S_k) \right|.$$

Therefore,

$$\begin{aligned} \nu(S) - \epsilon' &\leq \sum_{i=1}^n \sum_k \frac{1}{\|z_i\|} \left| P_{z_i}(E_1 \cdot S_k) \right| \\ &\leq \sum_k \sum_{i=1}^n F(E_1 \cdot S_k), \end{aligned}$$

and this is true for every finite sequence  $\{E_i\}$ ,

$$(5.17) \quad \leq \sum_k \nu(S_k).$$

Combining (5.16) and (5.17), we have,

$$\nu(S) = \sum_k \nu(S_k).$$

## CHAPTER II.

### INTEGRAL REPRESENTATION OF A RANDOM FUNCTION

WHOSE COVARIANCE IS A CARLEMAN KERNEL.

#### 1. Introduction.

Suppose  $X(t)$  is a second order process whose parameter  $t$  ranges over the finite closed interval,  $[a, b]$ . When the covariance function  $R(t, s)$  is continuous in the square,

$$\{ a \leq t \leq b; \quad a \leq s \leq b \} ,$$

a representation for  $X(t)$  in series has been obtained by several writers, (see e.g. Loève [10] and [3] for reference to Karhunen) by using the theory of integral equations with a continuous kernel. This representation, however, is not generally valid if  $a = -\infty$ ,  $b = +\infty$ . The reason for this breakdown is that  $R(t, s)$  does not now satisfy the conditions of the Hilbert-Schmidt theory. For example, if

$$R(t, s) = e^{-|t-s|} , \quad -\infty < t < \infty ,$$

the integral equation,

$$\phi(t) = \lambda \int_{-\infty}^{\infty} e^{-|t-s|} \phi(s) ds$$

is known to have a non-denumerable set of eigenvalues. In fact, every value of  $\lambda > \frac{1}{2}$  is an eigenvalue and the equation has the

solutions

$$e^{+i\alpha t}, \quad \text{where} \quad \lambda = \frac{1 + \alpha^2}{2}. \quad [1a]$$

To consider this case, Dolph and Woodbury have in a recent paper [3], employed the theory of singular integral equations developed by Carleman [1]. They have obtained a series representation for  $X(t)$  which makes use of random Hellinger integrals.

It is the aim of this chapter to derive a representation for  $X(t)$  in the form of a random integral. This, we shall do in the following sections, making extensive use of Carleman's generalization of Mercer's theorem on positive definite kernels.

## 2. The Carleman Kernel of Class I.

Let  $X(t)$ ,  $a \leq t \leq b$  be a second order process whose covariance function,  $R(t,s)$  satisfies the following conditions of Carleman:

$$(2.1) \quad R^2(t) = \int_a^b [R(t,s)]^2 ds$$

exists and is finite,

$$(2.2) \quad \lim_{t' \rightarrow t} \int_a^b [R(t,s) - R(t',s)]^2 ds = 0,$$

where both (2.1) and (2.2) hold for every  $t$  except on a set of

denumerable points  $\xi_1, \xi_2, \dots$ , which have only a finite number of limit points.

(2.3) It is possible to select a finite number of points,

$\eta_1, \eta_2, \dots, \eta_m$  from among the  $\xi_\nu$ , such that the integral,

$$\int_{I_\delta} [R(t)]^2 dt,$$

exists and is finite, for an arbitrary  $\delta > 0$ , for every domain  $I_\delta$  obtained by excluding from  $[a, b]$ , the intervals,

$$|t - \eta_\nu| < \delta \quad (\nu = 1, 2, \dots, m).$$

The conditions (2.1) - (2.3) which define a Carleman kernel are the original ones stipulated by Carleman and can be considerably simplified. For instance it is sufficient only to assume that (2.1) holds for almost all  $t$  in  $[a, b]$ , say for  $t \in T$ . If we define,

$$R_n(t, s) = R(t, s)$$

whenever  $t$  and  $s \in E_n$  and

$$R_n(t, s) = 0$$

elsewhere, where  $E_n$  is the set of points for which  $R(t) \leq n$ , then

$$\int_a^b \int_a^b [R_n(t,s)]^2 dt ds < \infty ,$$

and the Hilber-Schmidt theory is valid for the kernel  $R_n(t,s)$ .

(2.4)  $R(t,s)$  is of class I. That is, the equation,

$$\phi(t) = \lambda \int_a^b R(t,s) \phi(s) ds$$

has no solution

$$\phi(t) \in L^2(a,b)$$

other than the trivial one for non-real  $\lambda$ . It is known that this is equivalent to saying that the integral operator

$$H(\ ) = \int_a^b R(t,s) (\ ) ds$$

is self-adjoint.

(2.5) Let  $D$  be the domain  $T \times T$  where  $T$  is the set defined above of  $t$  values for which  $R(t)$  is finite. Then  $R(t,s)$  is continuous for  $(t,s)$  in  $D$ .

Conditions (2.1) and (2.4) do not seem to admit of any easy statistical interpretation but are sufficiently general as to

include a large class of covariances. Before proceeding further, we remark that here we have taken  $a, b$  to be finite. This is no restriction since, if the limits are infinite, they can be made finite by a change of scale. The important point is the non-fulfillment of the Hilbert-Schmidt assumptions.

