PROBABILISTIC BUCKLING BEHAVIOR OF STRUCTURES WITH RANDOM LACK-OF-FIT AND GEOMETRIC IMPERFECTIONS

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SUMMARY

Structures in service differ from the idealized perfect structure conceived and designed by an engineer. Inevitably the difference is caused by the presence of numerous defects, deviations and geometric lack-of-fit of component members in the actual structures. To serve its intended purpose in spite of the imperfections, a real structure must function in a manner reasonably close to the idealized structures envisaged by the engineer. Otherwise the effects of the actual structural imperfections must be properly considered in the design analyses.

For statically indeterminate structures, the presence of geometric lack-of-fit of the component members induces additional initial stresses to the nominal stresses caused by the applied and thermal load. It is well known that for structures which are sensitive to bifurcational buckling, the presence of initial stresses due to ill-fit reduces the buckling load capacity of the structure. On the other hand, geometric ill-fit of members occurs at random in magnitude and in spatial distribution. Therefore, to consider realistically the stability of structures with random initial stresses caused by lack-of-fit, the problem must be treated from a probabilistic point of view.

In this study, the buckling behavior of a structure with random lack-of-fit of members and geometric imperfections is represented by an appropriate finite element model. The probabilistic buckling behavior of the structure can be determined from the solution of the associating random eigenvalue problem. Generally, the probability distribution of the buckling load of such a structure cannot be determined exactly. In practice, however, the lower order statistical moments of the distribution can be determined approximately.

In this paper, four different approximate methods of finding the expectation and the variance of the random buckling load of a structure are presented. These methods are the so called perturbation method, hierarchy method, the Taylor series method and the Monte Carlo simulation. The theoretical bases of these methods are developed, and their efficiency and degree of accuracy are examined. Numerical results are presented for a number of simple structures for which exact solutions are available for comparison.

1. Introduction

For redundant structures, it is well known that the lack-of-fit of members induces initial stresses to the assembled structures. Depending on the initial stress state, it may or may not drastically reduce the buckling load capacity of the structure. Very often, large scale redundant structures are constructed from a large number of component members of a few typical "standard" lengths. In spite of the care and control exercised in the manufacturing and fabrication process, the "standard" members will not have exactly the same intended length. Furthermore the lack-of-fit of the members will occur randomly in magnitude and spatial distribution. For structures with a single redundancy, the absolute bound on the reduced buckling capacity can be assessed deterministically by assigning the same maximum and/or minimum lack-of-fit of the members to all the members. However, for multiple redundant structures, there is no way to evaluate such a deterministic bounds on the buckling load. The only rational approach to the problem is to treat the lack-of-fit of members as random, so that from the specified tolerance limits of error on member sizes, the buckling capacity of the assembled structure is evaluated probabilistically. Actually, from a designer's point of view, it may be more logical to specify a suitable tolerance limit on the member size error so that the probability of buckling of the structure due to the randomly distributed lack-of-fit of members is acceptably small. In any case, as far as it is known, such approaches have not been explored systematically.

In a different context, a number of investigators have previously studied the various aspects of random eigenvalues problems. Boyce [1], McDaniel [2], and Bernard and Bogdanoff [3] studied the buckling of columns with random imperfections by integrating the stochastic differential equation. Wirsching and Yao [4] used an analog computer to simulate the buckling behavior of columns subjected to random axial load, whereas Shinozuka and Astill [5] used Monte Carlo simulation technique to study the buckling behavior of beam columns containing random parameters. Haines [6] studied the random vibration of beams, and Fraser [7] and Amazigo, et al. [8], the buckling of columns with random imperfections using the so called heirarchy technique. For large scale structures, Collins and Thompson [9],[10] applied the linear perturbation technique to investigate the random matrix eigenvalue problem in which the elements are correlated. Also Hart and Collins [11] Hasselman and Hart [12] and Hart [13] using a technique of linear Taylor series expansion about the mean value to evaluate the lower order statistics of large scale random eigenvalue problems.

In this study, the buckling behavior of a rigid skeletal structure with random lack-of-fit of members is represented by an appropriate finite element model. In principle, the probabilistic buckling behavior of the structure can be determined from the solution of the associating random eigenvalue problem as demonstrated by Mak and Kelsey, using a simple hexagonal frame [14]. In practice, however, for practical large scale structures, the probability distribution of the buckling load of the imperfect structure can only be determined approximately in terms of lower order statistical moments of the distribution. In this paper, three different approximate methods of finding the mean value and the variance of the random

buckling load of a structure is presented. These are the perturbation method, hierarchy method, and the Monte Carlo simulation. The theoretical bases of these methods are presented.

