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PHOSPHORUS CYCLING IN LAKES

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ABSTRACT

A predictive model for phosphorus in lakes has been developed and verified. The model shows excellent agreement between observed and predicted average phosphorus concentrations for lakes with a wide range of hydraulic detention times (1 to 700 years) and mean depths (14 to 313 meters). At present the model should be applied only to lakes with oxic hypolimnetic waters. The model has been verified for conditions of constant nutrient loadings.

The model provides an explanation for the effects of mean lake depth on water quality noted by several limnologists. Vertical exchange of phosphorus forms across the thermocline and natural aggregation within the lake are important processes for the transport and deposition of phosphorus in lakes. The significance of these processes increases with lake depth.

Nomographs are presented which summarize model calculations and permit the model to be used for predictive purposes. Required information includes mean lake depth and areal hydraulic loading. Predictions of permissible phosphorus loadings can be made for lakes which meet the conditions assumed in formulating the model (e.g., oxic hypolimnetic waters).

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SUMMARY AND CONCLUSIONS

A model for phosphorus in lakes has been developed and verified. The model shows excellent agreement between observed and predicted average phosphorus concentrations for lakes with a wide range of hydraulic detention times (1 to 700 years) and mean depths (14 to 300 meters). In its present form the model is limited to lakes with oxic hypolimnetic waters. The minimum data required for predictions are the land-based phosphorus loading ($gmsP/m^2$-yr), the areal hydraulic loading (m/yr) and the mean lake depth (m). The model is useful for managing water quality in lakes, and provides an explanation for the morphometric effects of mean lake depth observed by several limnologists.

This research has led to the following conclusions:

1. Vertical exchange of phosphorus forms across the thermocline region and natural aggregation of particles within the lake waters appear to be important mechanisms for the transport and deposition of phosphorus in lakes. The significance of these processes increases with lake depth. They should be evaluated in other models for other particulate substances in lakes.
2. Deep lakes are able to assimilate higher nutrient loadings than shallow ones. This conceptual result is in agreement with the observations of Vollenweider (1968).
3. For lakes with similar mean depths, increased hydraulic loading permits increased phosphorus loading. This conceptual result is in agreement with other observations by Vollenweider (1974) based on

Lakes Tahoe and Zurich.

4. The model developed in this research is useful for predicting average seasonal phosphorus concentrations in lakes. It will not simulate rapid temporal changes which may be due to storms or other short-term variations in lake conditions. This is due primarily to the use of seasonal averages for the reaction coefficients.
5. The model is capable of predicting the rate at which a lake will respond to changes in nutrient inputs. Verification of these temporal predictions awaits sufficient data at present.

The model developed in this research can be used either directly or with some modifications to answer such questions as:

1. Given a desired phosphorus concentration in a lake, what phosphorus inputs can be permitted? What effluent guidelines are appropriate?
2. Given a proposed phosphorus loading, what lake conditions (phosphorus concentration) might result?
3. Given a proposed change in phosphorus loading, how long will it take for stable conditions in the lake to result?
4. What might be the effects on a lake of such management practices as hypolimnetic aeration, reservoir mixing, or alteration in the location, timing, and scale of fertilizer applications?

RECOMMENDATIONS

1. The model developed in this research can and should be used to analyse the causes of present conditions in many lakes and as a basis for evaluating alternative plans for managing water quality in these lakes.
2. Field studies which are conducted for water quality control in lakes must place emphasis on evaluating inputs of nutrients and water in addition to traditional studies of the lakes themselves. Data regarding nutrient inputs to lakes are almost nonexistent in North Carolina. Some information may become available from the National Eutrophication Survey. Additional measurements or estimates of nutrient inputs should be made if soundly-based measures for water quality control are to be formulated.
3. Research should be conducted to enlarge the range of application of the model. Particular emphasis should be given to lakes with anoxic hypolimnetic waters and to verification of the temporal characteristics of the model. The former is needed to model shallow lakes effectively. The latter is needed to permit reliable predictions about the rate at which changes will occur within a lake after changes in inputs occur.
4. Studies of the mechanisms responsible for mass transport across the thermocline region and of natural aggregation in lakes should be made. These will provide additional evidence for the reliability of the model and also assist in extending its application.