Let  $\{\lambda_\nu^{(n)}\}$  and  $\{\phi_\nu^{(n)}(t)\}$  be the eigenvalues and eigenfunctions respectively of the equation,

$$\phi^{(n)}(t) = \lambda^{(n)} \int_a^b R_n(t,s) \phi^{(n)}(s) ds .$$

Since  $R(t,s)$  is a function of non-negative type, so is  $R_n(t,s)$  by its definition. Hence all the  $\lambda_\nu^{(n)} \geq 0$ . Let,

$$\begin{aligned} \theta_n(t,s;\lambda) &= \sum_{0 < \lambda^{(n)} < \lambda} \phi_\nu^{(n)}(t) \phi_\nu^{(n)}(s) , & \text{if } \lambda > 0, \\ &= 0 , & \text{if } \lambda \leq 0 . \end{aligned}$$

Then  $\theta_n(t,s;\lambda)$  is a function of bounded variation with respect to  $\lambda$  in every finite interval,  $(\alpha, \beta)$ ,  $\alpha > 0$ . Carleman has shown that

$$\lim_{n \rightarrow \infty} \theta_n(t,s;\lambda) = \theta(t,s;\lambda) ,$$

exists for every real value of  $\lambda$  and for every point  $(t,s)$  in  $D$ .

In the next section, we state a number of properties of  $\theta(t,s;\lambda)$  and some auxiliary theorems relevant to our purpose. Most of these results are to be found in Carleman [1, 1a] and Stone [15], but the notation of the former author is adopted throughout this chapter.

### 3. Properties of $\theta(t,s;\lambda)$ and Some Auxiliary Theorems.

(3.1) For every  $(t,s)$  in  $D$ ,  $\theta(t,s;\lambda)$  is a function of bounded variation in  $\lambda$  in every finite  $\lambda$ -interval,  $(\alpha, \beta)$  with  $\alpha > 0$ .

(3.2)  $\theta(t,s;\lambda)$  is symmetric in  $t$  and  $s$  and continuous with respect to  $(t,s)$  in  $D$  for every fixed value of  $\lambda$ .

$$(3.3) \quad \Delta\theta(t,s;\lambda) = \theta(t,s;\lambda_2) - \theta(t,s;\lambda_1), \quad (\lambda_1 > 0),$$

is a function of non-negative type. That is,

$$\int_a^b \int_a^b \Delta\theta(t,s;\lambda) h(t) h(s) dt ds \geq 0,$$

for every  $h(t) \in L^2(a,b)$ .

(3.4) If  $\Delta_1, \Delta_2$  are disjoint intervals not having 0 as an end point, then

$$\int_a^b \Delta_1\theta(t,u;\lambda) \Delta_2\theta(u,s;\lambda) du = 0,$$



and

$$\int_a^b \Delta \theta(t, u; \lambda) \Delta \theta(u, s; \lambda) du = \Delta \theta(t, s; \lambda) .$$

The orthogonality property of  $\theta(t, s; \lambda)$  stated in (3.4) is of great importance in deriving our result. It is here that the necessity for restricting ourselves to kernels of class I becomes clear, for (3.4) does not necessarily hold if  $R(t, s)$  is not a kernel of class I.

$$(3.5) \quad \int_a^b R(t, u) \Delta \theta(u, s; \lambda) du = \int_{\Delta} \frac{1}{\lambda} d \theta(t, s; \lambda) .$$

(3.6) Since  $R(t, s)$  is of non-negative type and continuous in  $D$ ,

$$R(t, s) = \int_0^{\infty} \frac{1}{\lambda} d_{\lambda} \theta(t, s; \lambda) = \lim_{\substack{\epsilon \rightarrow +0 \\ \rho \rightarrow \infty}} \int_{\epsilon}^{\rho} \frac{1}{\lambda} d \theta(t, s; \lambda) .$$

(3.7) The following theorem will also be needed.

$$g_n(x) \quad (n = 1, 2, \dots) ,$$

is a sequence of square integrable functions over  $(a, b)$ , such that

$$(a) \quad \lim_{n \rightarrow \infty} g_n(x) = g(x),$$

almost everywhere, and

(b) for all  $n$ ,

$$\int_a^b [g_n(x)]^2 dx < C,$$

$C$  being a constant independent of  $n$ . Then if  $f(x)$  is square integrable over  $(a, b)$ ,

$$\lim_{n \rightarrow \infty} \int_a^b f(x) g_n(x) dx = \int_a^b f(x) g(x) dx.$$

This is a special case of Theorem IV\* in [1].

(3.8) Part of Loève's Fundamental Lemma: ([10], p 315).

If a random function,  $X_\tau(t)$ , ( $t \in A$ ) tends in quadratic mean to  $X(t)$  as  $\tau \rightarrow \tau_0$ , the covariance of  $X(t)$  is given by

$$\Gamma(t, t') = \lim_{\tau \rightarrow \tau_0} E [X_\tau(t) X_\tau(t')] \quad (t, t' \in A).$$

(We are concerned here with only real-valued random variables.)

4. Definition of the Random Variable  $\Delta \sigma(t, \lambda)$ .

Define the random variable,

$$\Delta \omega(t, \lambda) = \int_a^b X(u) \cdot \Delta \theta(t, u; \lambda) du,$$

where the Karhunen integral on the right side exists if

$$\int_a^b \int_a^b R(u, v) \Delta \theta(t, u; \lambda) \Delta \theta(s, v; \lambda) du dv ,$$

exists.  $\Delta$  is a finite  $\lambda$ -interval.