2. Method of Analysis

The probabilistic buckling behavior of structure due to externally applied deterministic load and random lack-of-fit of members can be defined as a random eigenvalue problem as follows:

$$[\underset{\sim}{K} + \lambda \underset{\sim}{K}_g + \mu \underset{\sim}{K}_e]\underset{\sim}{q} = 0 \qquad (1)$$

in which $\underset{\sim}{q}$ is a nodal displacement vector, and $\underset{\sim}{K}$ and $\underset{\sim}{K}_g$ are the usual assembled elastic and geometric stiffness matrices commonly employed in the displacement method of analysis of structures [15]. In this case, μK_e can be considered as a modification of the geometric stiffness matrix caused by the lack-of-fit of members. Eq. 1 can be explained from the following physical argument. Due to external load, if initial member forces N_i are present in the i th member, in addition to forces $\underset{\sim}{K}$, forces $\lambda \underset{\sim}{K}_g$ are required to produce nodal displacement $\underset{\sim}{q}$. Furthermore, if the structure is not perfectly fit, additional member forces S_i are induced in the i th member by the lack-of-fit of the members. It is, therefore, necessary to require additional $\mu \underset{\sim}{K}_e$ forces to maintain the same $\underset{\sim}{q}$. In general, $\lambda \underset{\sim}{K}_g$ is related to the elemental geometric stiffness $(\underset{\sim}{kg})_i$ and N_i, according to the following:

$$\lambda \underset{\sim}{K}_g = \underset{\sim}{\beta}^T \begin{bmatrix} N_1(\underset{\sim}{k}_g)_1 & & \\ & \ddots & \\ & & N_i(\underset{\sim}{k}_g)i \\ & & & \ddots \end{bmatrix} \underset{\sim}{\beta} \qquad (2)$$

in which $\underset{\sim}{\beta}$ is a connectivity matrix. Similarly, $\mu \underset{\sim}{K}_e$ must have the following form:

$$\mu \underset{\sim}{K}_e = \underset{\sim}{\beta}^T \begin{bmatrix} S_1(\underset{\sim}{k}_g)_1 & & \\ & \ddots & \\ & & S_i(\underset{\sim}{kg})i \\ & & & \ddots \end{bmatrix} \underset{\sim}{\beta} \qquad (3)$$

in which S_i can be found by the following relation [15]:-

$$\underset{\sim}{S}_i = \underset{\sim}{k}_i \left\{ \underset{\sim}{\beta}_i \underset{\sim}{K}^{-1} \underset{\sim}{\beta}_i^T \underset{\sim}{k}_i - \underset{\sim}{I} \right\} \{e\}_i \qquad (4)$$

Therefore S_i is a random variable depending on the probability distribution of the random lack-of-fit of member e_i. Mathematically, $\mu \underset{\sim}{K}_e$ can be interpreted as a random perturbation from the base operators $\underset{\sim}{K}$ and $\underset{\sim}{K}_g$, so that λ and $\underset{\sim}{q}$ are the random eigenvalues and random eigenvectors of the problem defined by eq. (1). The desired solutions of eq. (1) are therefore the probability distribution of λ and $\underset{\sim}{q}$. For most practical problems, such probability distribution of

λ and $\underset{\sim}{q}$ are difficult to determine. Nevertheless, a great deal of the statistical information can be obtained from the lower order moments of the distribution such as the mean and variance of λ and $\underset{\sim}{q}$. These lower order moments can be estimated by the following approximate methods; perturbation, truncated hierarchy, and Monte Carlo simulation.