INTRODUCTION

This report is a presentation of an input-output model for phosphorus in lakes. The research has been undertaken for two purposes: (1) to formulate and test a model for guiding certain decisions about the management of water quality in lakes, and (2) to determine causes for the inverse relationships between several lake conditions and mean lake depth observed by some limnologists.

No attempt is made here to present a comprehensive review of the vast limnological literature. Several important and imaginative modeling efforts are not included. The limnological investigations which are reviewed here are considered because they provide a base for the present work. Some are essentially empirical. Some are primarily conceptual. All are quantitative.

Empirical Models

Rawson (1955) considers that the factors affecting lake productivity can be conveniently classified into three groups: edaphic (those dealing with nutrient supply), climatic (those dealing with energy supply), and morphometric (those dealing with hydrology and the shape of a lake basin). Evidence for the importance of morphometry as an important factor in the productivity of large lakes is presented. Measurements (mass of organisms per unit of lake surface area) of the standing crop of net plankton, the bottom fauna, and annual commercial fish production were obtained for a number of lakes in western Canada and northern United States. All of these

variables show strong inverse relationships with mean lake depth, regarded by Rawson as the best single index to morphometric conditions. Shallow lakes are shown to produce more plankton, bottom organisms, and fish than deep lakes. Empirical mathematical relationships for this effect are derived. No causes for these relationships are proposed.

Sakamoto (1966) notes inverse correllations between chlorophyll measurements during circulation periods and the mean depths of twelve lakes in central Japan. Deep lakes are observed to contain lower concentrations of chlorophyll (mg/m^3) in their euphotic zones and lower total amounts of chlorophyll in their water columns (mg/m^2) than shallow lakes. A close relationship is noted between chlorophyll content and total phosphorus or total nitrogen contents in several lakes. Nutrients and light are considered in a model developed to explain these observations. Sakamoto indicates that the total phosphorus content controls the production of phytoplankton in these lakes. For deep oligotrophic and mesotrophic lakes, light limitation is also proposed as a significant factor during circulation periods.

Vollenweider (1968) has built on these works by evaluating both mean lake depth and the input of land-based nutrients as factors in determining the trophic state of lakes. Using data from forty lakes in Europe and North America, he establishes an empirical relationship between phosphorus loading (e.g., gms P/m^2-yr), mean lake depth, and the trophic state of these lakes. For a given trophic level, deep lakes are shown to be capable of assimilating higher nutrient loadings than shallow ones. Based on this empirical analysis, permissible loading levels (loadings which maintain an

oligotrophic state) and dangerous loading levels (those which can cause a transition to an eutrophic state) are determined for lakes with mean depths from 5 to 200 meters. For example, the permissible phosphorus loading for a lake with an average depth of 10 m is given as 0.10 gms P/m^2-yr, and a lake with a depth of 200 m is considered capable of assimilating 0.60 gms P/m^2-yr while remaining in an oligotrophic state.

Conceptual Models

Stumm and Morgan (1962) have applied stoichiometric concepts to lake eutrophication. It can be shown that the introduction of 1 mg of P into a lake has the potential in a single pass through the phosphorus cycle of producing 115 mg of algae in the epilimnion and then causing a consumption of 140 mg of oxygen if these algae are mineralized in the hypolimnion. These authors illustrate the inability of conventional biological wastewater treatment to solve problems of cultural eutrophication, stress the role of phosphorus as a pollutant, and emphasize the direct relationship between man's activities and lake productivity.

O'Connor and Mueller (1970) have successfully applied a box-model approach to the Great Lakes, considering the temporal and spacial distribution of a conservative material (chlorides) throughout the Great Lakes system. In this approach the lake system is treated as a number of interconnected subsystems; a separate mass balance is written for each subsystem. O'Connor and Mueller consider each of the Great Lakes as a single box that is completely mixed (a continuous stirred-tank reactor, or CSTR), and treat the Great Lakes as a group of CSTRs in series. Based on an analysis of water and chloride inputs together with lake quality data over a sixty-

year period, they are able to demonstrate the applicability of this approach to the chloride problem in the Great Lakes and to make predictions about the consequences of several alternative management plans for controlling chloride levels in these lakes in the future.