$$\begin{aligned} (4.1) \quad \mathbb{E} [X(t) \Delta \omega(s, \lambda)] &= \int_a^b R(t, u) \Delta \theta(s, u; \lambda) du \\ &= \int_{\Delta} \frac{1}{\lambda} d_{\lambda} \theta(t, s; \lambda) , \end{aligned}$$

from (3.5). Now

$$\begin{aligned} \mathbb{E} [\Delta \omega(t, \lambda) \cdot \Delta \omega(s, \lambda)] &= \int_a^b \int_a^b R(u, v) \Delta \theta(t, u; \lambda) \Delta \theta(s, v; \lambda) dudv \\ &= \int_a^b \Delta \theta(t, u; \lambda) \cdot \int_{\Delta} \frac{1}{\lambda} d_{\lambda} \theta(u, s; \lambda) du \\ (4.2) \quad &= \int_{\Delta} \frac{1}{\lambda} d_{\lambda} \theta(t, s; \lambda) . \end{aligned}$$

Proof of the last assertion in (4.2):

Divide  $\Delta$  into  $n$  sub-intervals  $\Delta_i$  ( $i = 1, \dots, n$ ).

and let  $|\Delta_i|$  denote the length of  $\Delta_i$ . Then,

$$\int_{\Delta} \frac{1}{\lambda} d\theta(u, s; \lambda) = \lim_{\max |\Delta_i| \rightarrow 0} \sum_{i=1}^n \frac{1}{\lambda_i + \epsilon_i} \Delta_i \theta(u, s; \lambda).$$

Here  $s$  is fixed and  $\epsilon \in T$ .

Write

$$J_n(u) = \sum_{i=1}^n \frac{1}{\lambda_i + \epsilon_i} \Delta_i \theta(u, s; \lambda)$$

$$\begin{aligned} \int_a^b \int J_n(u) \overline{J}^2 du &= \sum_{i, j=1}^n \frac{1}{(\lambda_i + \epsilon_i)(\lambda_j + \epsilon_j)} \int_a^b \Delta_i \theta(u, s; \lambda) \Delta_j \theta(u, s; \lambda) du \\ &= \sum_{i=1}^n \frac{1}{(\lambda_i + \epsilon_i)^2} \Delta_i \theta(s, s; \lambda) \end{aligned}$$

Hence

$$\lim_{n \rightarrow \infty} \int_a^b \int J_n(u) \overline{J}^2 du = \int_{\Delta} \frac{1}{\lambda^2} d\theta(s, s; \lambda) < \infty.$$

Therefore,

$$\int_a^b [J_n(u)]^2 du < C ,$$

for all  $n$ , the constant  $C$  being independent of  $n$ . Also,

$$\lim_{n \rightarrow \infty} J_n(u) \text{ exists} = \int_{\Delta} \frac{1}{\lambda} d\theta(u, s; \lambda) ,$$

for almost all  $u$ . Write,

$$f(u) = \Delta \theta(t, u; \lambda) .$$

( $t$  as well as  $s$  is kept fixed in  $T$ .) Since

$$\int_a^b [\Delta \theta(t, u; \lambda)]^2 du < \infty ,$$

for  $t$  in  $T$ , we may apply (3.7) and obtain,

$$\lim_{n \rightarrow \infty} \int_a^b f(u) J_n(u) du = \int_a^b f(u) \left[ \lim_{n \rightarrow \infty} J_n(u) \right] du .$$

The right hand side is

$$= \int_a^b \Delta \theta(t, u; \lambda) \int_{\Delta} \frac{1}{\lambda} d\theta(u, s; \lambda) du .$$

The left hand side is

$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} \left[ \sum_{i=1}^n \frac{1}{\lambda_i + \epsilon_i} \int_a^b \Delta \theta(t, u; \lambda) \Delta_i \theta(u, s; \lambda) du \right] \\
 &= \lim_{n \rightarrow \infty} \left[ \sum_{i=1}^n \frac{1}{\lambda_i + \epsilon_i} \int_a^b \Delta_i \theta(t, u; \lambda) \Delta_i \theta(u, s; \lambda) du \right]
 \end{aligned}$$

from (3.4)

$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} \left[ \sum_{i=1}^n \frac{1}{\lambda_i + \epsilon_i} \Delta_i \theta(t, s; \lambda) \right] \\
 &= \int_{\Delta} \frac{1}{\lambda} d\theta(t, s; \lambda) = \Delta \Gamma(t, s; \lambda),
 \end{aligned}$$

say.

By an exactly similar argument we have,

$$\mathbb{E} \left[ \Delta_1 \omega(t, \lambda) \Delta_2 \omega(s, \lambda) \right] = \int_a^b \Delta_1 \theta(t, u; \lambda) \int_{\Delta_2} \frac{1}{\lambda} d\theta(u, s; \lambda) du = 0,$$

if  $\Delta_1$  and  $\Delta_2$  are disjoint intervals (not having 0 as an end point).

Now consider any interval  $\Delta$ ,  $\alpha \leq \lambda \leq \beta$ , where

$\beta > \alpha > 0$ . Let  $\mathcal{T}$  denote a partition,

$$\alpha = \lambda_0 < \lambda_1 < \dots < \lambda_n = \beta . \text{ Write}^3$$

$$\sum_{\mathcal{T}} \phi \Delta \omega(t, \lambda) = \sum_{i=1}^n \phi(\lambda'_i) \Delta_i \omega(t, \lambda) ,$$

where  $\lambda_{i-1} \leq \lambda'_i \leq \lambda_i$ . Let  $m(\mathcal{T})$  denote the maximum length of the subintervals  $\Delta_i$ . Since  $\phi(\lambda)$  is uniformly continuous in  $(\alpha, \beta)$  we can choose a  $\delta > 0$ , such that

$$\left| \phi(\lambda_i) - \phi(\lambda_{i-1}) \right| < \epsilon ,$$

when  $\left| \lambda_i - \lambda_{i-1} \right| < \delta$ , ( $i=1, \dots, n$ ). Make the divisions so small that  $m(\mathcal{T}) \leq \delta$ . Define the function

$$\phi_{\mathcal{T}}'(\lambda) = \phi(\lambda'_i) ,$$

if  $\lambda_{i-1} \leq \lambda \leq \lambda_i$ . We have then

$$\left| \phi_{\mathcal{T}}'(\lambda) - \phi(\lambda) \right| < \epsilon ,$$

for  $\alpha \leq \lambda \leq \beta$ , since  $m(\mathcal{T}) \leq \delta$ . If  $\mathcal{T}_0$  is a "finer" subdivision of  $(\alpha, \beta)$  whose points of division include those of  $\mathcal{T}$ ,