2.1 Perturbation Method

In general, the parameter μ in eq. (1) is related to the induced initial member force S and can be expressed in terms of the random lack-of-fit of member e. For sufficiently small $|\mu|$, $\mu\underset{\sim}{K}_e$ is a small perturbation from $\underset{\sim}{K}$ and $\underset{\sim}{K}_g$. Therefore both λ and $\underset{\sim}{q}$ can be expanded into the following power series of the small parameter μ :

$$\lambda = \sum_{i=0}^{\infty} \lambda_i \mu^i \tag{5}$$

$$\underset{\sim}{q} = \sum_{i=0}^{\infty} \underset{\sim}{q}_i \mu^i \tag{6}$$

To determine λ_i and $\underset{\sim}{q}_i$, eqs. (5) and (6) are substituted into eq. (1). Collecting the like coefficient of each power of μ, the following is obtained:

$$(\underset{\sim}{K}\underset{\sim}{q}_0 + \lambda_0\underset{\sim}{K}_g \underset{\sim}{q}_0) + \mu (\underset{\sim}{K}_e\underset{\sim}{q}_0 + \underset{\sim}{K}\underset{\sim}{q}_1 + \lambda_0\underset{\sim}{K}_g \underset{\sim}{q}_1 + \lambda_1\underset{\sim}{K}_g \underset{\sim}{q}_0)$$

$$+ \mu^2 (\underset{\sim}{K}_e \underset{\sim}{q}_1 + \underset{\sim}{K} \underset{\sim}{q}_2 + \lambda_0\underset{\sim}{K}_g \underset{\sim}{q}_2 + \lambda_1\underset{\sim}{K}_g \underset{\sim}{q}_1 + \lambda_2\underset{\sim}{K}_g \underset{\sim}{q}_0) + \ldots = 0 \tag{7}$$

In order to satisfy eq. (1), the bracketed terms in eq. (7) must vanish identically, i.e.,

$$(\underset{\sim}{K} + \lambda_0 \underset{\sim}{K}_G) \underset{\sim}{q}_0 = 0 \tag{8}$$

$$\underset{\sim}{K}_e \underset{\sim}{q}_0 + \underset{\sim}{K} \underset{\sim}{q}_1 = -\lambda_0 \underset{\sim}{K}_g \underset{\sim}{q}_1 - \lambda_1 \underset{\sim}{K}_g \underset{\sim}{q}_0 \tag{9}$$

$$\underset{\sim}{K}_e \underset{\sim}{q}_1 + \underset{\sim}{K} \underset{\sim}{q}_2 = -\lambda_0 \underset{\sim}{K}_g \underset{\sim}{q}_2 - \lambda_1\underset{\sim}{K}_g \underset{\sim}{q}_1 - \lambda_2\underset{\sim}{K}_g \underset{\sim}{q}_0 \tag{10}$$

from eq. (8), it is observed that λ_0 and $\underset{\sim}{q}_0$ are the eigenvalues and eigenvector of the unperturbed problem, i.e., they are the buckling load and buckling mode of the perfect structure and can be obtained routinely. On the other hand, λ_1, can be found by forming the scaler product of eq. (9) with $\underset{\sim}{q}_0$ so that

$$(\underset{\sim}{K}\underset{\sim}{q}_1, \underset{\sim}{q}_0) + \lambda_0 (\underset{\sim}{K}_g \underset{\sim}{q}_1, \underset{\sim}{q}_0) = -(\underset{\sim}{K}_e \underset{\sim}{q}_0, \underset{\sim}{q}_0) - \lambda_1 (\underset{\sim}{K}_g \underset{\sim}{q}_0, \underset{\sim}{q}_0) \tag{11}$$

From the orthogonality condition of $\underset{\sim}{q}_1$, the left-hand side of eq. (11) vanishes so that

$$\lambda_1 = \frac{-(\underset{\sim}{K}_e \underset{\sim}{q}_o , \underset{\sim}{q}_o)}{(\underset{\sim}{K}_g \underset{\sim}{q}_o, \underset{\sim}{q}_o)} \tag{12}$$

Keeping only the linear terms in eq. (5), then

$$\lambda = \lambda_o - \mu \frac{(\underset{\sim}{K}_e \underset{\sim}{q}_o, \underset{\sim}{q}_o)}{(\underset{\sim}{K}_g \underset{\sim}{q}_o, \underset{\sim}{q}_o)} \tag{13}$$

Therefore the expected value of the random variable λ becomes

$$E[\lambda] = \lambda_o + \frac{E[\mu]}{(\underset{\sim}{K}_g \underset{\sim}{q}_o, \underset{\sim}{q}_o)} E[\underset{\sim}{K}_e \underset{\sim}{q}_o, \underset{\sim}{q}_o] \tag{14}$$

and the variance of λ becomes:

$$\text{Var}[\lambda] = \frac{\mu^2}{(\underset{\sim}{K}_g \underset{\sim}{q}_o, \underset{\sim}{q}_o)^2} \text{Var}[\underset{\sim}{K}_e \underset{\sim}{q}_o, \underset{\sim}{q}_o] \tag{15}$$

It is noted that if the imperfection e has a zero mean so that $E[\mu] = 0$, then $E[\lambda]$ equals to the eigenvalue of the perfect structure, λ_o in eq. (8).