Phosphorus cannot be modeled as a conservative substance in lakes. Vollenweider (1969, 1974) has applied the one-box modeling approach to phosphorus in lakes with considerable imagination and success by considering that phosphorus is removed from the lake by a first-order reaction (sedimentation) which occurs throughout the lake. For steady-state conditions Vollenweider (1969) obtains the following equation:

$$[\overline{TP}] = \frac{L_a}{Q_a + \sigma \bar{Z}} \qquad (1)$$

Here L_a is the areal phosphorus loading (gms/m^2-yr), Q_a is the areal hydraulic loading (m^3/m^2-yr or m/yr), $\bar{Z}$ is the mean lake depth (m), σ is a sedimentation rate coefficient (yr^{-1}), and $[\overline{TP}]$ is the steady-state concentration of total phosphorus everywhere within the well-mixed lake and in the lake outflow.

Examination of Eq. 1 indicates that $[\overline{TP}]$ depends upon the inputs of phosphorus and water, the sedimentation coefficient of particulate phosphorus, and the mean lake depth. This is in general agreement with the empirical analyses of Rawson, Sakamoto, and Vollenweider discussed previously, if one assumes that trophic state and the standing crop of plankton are directly dependent on the phosphorus concentration in a lake. Comparison (Vollenweider, 1969) between predictions based on Eq. 1 and

data from several lakes was unsatisfactory, and semi-empirical modifications have been made (Vollenweider, 1969, 1974).

One of the assumptions made by Vollenweider in deriving Eq. 1 will be examined here. Removal of phosphorus by sedimentation from the lake is assumed to occur throughout the lake. The sedimentation rate (gms/yr) is represented as σ V [TP], where V is the total volume of the lake. A more realistic assumption used by Imboden (1973) and in this research is to treat sedimentation as an output of phosphorus at the lake bottom, as follows:

$$\text{Sedimentation Rate} = gA_s\ [TP] \tag{2}$$

Here A_s is the area of the sediment-water interface (m^2) and g is a sedimentation rate coefficient (m/yr) which is affected by the settling velocity of the particles being removed. However, when used in the one-box model of Vollenweider, this assumption results in a relationship for $[\overline{TP}]$ which is independent of depth:

$$[\overline{TP}] = \frac{L_a}{Q_a + g} \tag{3}$$

Hence, the inverse relationship between phosphorus concentration and lake depth predicted by Eq. 1 derives from a debatable assumption about sedimentation in lakes and does not provide a suitable cause-effect relationship for the empirical observations of Rawson, Sakamoto, and Vollenweider.

A model which is similar to that developed in this research has been formulated by Imboden (1973, 1974). This research has been influenced considerably by Imboden's analysis and results. However, Imboden's model leads to predictions that deep lakes will have higher concentrations of

phosphorus in their euphotic zones than shallow ones (Imboden, 1973). Such predictions are not in agreement with the observations of Rawson, Sakamoto, and Vollenweider. This discrepancy will be discussed subsequently.

PROPOSED MODEL

The model lake considered here has a summer stratification lasting six months and is unstratified (homothermal) during a winter circulation. A two-box model is used for the summer period; subsystems are the epilimnion and the hypolimnion. A one-box model is assumed for the winter, although only a portion of the lake may be in the euphotic zone. The models are coupled at the vernal and autumnal overturns. All boxes are well-mixed (CSTRs), so that horizontal concentration gradients are negligible. The effects of the littoral zone are neglected, and the hypolimnion is assumed to remain oxic at all times. Inputs of phosphorus from the sediments to the lake waters are not considered. All external inputs of phosphorus and water are made to the epilimnion during the summer stratification; the lake discharge is also withdrawn from this zone. All external phosphorus inputs are assumed to be biochemically active. Two phosphorus compartments are used, soluble orthophosphate (OP) and particulate phosphorus (PP). Soluble organic phosphorus is neglected. This two-compartment assumption for phosphorus, used successfully by Imboden (1973, 74), may be useful for modeling P over time scales of months and seasons. Schematic representations of the summer and winter models are presented in Figures 1A and B.