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3. The notation used here is taken from [11], p. 68.

$$\sum_{\mathcal{I}'_0} \phi_{\mathcal{I}'} \Delta \omega(t, \lambda) = \sum_{\mathcal{I}} \phi \Delta \omega(t, \lambda) .$$

Hence,

$$\begin{aligned} & \sum_{\mathcal{I}} \phi \Delta \omega(t, \lambda) - \sum_{\mathcal{I}'_0} \phi \Delta \omega(t, \lambda) \\ &= \sum_{\mathcal{I}'_0} \phi_{\mathcal{I}'} \Delta \omega(t, \lambda) - \sum_{\mathcal{I}'_0} \phi \Delta \omega(t, \lambda) \\ &= \sum_{\mathcal{I}'_0} (\phi_{\mathcal{I}'} - \phi) \Delta \omega(t, \lambda) . \end{aligned}$$

Therefore,

$$\begin{aligned} & \mathbb{E} \left[ \sum_{\mathcal{I}} \phi \Delta \omega(t, \lambda) - \sum_{\mathcal{I}'_0} \phi \Delta \omega(t, \lambda) \right]^2 \\ &= \sum_{\mathcal{I}'_0} (\phi_{\mathcal{I}'} - \phi)^2 \Delta \Gamma(t, t; \lambda) , \end{aligned}$$

$\longrightarrow 0$  as  $m, n \rightarrow \infty$  in such a way that  $m(\mathcal{I}'), m(\mathcal{I}'_0) \rightarrow 0$ .

Therefore for every  $t$  in  $T$ ,

$$\sum_{\mathcal{I}} \phi \Delta \omega(t, \lambda)$$



tends in quadratic mean to a random variable, which we write as

$$\int_{\Delta} \phi(\lambda) d\omega(t, \lambda) .$$

Taking

$$\phi(\lambda) = \sqrt{\lambda} ,$$

define,

$$\Delta \sigma(t, \lambda) = \int_{\Delta} \sqrt{\lambda} d\omega(t, \lambda) .$$

Now

$$E[\Delta \sigma(t, \lambda) \cdot \Delta \sigma(s, \lambda)] = E\left[ \int_{\Delta} \sqrt{\lambda} d\omega(t, \lambda) \cdot \int_{\Delta} \sqrt{\lambda} d\omega(s, \lambda) \right]$$

$$= \lim_{n \rightarrow \infty} \left[ \sum_{i=1}^n \lambda_i E[\Delta_i \omega(t, \lambda) \Delta_i \omega(s, \lambda)] \right] ,$$

{ by using (3.8) and because,

$$E[\Delta_i \omega(t, \lambda) \Delta_j \omega(s, \lambda)] = 0, \quad \text{for } i \neq j . \}$$

$$= \lim_{n \rightarrow \infty} \left[ \sum_{i=1}^n \lambda_i \cdot \int_{\Delta_i} \frac{1}{\mu} d\theta(t, s; \mu) \right] .$$

Since  $\theta(t,s;\mu)$  is a function of bounded variation of  $\mu$ , we may write

$$\int_{\Delta_1} \frac{1}{\mu} d\theta(t,s;\mu) = \int_{\Delta_1} \frac{1}{\mu} d\theta^+(t,s;\mu) - \int_{\Delta_1} \frac{1}{\mu} d\theta^-(t,s;\mu),$$

where,

$$\theta^+(t,s;\mu) \quad \text{and} \quad \theta^-(t,s;\mu)$$

are non-decreasing functions of  $\mu$ . Then,

$$\begin{aligned} \sum_{i=1}^n \lambda_i \int_{\Delta_1} \frac{1}{\mu} d\theta(t,s;\mu) \\ &= \sum_{i=1}^n \lambda_i \int_{\Delta_1} \frac{1}{\mu} d\theta^+(t,s;\mu) - \sum_{i=1}^n \lambda_i \int_{\Delta_1} \frac{1}{\mu} d\theta^-(t,s;\mu) \\ &= A_n - B_n, \end{aligned}$$

say.

$$A_n = \sum_{i=1}^n \frac{\lambda_i}{\lambda_i + \epsilon_i} \Delta_1 \theta^+(t,s;\lambda)$$

by the mean value theorem.

$$= \Delta \theta^+(t, s; \lambda) - \sum_{i=1}^n \frac{\epsilon_i}{\lambda_i + \epsilon_i} \Delta_i \theta^+(t, s; \lambda)$$

$$\epsilon_i \ll |\Delta_i| \ll \max |\Delta_i| = \eta, \text{ say. Hence,}$$

$$\left| \sum_{i=1}^n \frac{\epsilon_i}{\lambda_i + \epsilon_i} \Delta_i \theta^+(t, s; \lambda) \right| \ll \frac{\eta}{\alpha} \sum_{i=1}^n |\Delta_i \theta^+(t, s; \lambda)|$$

$$= \frac{\eta}{\alpha} \Delta \theta^+(t, s; \lambda), \text{ since } \Delta_i \theta^+ \text{ is non-negative.}$$