2.2 Hierarchy Method

Another very powerful approximate technique to solve the random eigenvalue problem is the truncated hierarchy method. The advantage of this method over the perturbation technique is that the random parameter of the problem does not need to be restricted to small perturbation about the mean value. However, it is worthy to note that in the previous method, the solution of the problem is first expressed in terms of the random coefficients of the problem so that the statistical implication of the solution is evaluated "honestly" (though honesty does not necessarily imply accuracy). Whereas the hierarch method is really a "dishonest" method [16] [17]. Basically, the hierarchy method reduces the random problem into a deterministic one by first averaging the governing stochastic equations. By making appropriate statistical independence assumptions for the averaged equations, the statistical moment of the solution is determined directly (hence dishonestly).

To apply the hierarchy method to the random eigenvalue problem, the mean value operator $E[\ ]$, is applied to all the random quantities in eq. (1) to obtain the following:

$$K\, E[\underset{\sim}{q}] + \underset{\sim}{K}_g\, E[\lambda\, \underset{\sim}{q}] + \underset{\sim}{K}_e\, E[\mu\, \underset{\sim}{q}] = 0 \tag{16}$$

As it stands, eq. (16) cannot be solved because it contains $E[\underset{\sim}{q}]$, as well as the higher joint moments of $E[\mu\, \underset{\sim}{q}]$ and $E[\lambda\, \underset{\sim}{q}]$. Additional equations must be provided so that the higher moment of $\mu\underset{\sim}{q}$ and $\lambda\underset{\sim}{q}$ can be expressed in terms of the lower order moments. Generally, these equations will involve still higher moments of the random variables so that this procedure will lead to an infinite hierarchy of

equations. In practical application, the infinite process can be truncated by some appropriate closure assumptions. Usually these assumptions involve a systematic replacement of higher moments in terms of lower order ones. For eq. (16) a simple closure assumption is:

$$E[\mu \underset{\sim}{q}] = E[\mu]\, E[\underset{\sim}{q}]$$
$$E[\lambda \underset{\sim}{q}] = E[\lambda]\, E[\underset{\sim}{q}] \qquad (17)$$

Substituting eq. (17) into eq. (16), the following deterministic eigenvalue problem is obtained:

$$(\underset{\sim}{K} + E[\mu]\underset{\sim}{K}_e)\ E[\underset{\sim}{q}] = E[\lambda]\ \underset{\sim}{K}_g\ E[\underset{\sim}{q}] \qquad (18)$$

in which $E[\lambda]$ and $E[q]$ can be determined routinely. If the random lack-of-fit of the members have a zero mean, then $E[\mu] = 0$ so that $E[\lambda]$ and $E[\underset{\sim}{q}]$ become simply the buckling load and buckling modes of a perfect structure or $E[\lambda] = \lambda_o$, $E[q] = \underset{\sim}{q}_o$. This agrees with the results predicted by the first order perturbation technique indicated by eq. (14).

To estimate the variance of λ, $\mathrm{Var}[\lambda]$, eq. (1) is multiplied by λ and then μ, both equations are then averaged to obtain the following:

$$\underset{\sim}{K}\, E[\lambda\, \underset{\sim}{q}] + \underset{\sim}{K}_g E[\lambda^2 \underset{\sim}{q}] + \underset{\sim}{K}_e\, E[\lambda\mu \underset{\sim}{q}] = 0 \qquad (19)$$

$$\underset{\sim}{K}\, E[\mu\, \underset{\sim}{q}] + \underset{\sim}{K}_g E[\lambda\mu \underset{\sim}{q}] + \underset{\sim}{K}_e\, E[\mu^2 \underset{\sim}{q}] = 0 \qquad (20)$$