For the summer model (Fig. 1A), four interdependent linear differential equations are obtained by formulating mass balances for each phosphorus compartment in each box. For OP in the epilimnion,

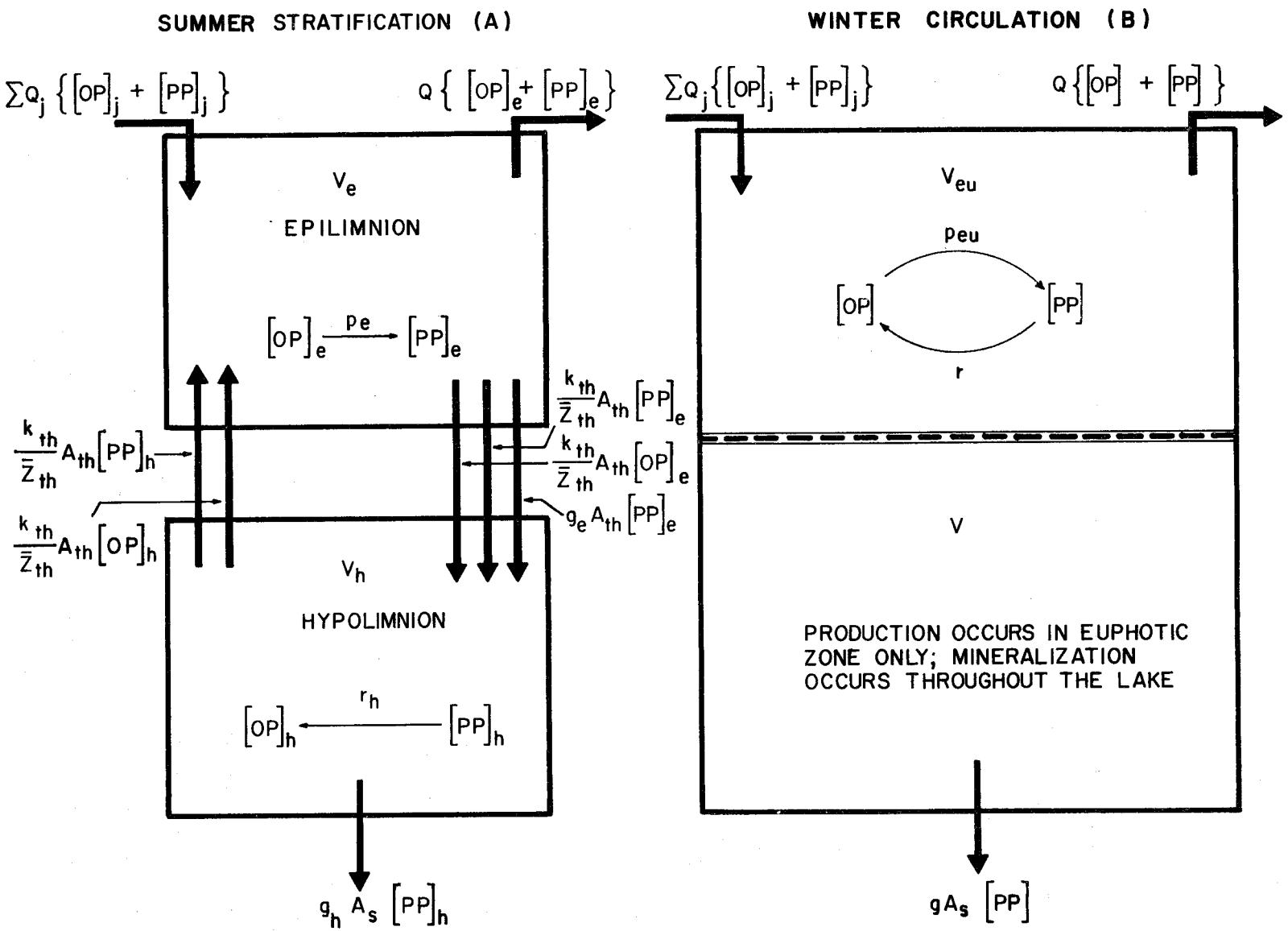

FIGURE I: SCHEMATIC OUTLINE OF PROPOSED MODEL

$$V_e \frac{d[OP]_e}{dt} = \Sigma Q_j [OP]_j - Q[OP]_e - p_e V_e [OP]_e + \frac{k_{th}}{\bar{Z}_{th}} A_{th} [OP]_h$$

net rate of change | input: loading from land-based sources | output: hydraulic discharge | reaction: net production | input: vertical exchange from hypolimnion

$$- \frac{k_{th}}{\bar{Z}_{th}} A_{th} [OP]_e$$

output: vertical exchange to hypolimnion (4)

For PP in the epilimnion:

$$V_e \frac{d[PP]_e}{dt} = \Sigma Q_j [PP]_j - Q[PP]_e + p_e V_e [OP]_e - g_e A_{th} [PP]_e$$

net rate of change | input: loading from land-based sources | output: hydraulic discharge | reaction: net production | output: settling to hypolimnion

$$+ \frac{k_{th}}{\bar{Z}_{th}} A_{th} [PP]_h - \frac{k_{th}}{\bar{Z}_{th}} A_{th} [PP]_e$$

input: vertical exchange from hypolimnion | output: vertical exchange to hypolimnion (5)

For OP in the hypolimnion,

$$V_h \frac{d[OP]_h}{dt} = r_h V_h [PP]_h + \frac{k_{th}}{\bar{Z}_{th}} A_{th} [OP]_e - \frac{k_{th}}{\bar{Z}_{th}} A_{th} [OP]_h$$

net rate of change | reaction: decomposition | input: vertical exchange from epilmnion | output: vertical exchange to epilimnion (6)

For PP in the hypolimnion,

$$V_h \frac{d[PP]_h}{dt} = g_e A_{th} [PP]_e - g_h A_s [PP]_h - r_h V_h [PP]_h$$

net rate of change | input: settling from epilimnion | output: settling to sediments | reaction: decomposition

$$+ \frac{k_{th}}{\bar{Z}_{th}} A_{th} [PP]_e \quad - \frac{k_{th}}{\bar{Z}_{th}} A_{th} [PP]_h \qquad (7)$$