$\rightarrow 0$  as  $n \rightarrow \infty$  in such a way that  $\eta \rightarrow 0$ . Therefore,

$$\lim_{n \rightarrow \infty} A_n = \Delta \theta^+(t, s; \lambda),$$

and similarly,

$$\lim_{n \rightarrow \infty} B_n = \Delta \theta^-(t, s; \lambda).$$

Finally we obtain,

$$\lim_{n \rightarrow \infty} \left[ \sum_{i=1}^n \frac{1}{\lambda_i} \int_{\Delta_i} \frac{1}{\mu} d\theta(t, s; \mu) \right] = \Delta \theta(t, s; \lambda).$$

Thus we have established the relation,

$$E \left[ \Delta \sigma(t, \lambda). \quad \Delta \sigma(s, \lambda) \right] = \Delta \theta(t, s; \lambda).$$

By a similar argument, if  $\Delta_1$  and  $\Delta_2$  are disjoint  $\lambda$ -intervals, we get

$$(4.4) \quad \mathbb{E}[\Delta_1 \sigma(t, \lambda) \Delta_2 \sigma(s, \lambda)] = 0 .$$

It is not surprising that we should obtain relation (4.3), for from (3.2) and (3.3)  $\Delta \theta(t, s; \lambda)$  is a symmetric function of non-negative type and is hence the covariance function of a random function  $\Delta \sigma(t, \lambda)$ . This is precisely what we have in (4.3), but this argument alone would not have given us the orthogonality relation (4.4) .

#### 5. Integral Representation for $X(t)$ .

Exactly as at the beginning of Section 4, we define the stochastic integral,

$$\int_{\alpha}^{\beta} \phi(\lambda) d\sigma(t, \lambda) = U_{\phi}(t; \alpha, \beta)$$

for every  $t$  in  $T$ . The covariance

$$\mathbb{E}[U_{\phi}(t; \alpha, \beta) U_{\phi}(s; \alpha, \beta)]$$

is given by Loève's fundamental lemma (3.8).

$$E[U_\phi(t; \alpha, \beta) U_\phi(s; \alpha, \beta)]$$

$$= \lim_{n \rightarrow \infty} \left[ E \left[ \sum_{i=1}^n \phi(\lambda_i) \Delta_{i\sigma}(t, \lambda) \cdot \sum_{j=1}^n \phi(\lambda_j) \Delta_{j\sigma}(s, \lambda) \right] \right]$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \phi^2(\lambda_i) \cdot E[\Delta_{i\sigma}(t, \lambda) \Delta_{i\sigma}(s, \lambda)] \text{ by (4.4),}$$

$$= \lim_{n \rightarrow \infty} \sum_{i=1}^n \phi^2(\lambda_i) \Delta_{i\theta}(t, s; \lambda) ,$$

$$(5.1) \quad = \int_{\alpha}^{\beta} \phi^2(\lambda) d_{\lambda} \theta(t, s; \lambda) ,$$

Taking

$$\phi(\lambda) = \frac{1}{\sqrt{\lambda}} ,$$

and writing,

$$(5.2) \quad U(t; \alpha, \beta) = \int_{\alpha}^{\beta} \frac{1}{\sqrt{\lambda}} d\sigma(t, \lambda) ,$$

we obtain

$$(5.3) \quad \mathbb{E} \left[ U(t; \alpha, \beta) U(s; \alpha, \beta) \right] = \int_{\alpha}^{\beta} \frac{1}{\lambda} d\theta(t, s; \lambda),$$

from (5.1) .

Let  $0 < \alpha' < \alpha$  and  $\beta' > \beta$ ,

$$\begin{aligned} & \mathbb{E} \left[ \bar{U}(t; \alpha', \beta') - U(t; \alpha, \beta) \right]^2 \\ &= \int_{\alpha'}^{\alpha} \frac{1}{\lambda} d\theta(t, t; \lambda) + \int_{\beta}^{\beta'} \frac{1}{\lambda} d\theta(t, t; \lambda), \end{aligned}$$

$\rightarrow 0$  as  $\alpha, \alpha' \rightarrow +0$  and  $\beta, \beta' \rightarrow +\infty$ , since

$$\int_0^{\infty} \frac{1}{\lambda} d\theta(t, s; \lambda)$$

exists as an improper integral for every  $(t, s)$  in  $D$  and equals  $R(t, s)$ . Hence,

$$\begin{array}{l} \text{l. q. m.} \\ \alpha \rightarrow +0 \\ \beta \rightarrow +\infty \end{array} \quad U(t; \alpha, \beta) = U(t)$$

exists, (for every  $t$  in  $T$ ) and we may write,

$$(5.4) \quad U(t) = \int_0^{\infty} \frac{1}{\sqrt{\lambda}} d\sigma(t, \lambda) .$$

Also,

$$(5.5) \quad E[U(t) U(s)] = \int_0^{\infty} \frac{1}{\lambda} d\theta(t, s; \lambda) .$$

Remembering the manner in which  $\Delta\sigma(t, \lambda)$  was defined, we obtain by repeating the procedure used in the preceding section,

$$(5.6) \quad E[X(t) \Delta\sigma(s, \lambda)] = \int_{\Delta}^{\beta} \frac{1}{\sqrt{\lambda}} d\theta(t, s; \lambda) .$$