To solve eqs. (19) and (20), improved estimate of the second hierarchy can be made by the cumulant discard method [6] in which the n th hierarchy neglects the following quantity

$$\left\{ \frac{\partial^{n+1}}{\partial\xi_1 \cdots \partial\xi_n \partial\eta}\ \ln\left\{\left[\exp\left(i \sum_{k=1}^{n} \xi_i \beta_i + i\eta\,(u(x) - E[u(x)])\right)\right]\right\}\right. \qquad (21)$$
$$\xi_1 = 0, \ \ldots\ \xi_n = 0, \ \eta = 0$$

It can be shown that, from eq. (21)

$$E[\mu\, \beta_1] = E[u]\, E[\beta_1]$$
$$E[\mu\, \beta_1 \beta_2] = E[\beta_1 \beta_2]\, E[u] + E[u\beta_1]\, E[\beta_2] + \qquad (22)$$
$$E[u\beta_2]\, E[\beta_1] - 2E[u]\, E[\beta_1]\, E[\beta_2]$$

From eq. (22), the following closure assumptions are obtained:

$$E[\lambda\, \underset{\sim}{q}] = E[\lambda]\, E[\underset{\sim}{q}] \qquad (23)$$

$$E[\mu\, \underset{\sim}{q}] = E[\mu]\, E[\underset{\sim}{q}] \qquad (24)$$

$$E[\lambda^2 \underset{\sim}{q}] = E[\lambda^2]\, E[\underset{\sim}{q}] + 2E[\lambda]\, E[\lambda \underset{\sim}{q}] - 2E[\underset{\sim}{q}]\, E[\lambda]^2 \tag{25}$$

$$E[\mu^2 \underset{\sim}{q}] = E[\mu^2]\, E[q] + 2E[\mu]\, E[\mu \underset{\sim}{q}] - 2E[\underset{\sim}{q}]\, E[\mu]^2 \tag{26}$$

$$E[\lambda\mu \underset{\sim}{q}] = E[\lambda\mu]\, E[\underset{\sim}{q}] + E[\lambda \underset{\sim}{q}]\, E[\mu] + E[\mu \underset{\sim}{q}]\, E[\lambda] - 2E[\lambda]\, E[\mu]\, E[\underset{\sim}{q}] \tag{27}$$

Substituting eqs. (24) and (26) into eq. (20), the following equation is obtained:

$$\underset{\sim}{K}\, E[\mu]\, E[\underset{\sim}{q}] + \underset{\sim}{K}_g\, E[\lambda\mu \underset{\sim}{q}] + \underset{\sim}{K}_e \left\{ E[\mu^2]\, E[q] + 2E[\mu]\, E[\mu \underset{\sim}{q}] - 2E[q] E[\mu]^2 \right\} = 0 \tag{28}$$

Again if $E[\mu] = 0$, eq. (28) becomes

$$E[\lambda\mu \underset{\sim}{q}] = -\underset{\sim}{K}_g^{-1} \underset{\sim}{K}_e \left\{ E[\mu^2]\, E[\underset{\sim}{q}] \right\} \tag{29}$$

Substituting eqs. (23), (25) and (29) into eq. (19), the following equation is obtained:

$$\left(\underset{\sim}{K} + \underset{\sim}{K}_g \left\{ \frac{E[\lambda^2]}{E[\lambda]} \right\} - \underset{\sim}{K}_e\, \underset{\sim}{K}_g^{-1}\, \underset{\sim}{K}_e \left\{ \frac{E[\mu^2]}{E[\lambda]} \right\} \right) E[\underset{\sim}{q}] = 0 \tag{30}$$