input: vertical exchange from epilmnion | output: vertical exchange to epilimnion

The subscripts e and h refer to the epilimnion and hypolimnion, th and s denote the thermocline region and the sediment-water interface, [OP] and [PP] refer to the concentrations of orthophosphate and particulate phosphorus, p and r are rate coefficients for net production and decomposition, k is a vertical exchange coefficient (L^2T^{-1}) which includes the effects of molecular and turbulent diffusion, internal waves, erosion of the hypolimnion and other fluid processes on the transfer of heat and materials across the thermocline, $\bar{Z}$ is a mean depth, V is the volume of a box, A is an interfacial area, Q_j is a land-based volumetric rate of inflow of water, Q is the volumetric rate of lake discharge, and g is an effective settling velocity.

Equations 4 to 7 are mathematical descriptions of mass balances. These may also be described in words. For example, a mass balance of OP in the epilimnion (Eqn. 4) is a statement that rate of change or accumulation is the sum of input, output, and reaction rates. More specifically,

the net rate of change ($V_e d[OP]_e/dt$) is equal to the rate of input of OP from land-based sources ($\Sigma Q_j [OP]_i$) plus the rate of input of OP from the hypolimnion by vertical exchange across the thermocline ($(k_{th}/\bar{Z}_{th}) \cdot A_{th}[OP]_h$) minus the rate of output of OP to downstream areas in the lake discharge ($Q[OP]_e$) minus the rate of output of OP to the hypolimnion by vertical exchange ($(k_{th}/\bar{Z}_{th})\ A_{th}\ [OP]_e$) minus the rate of utilization of OP for net production of particulate phosphorus ($p_e V_e [OP]_e$).