By utilizing (5.6) we have

$$(5.7) \quad E[X(t) U(t; \alpha, \beta)] = \int_{\alpha}^{\beta} \frac{1}{\lambda} d\theta(t, t; \lambda) .$$

Making  $\alpha \rightarrow +0$ ,  $\beta \rightarrow +\infty$  in (5.7) we get,

$$(5.8) \quad E[X(t) \cdot U(t)] = \int_0^{\infty} \frac{1}{\lambda} d\theta(t, t; \lambda) .$$

Finally, for every  $t$  in  $T$ ,

$$\begin{aligned} \mathbb{E}[X(t) - U(t)]^2 &= R(t,t) - 2 \mathbb{E}[X(t) U(t)] + \mathbb{E}[U(t)]^2 \\ &= R(t,t) - \int_0^{\infty} \frac{1}{\lambda} d\theta(t,t;\lambda) , \end{aligned}$$

from (5.5) and (5.8)

$$= 0$$

because of Carleman's representation theorem (3.6).

Thus we have the integral representation

$$(5.9) \quad X(t) = \int_0^{\infty} \frac{1}{\sqrt{\lambda}} d_{\lambda} \sigma(t, \lambda) ,$$

for every  $t$  in  $T$ .

It must be remarked that the theorem of Carleman quoted in (3.6) assumes that  $R(t,s)$  fulfills the condition,

$$\int_a^b |g(t)| R(t) dt ,$$

is convergent for every  $g(t) \in L^2(a,b)$  .



CHAPTER III.

FUNCTIONALS OF A MARKOV PROCESS - THE DISCONTINUOUS CASE.

1. Definitions and Notation.

$X(t)$ ,  $(T_1 \leq t \leq T_2)$  is a Markov process with the transition probabilities,

$$F(t, x; \tau, e) = \Pr [X(\tau) \in e / X(t) = x],$$

where

$$\tau > t \quad \text{and} \quad e \in \mathcal{B}$$

which is a  $\sigma$ -algebra of subsets of  $\mathcal{X}$ .  $\mathcal{X}$  itself may be taken to be the real line, though our theory applies to more general spaces.

Following Fortet [5], we give below the definition of an additive functional of  $X(t)$ . The terminology used below is the same as that in [5].

To every interval  $(t, \tau)$  ( $t < \tau$ ) let there correspond a real numerical random variable  $L(t, \tau)$  which has the following properties:

$$(1.1) \quad L(t, \tau) = L(t, t') + L(t', \tau),$$

for every  $t < t' < \tau$ ; if we write

$$\mathcal{L}(t) = L(t_0, t),$$

$t_0$  being some fixed value,  $\mathcal{L}(t)$  is a random function of  $t$  and  $L(t, \tau)$  is the increment of  $\mathcal{L}(t)$  for the interval  $(t, \tau)$ .

(1.2) For every  $t, \tau, t'$ , with  $t < \tau$  and  $t' < t$  or  $t' > \tau$ , and for every  $x$  and  $\xi$ ,  $L(t, \tau)$  and  $X(t')$  are independent, given that  $X(t) = x$  and  $X(\tau) = \xi$ . This means, that if

$$G(t, x; \tau, \xi; \alpha) = \Pr \left[ L(t, \tau) < \alpha \mid X(t) = x, X(\tau) = \xi \right]$$

and

$$\begin{aligned} G(t', x'; t, x; \tau, \xi; \alpha) \\ = \Pr \left[ L(t, \tau) < \alpha \mid X(t) = x, X(\tau) = \xi; X(t') = x' \right] \end{aligned}$$

then,

$$G(t', x'; t, x; \tau, \xi; \alpha) = G(t, x; \tau, \xi; \alpha).$$

(1.3) For every  $t, t', \tau$  with  $t < t' < \tau$ , and for every  $x, x', \xi$ ,  $L(t, t')$  and  $L(t', \tau)$  are independent random variables, given that

$$X(t) = x, \quad X(t') = x', \quad X(\tau) = \xi.$$

We shall then say that  $L(t, \tau)$  is an additive functional of  $X(t)$ .

Notations.

$$\psi(t, x; \tau, \xi; v) = \mathbb{E} \left[ e^{ivL(t, \tau)} \mid X(t) = x, X(\tau) = \xi \right]$$

$$\phi(t, x; \tau; v) = \mathbb{E} \left[ e^{ivL(t, \tau)} \mid X(t) = x \right]$$

$$= \int_{\mathfrak{E}} \psi(t, x; \tau, \xi; v) d_{\xi} F(t, x; \tau, e).$$

Define

$$\mu(t, x; \tau, e; v) = \int_e \psi(t, x; \tau, x'; v) d_{x'} F(t, x; \tau, e')$$

$\mu$  is not a characteristic function (c.f.), but

$$\mu(t, x; \tau, \mathcal{X}; v) = \phi(t, x; \tau; v)$$

is. Fortet, in the paper cited above, has derived the following functional equation for  $\mu$ .

$$(1.4) \quad \mu(t, x; \tau, e; v) = \int_{\mathcal{X}} \mu(t', x'; \tau, e; v) \psi(t, x; t', x'; v) d_{x'} F(t, x; t', e')$$

or

$$(1.5) \quad = \int_{\mathcal{X}} \mu(t', x'; \tau, e; v) d_{x'} \mu(t, x; t', e').$$

The form (1.5) is interesting, as it resembles the equation satisfied by the transition probabilities:

$$(1.6) \quad F(t, x; \tau, e) = \int_{\mathcal{X}} F(t', x'; \tau, e) d_{x'} F(t, x; t', e').$$

The study of additive functionals of  $X(t)$  when  $X(t)$  is a continuous Markov process has been carried out by Fortet in the paper referred to above. In what follows, we shall be concerned with a purely discontinuous process  $X(t)$ . (see Feller, [4]).