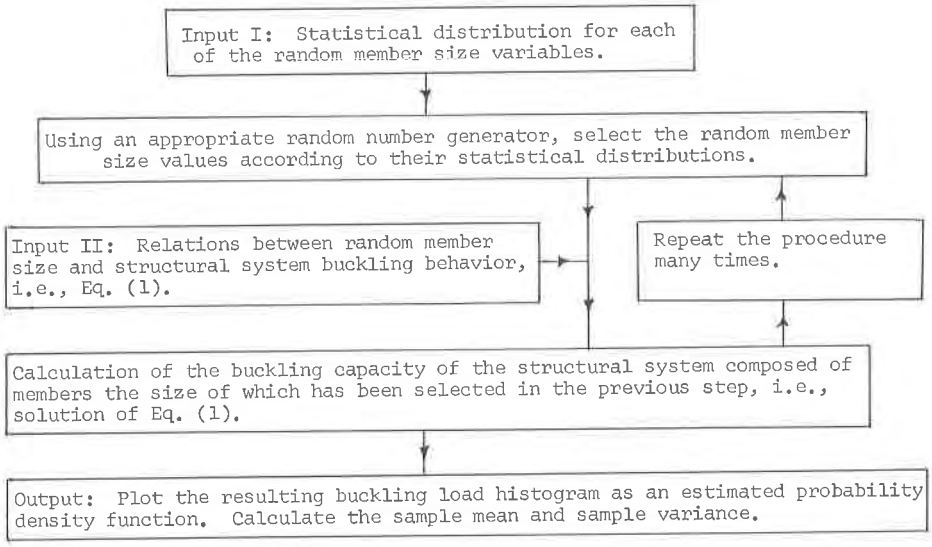
For a given $E[\mu^2]$, and $E[\lambda]$ and $E[\underset{\sim}{q}]$ as determined from eq. (18), $E[\lambda^2]$ can be extracted from the eigenvalue of eq. (30). From the definition of variance, $Var[\lambda]$ is simply:

$$Var[\lambda] = E[\lambda^2] - E[\lambda]^2 \tag{31}$$

2.3 Monte Carlo Simulation

Monte Carlo simulation is also commonly referred to as synthetic sampling. It has been used extensively to study complex system performance from the statistical data of component members. Generally it relies on a high-speed digital computer and an efficient algorithm to randomly select the component members from the available sample population and to analyze the assembled, synthesized system. With the advent of the matrix and finite element methods, the deterministic stability analysis of large scale, complex structural systems does not present any great difficulty. For a given structural system with a given applied load the elastic and geometric stiffness matrices $\underset{\sim}{K}$ and $\underset{\sim}{K}_g$ in eq. (1) can be assembled in the usual manner. From the given probability distribution of the lack-of-fit of the members, the member size of a sample structure is selected by an appropriate random number generator [18] so that the corresponding initial stress matrix $\mu\underset{\sim}{K}_e$ of the structure can be evaluated. The resulting eigenvalue problem is completely deterministic and can be solved in a routine manner. By repeating the synthetic

sampling and solution process for a sufficiently large number of times, a histogram as an approximate probability density function for the buckling load of the structure is obtained. The statistical moments of the histogram such as the sample mean and sample variance can be calculated easily. The entire simulation procedure is outlined in the following flow chart:



Because Monte Carlo simulation involves the generation of random numbers, the results are subjected to statistical fluctuations, and the estimated probability density will have an associated error band. Generally the larger the number of synthetic samples in the simulation, the more precise will be the histogram to approximate the actual probability density function. In practice, however, the allowable error band, expressed in terms of percentage of confidence limits for the estimated parameters, (in this case, the sample moments), is initially specified. The required number of Monte Carlo trials can then be determined from the usual statistical methods for estimates of parameters, [19]. For instance, if $(1 - \alpha)$ is the desired probability or confidence limit that the sample mean does not differ from the real mean by more than $\pm \xi$, then the required number of Monte Carlo trials, n is found approximately by [19]:

$$n = \left\{ \frac{\varphi(1 - \frac{\alpha}{2})\sigma'}{\xi} \right\}^2 \qquad (32)$$

in which $\varphi(1 - \frac{\alpha}{2})$ is the $(1 - \frac{\alpha}{2})$ (100) percent point of a standard normal distribution and σ' is an initial estimate of the sample standard deviation.

If the buckling load of each sample structure is λ_i, then the sample average of the buckling load of the structure, $\bar{\lambda}$, is simply

$$\bar{\lambda} = \frac{1}{n} \sum_{i=1}^{n} \lambda_i \tag{33}$$

and the sample variance of λ_i, s^2 is:

$$s^2 = \frac{1}{n} \sum_{i=1}^{n} (\lambda_i - \bar{\lambda})^2 \tag{34}$$

3. Conclusion

Three approximate methods are presented which can be used to estimate the lower order moments of the probability distribution of the buckling capacity of multiple redundant structures with random lack-of-fit or geometric imperfection. Among the three methods, Monte Carlo simulation requires a large amount of computation in order to obtain reasonably reliable estimates. The perturbation technique is limited to structures with small random imperfections from the mean perfect state, whereas the hierarchy technique is not restricted to small perturbations. It is noted these techniques can be applied to solve nonlinear random buckling problem of shells with geometric imperfections.

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