For the winter model (Fig. 1B), two linear differential equations are obtained from mass balances of the two phosphorus compartments. For OP throughout the lake,

$$V \frac{d[OP]}{dt} = \Sigma Q_j [OP]_j - Q[OP] - p_{eu}\ V_{eu}[OP] + rV[PP] \quad (8)$$

net rate of change	input: loading from land-based sources	output: hydraulic discharge	reaction: production in euphotic zone	reaction: decomposition

For PP throughout the lake,

$$V \frac{d[PP]}{dt} = \Sigma Q_j [PP]_j - Q[PP] + p_{eu}\ V_{eu}\ [OP] - rV[PP] - gA_s\ [PP] \quad (9)$$

net rate of change	input: loading from land-based sources	output: hydraulic discharge	reaction: production in euphotic zone	reaction: decomposition	output: settling to sediments

The subscript eu denotes the euphotic zone.

These models are joined by boundary conditions at the fall overturn and the spring stratification. At the fall overturn,

$$[OP] \text{ just after mixing} = \frac{[OP]_e V_e + [OP]_h V_h}{V} \tag{10}$$

$$[PP] \text{ just after mixing} = \frac{[PP]_e V_e + [PP]_h V_h}{V} \tag{11}$$

At the onset of the spring stratification, the concentrations of PP in the epilimnion and the hypolimnion are set equal to the concentration of PP throughout the lake at the end of the winter circulation ($[PP]_e = [PP]_h = [PP]$). Similarly, $[OP]_e = [OP]_h = [OP]$.

With such a coupled seasonal model, steady-state solutions (e.g., time-invariant solutions obtained by setting the left-hand sides of Eq. 4-9 equal to zero) cannot result. Under constant inflows of phosphorus and water, numerical solution of Eqns. 4-9 yields a repetitive or stable annual cycle for a lake (Fig. 2). Lake coefficients (r, p_e, k_{th}, etc.) are assumed constant over the appropriate period (summer stratification or winter circulation). A detailed description of techniques used for the numerical solution is presented elsewhere (Snodgrass, 1974).

The selection of relationships to represent various physical, chemical, and biological processes will be summarized. The net rate of production of particulate phosphorus (phytoplankton, etc.) during summer stratification is described by $p_e V_e [OP]_e$ (mass/time). Relationships which are more theoretically satisfying have been used by several others. This expression has been selected because it is simple, it is linear,