2. The Purely Discontinuous Case.Assumptions on X(t):

- (a) For every fixed t and x,

$$F(t,x; \tau, \xi) = \int_0^{\tau-t} [1 - p(t,x) \cdot (\tau - t)] \cdot \delta(\xi - x) \\ + (\tau - t) \cdot p(t,x) \cdot P(t,x, \xi) + O(\tau - t);$$

where  $\tau > t$  and,

$$\delta(\xi - x) = 0 \text{ if } \xi < x, 1 \text{ if } \xi \geq x.$$

- (b)
- $p(t,x)$
- is non-negative, continuous in t, a measurable function of x and is bounded in every finite t interval.

- (c)
- $P(t,x, \xi)$
- is, for fixed t and x, a distribution function in
- $\xi$
- , continuous in t, and a measurable function of x.

These are the usual hypotheses made in the derivation of Feller's equation [4].

Assumptions on L(t, \tau).

- (a) For fixed t, x, and
- $\xi$
- ,

$$\lim_{\Delta t \rightarrow +0} \psi(t,x; t + \Delta t, \xi) =$$

$$\lim_{\Delta t \rightarrow +0} \psi(t,x; t - \Delta t, \xi) = H(t,x; \xi)$$

exists and is finite. (We have momentarily suppressed v which is kept fixed throughout the following argument.)

(β) For fixed  $t, x,$

$$\begin{aligned} & \lim_{\Delta t \rightarrow +0} \frac{\Psi(t, x; t + \Delta t, x) - 1}{\Delta t} \\ &= \lim_{\Delta t \rightarrow +0} \frac{\Psi(t, x; t - \Delta t, x) - 1}{\Delta t} = G(t, x, x) \end{aligned}$$

exists and is finite. From (α) and (β) it follows that

$$H(t, x, x) = 1.$$

We shall return to consider conditions (α) and (β) later on.

From equation (1.4) we have, ( $\Delta u > 0$ )

$$\begin{aligned} (2.1) \quad & \mu(u, y; \tau, e) \\ &= \int_{\mathfrak{X}} \mu(u + \Delta u, x'; \tau, e) \Psi(u, y; u + \Delta u, x') \\ & \quad \cdot d_{x'} P(u, y; u + \Delta u, e') \\ &= [1 - p(u, y) \cdot \Delta u] \cdot \mu(u + \Delta u, y; \tau, e) \cdot \Psi(u, y; u + \Delta u, y) \\ & \quad + p(u, y) \Delta u \int_{\mathfrak{X}} \mu(u + \Delta u, x'; \tau, e) \Psi(u, y; u + \Delta u, x') \\ & \quad \cdot d_{x'} P(u, y, x') + O(\Delta u). \end{aligned}$$

Hence

$$\begin{aligned}
 (2.2) \quad & \frac{\mu(u + \Delta u, y; \tau, e) \Psi(u, y; u + \Delta u, y) - \mu(u, y; \tau, e)}{\Delta u} \\
 & = p(u, y) \mu(u + \Delta u, y; \tau, e) \Psi(u, y; u + \Delta u, y) - p(u, y) \\
 & \quad \cdot \int_{\mathfrak{X}} \mu(u + \Delta u, x'; \tau, e) \Psi(u, y; u + \Delta u, x') d_{\mathfrak{X}, P(u, y, x')} \\
 & \quad + o(i)
 \end{aligned}$$

The left-hand side of (2.2) can be written as

$$\begin{aligned}
 & \mu(u + \Delta u, y; \tau, e) \cdot \left[ \frac{\Psi(u, y; u + \Delta u, y) - 1}{\Delta u} \right] \\
 & + \frac{\mu(u + \Delta u, y; \tau, e) - \mu(u, y; \tau, e)}{\Delta u}
 \end{aligned}$$

Making  $\Delta u \rightarrow 0$ , (a similar argument holds if  $\Delta u < 0$ ), because of conditions ( $\alpha$ ) and ( $\beta$ ), we find that  $\frac{\partial \mu}{\partial t}$  exists and we obtain,

$$\begin{aligned}
 & \frac{\partial \mu(t, x; \tau, e; v)}{\partial t} + G(t, x, x; v) \cdot \mu(t, x; \tau, e; v) = p(t, x) \mu(t, x; \tau, e; v) \\
 & - p(t, x) \int_{\mathfrak{X}} H(t, x, x'; v) \mu(t, x'; \tau, e; v) d_{\mathfrak{X}, P(t, x, x')}.
 \end{aligned}$$

Thus  $\mu$  satisfies the integro-differential equation

$$\begin{aligned}
 (2.3) \quad & \frac{\partial \mu}{\partial t} = [p(t, x) - G(t, x, x; v)] \cdot \mu(t, x; \tau, e; v) - p(t, x) \\
 & \cdot \int_{\mathfrak{X}} H(t, x, x'; v) \mu(t, x'; \tau, e; v) d_{\mathfrak{X}, P(t, x, x')}.
 \end{aligned}$$

3. On the Conditions (α) and (β).

We know that

$$\begin{aligned}
 \phi(t, x; t + \Delta t) &= \int_{\mathfrak{X}} \psi(t, x; t + \Delta t, \xi) \cdot d_{\xi} F(t, x; t + \Delta t, e) \\
 &= [1 - p(t, x) \cdot \Delta t] \psi(t, x; t + \Delta t, x) \\
 &\quad + p(t, x) \Delta t \int_{\mathfrak{X}} \psi(t, x; t + \Delta t, \xi) d_{\xi} P(t, x, \xi) \\
 &\quad + o(\Delta t) .
 \end{aligned}$$

Hence,

$$\begin{aligned}
 (3.1) \quad \frac{\phi(t, x; t + \Delta t) - 1}{\Delta t} &= \frac{\psi(t, x; t + \Delta t, x) - 1}{\Delta t} - p(t, x) \psi(t, x; t + \Delta t, x) \\
 &\quad + p(t, x) \int_{\mathfrak{X}} \psi(t, x; t + \Delta t, \xi) d_{\xi} P(t, x, \xi) \\
 &\quad + o(1) .
 \end{aligned}$$

From (3.1) it follows that we may replace (β) by the condition

$$(\beta^*): \quad \lim_{\Delta t \rightarrow 0} \frac{\phi(t, x; t + \Delta t) - 1}{\Delta t} = A(t, x)$$

exists and is finite.

4. Applications.

(1) Take

$$L(t, \tau) = X(\tau) - X(t),$$

 $\tau > t$ . Then

$$\psi(t, x; \tau, \xi; v) = e^{iv(\xi - x)}$$

$$H(t, x; \xi; v) = e^{iv(\xi - x)}.$$

Since

$$\psi(t, x; t + \Delta t, x; v) = e^{iv(x - x)} = 1,$$

$$G(t, x, x; v) = 0.$$

Equation (2.3) becomes,

$$(4.1) \quad \frac{\partial \mu}{\partial t} = p(t, x) \mu$$

$$- p(t, x) \int_{-\infty}^{\xi} e^{iv(x' - x)} \mu(t, x'; \tau, e; v) d_x P(t, x, x').$$

If we now put

$$F(t, x; \tau, \xi) = F(t, x; \tau, e),$$

where

$$e = (-\infty < x < \xi),$$



and if we assume that  $\frac{\partial F}{\partial t}$  exists for some fixed  $\xi$ ,  $\tau$ , then equation (4.1) holds for  $v=0$  and we have for any  $t, x$ , and for the values of  $\xi$ ,  $\tau$ , considered,

$$(4.2) \quad \frac{\partial F(t, x; \tau, \xi)}{\partial t} = p(t, x) \int F(t, x; \tau, \xi) - \int_{-\infty}^{\infty} F(t, x'; \tau, \xi) d_x P(t, x, x'),$$

which is Feller's equation.

(ii) Let  $V(t; x)$  be any function of  $t$  and  $x$ ,  $(-\infty < x < \infty)$ . If

$$L(t, \tau) = V \int \tau; X(\tau) - V \int t; X(t),$$

$\tau > t$ , then  $L$  is an additive functional of  $X(t)$ . Assume further that  $V$  is a continuous function of  $t$  and that  $\frac{\partial V}{\partial t}$  exists for fixed  $x$  and is finite.

The conditional c.f. of  $L(t, t+\Delta t)$ , given  $X(t) = x$ , and  $X(t+\Delta t) = \xi$ , is

$$\psi(t, x; t+\Delta t, \xi; v) = e^{iv \int V(t+\Delta t, \xi) - V(t, x)}.$$

Therefore,

$$H(t, x, \xi; v) = e^{iv \int V(t, \xi) - V(t, x)}.$$

Also,

$$\frac{V(t, x; t+\Delta t, x; v) - 1}{\Delta t} = iv \int \frac{V(t+\Delta t, x) - V(t, x)}{\Delta t} \int - \frac{\theta v^2}{2} \frac{\int V(t+\Delta t, x) - V(t, x) \int^2}{\Delta t},$$

where  $|\theta| < 1$ . The last term on the right hand side  $\rightarrow 0$ , as  $\Delta t \rightarrow 0$  and we get,

$$G(t, x, x; v) = iv \frac{\partial V(t, x)}{\partial t}.$$

Thus conditions  $(\alpha)$  and  $(\beta)$  are both satisfied and  $\mu$  in this case satisfies the equation

$$(4.3) \quad \frac{\partial \mu}{\partial t} = \int p(t, x) - iv \frac{\partial V(t, x)}{\partial t} \int \mu$$

$$- p(t, x) \int_{-\infty}^{\infty} e^{iv \int V(t, \xi) - V(t, x) \int} \cdot \mu(t, \xi; \tau, e v).$$

$$\cdot d_{\xi} P(t, x, \xi).$$

Note that for  $v=0$ , we again obtain Feller's equation.

### 5. Another Form for Equation (2.3).

In some applications it may be desirable to have (2.3) in a form which involves  $A(t, x; v)$ . (see section 3.) In equation (3.1), making  $\Delta t \rightarrow 0$ , we obtain

$$A(t, x) = G(t, x, x) - p(t, x) + p(t, x) \cdot \int_{\mathcal{X}} H(t, x; \xi) d_{\xi} P(t, x, \xi).$$

Substituting in (2.3), we have,

$$\begin{aligned}
 (5.1) \quad \frac{\partial \mu}{\partial t} + A(t, x; v) \mu(t, x; \tau, e; v) \\
 = p(t, x) \int_x^{\cdot} H(t, x, x'; v) [\mu(t, x; \tau, e; v) - \mu(t, x'; \tau, e; v)] \cdot \\
 \cdot d_x P(t, x, x') .
 \end{aligned}$$

Many interesting problems remain to be considered. For instance, one might consider cases which are neither continuous nor purely discontinuous. Also, one could consider applications of this theory which are different from the classical ones. We hope to return to these questions at a later date.

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