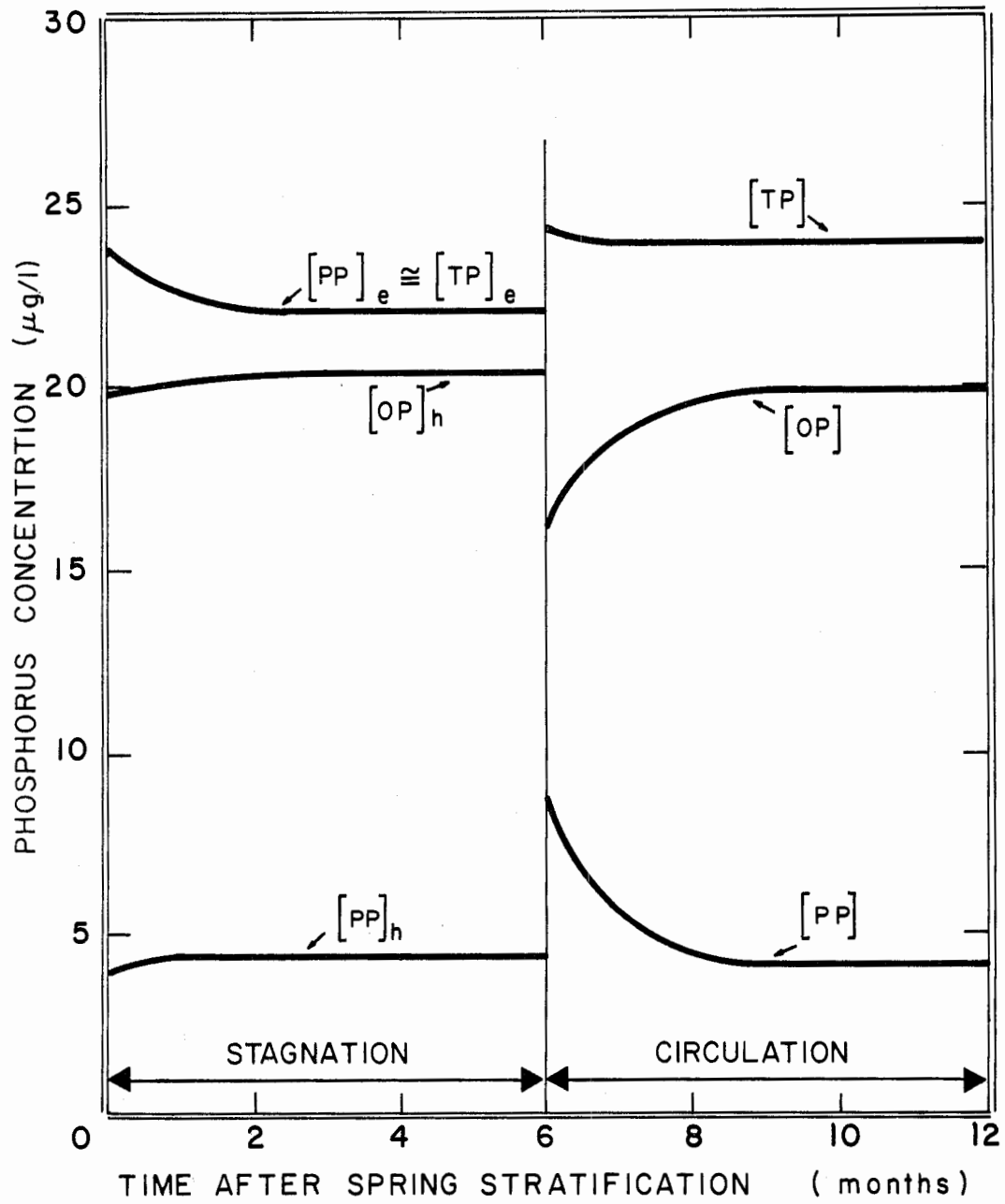

FIGURE 2: PREDICTED STABLE ANNUAL CYCLE FOR LAKE ONTARIO

and it works. A similar expression has been used by Imboden. Values p_e ranging from 0.1 days $^{-1}$ to 5.0 days $^{-1}$ were tested to cover the range suggested by the literature. A value of 2.0 days $^{-1}$ was selected. In a similar manner, gross production of particulate phosphorus during the winter circulation is described by p_{eu} V [OP] (mass/time). A range of p_{eu} from 0.01 days $^{-1}$ to 0.5 days $^{-1}$ was tested and a value of 0.06 days $^{-1}$ was selected.

The rate of decomposition of particulate phosphorus in the hypolimnion during the summer stratification is given by $r_h V_h$ $[PP]_h$ (mass/time). This first-order decomposition rate has been used by others (Jewell and McCarty, 1971, Imboden, 1973, 1974). For the winter circulation a similar expression was used to describe PP throughout the lake, viz., rV[PP] (mass/time). Values of r_h and r from 0.003 days $^{-1}$ to 0.07 days $^{-1}$ were tested; this range encompasses most reported results. A value of 0.03 days $^{-1}$ was selected for both coefficients.

Transport across the thermocline during the summer stratification is described by a Fickian relationship. For example, the rate of transport of OP from the hypolimnion to the epilimnion by vertical exchange processes is represented by $(k_{th}/\bar{Z}_{th})$ A_{th} $[OP]_h$ (mass/time). It is sometimes useful to consider the flux of material (mass/area-time) across the thermocline region. For example, this is described by $(k_{th}/\bar{Z}_{th})$. $[OP]_h$ when considering transport of OP from the hypolimnion to the epilimnion. An extended discussion of vertical exchange is presented subsequently. In a preliminary analysis, values of k_{th} = 2.1 m^2/day and $\bar{Z}_{th}$ = 5 m were used.

Sedimentation of particulate phosphorus is described by the product of a settling coefficient (g, m/day), a cross-sectional area (m^2) and a PP concentration (mass/m^3). For example, the rate of sedimentation of PP from the hypolimnion to the lake sediments during the summer stratification is given by $g_h A_s [PP]_h$ (mass/time). A similar approach is used by Imboden (1973, 1974). Values of the settling coefficients reported in the literature range from 0 to over 20 m/day. In this research, values from 0.05 m/day to 0.5 m/day were tested. In a preliminary analysis, values of g_e = 0.1 m/day, g_h = 0.246 m/day, and g = 0.27 m/day were used. Additional discussion of sedimentation is presented subsequently.

The effects of mean lake depth on phosphorus concentrations in lakes were calculated using the following coefficients:

$$p_e = 2.0/\text{day},\ p_{eu} = 0.06/\text{day},\ r_h = 0.03/\text{day},$$

$$r = 0.03/\text{day},\ g_e = 0.1\text{m/day},\ g_h = 0.246\ \text{m/day},$$

$$g = 0.27\text{m/day},\ \bar{Z}_{eu} = 10\text{m},\ \bar{Z}_{th} = 5\text{m},\ \bar{Z}_e = 10\text{m},$$

$$Q_a = 0.031\text{m/day},\ k_{th} = 2.1\text{m}^2/\text{day}.$$

These coefficients were selected by a preliminary analysis of data for Lake Ontario. However, phosphorus concentrations in both the epilimnion and the hypolimnion are calculated to increase with increasing lake depth for both steady-state and stable-cycle conditions. Similar calculated results are also reported by Imboden (1973). Model predictions are thus inconsistent with limnological observations.

While not in agreement with the observations of Rawson, Sakamoto, and Vollenweider, these calculated results do have a reasonable basis. Particulate phosphorus in the hypolimnion is removed simultaneously by mineralization to OP (rate = $r_h[PP]_h$) and by sedimentation at the lake bottom (rate = $gA_s[PP]_h/V_h \cong (g/\bar{Z}_h)[PP]_h$, where $\bar{Z}_h$ is the average depth of the hypolimnion). Mineralization and sedimentation are thus competitive parallel reactions. Deep lakes provide more opportunity for mineralization of PP than shallow ones (compare r_h with $g/\bar{Z}_h$ increases), so that phosphorus is retained in solution rather than being removed to the sediments. Higher $[OP]_h$ increases the upward transport of orthophosphate to the epilimnion (if $k_{th}/\bar{Z}_{th}$ is independent of depth) and leads to higher $[PP]_e$.

In order to decide how the model should be modified so that predictions will be consistent with observations, let us consider two lakes receiving identical inflows of phosphorus, light, and water. One lake is deep and the other is shallow. The shallow lake is observed to contain more net plankton, fish, chlorophyll, and PP than the deep lake. This could arise for at least two reasons. First, the deep lake may not transform OP to PP as efficiently as the shallow lake. Light may be limiting. This has been proposed by Sakamoto (1966) and by Lorenzen and Mitchell (1973). Second, the deep lake may remove particulate phosphorus to the sediments more rapidly than the shallow lake by reactions that are not described adequately in the model, or by reactions which are not included in the model.

Phosphorus that enters a lake will eventually leave it. If light is limiting, some P will not be incorporated into biomass and so can leave the lake in the outflow as OP. However, observations suggest that OP is reduced

to very low levels in the epilimnetic waters of most lakes. For example, Thomas (1969) reports trace concentrations of OP at the summer minimum in 38 Central European lakes having wide ranges of mean depth and trophic state. Hence, attention is directed here toward possible mechanisms for discharging PP to the lake sediments. Two are evaluated here. First, a depth-dependent exchange of P across the thermocline is developed, based primarily on empirical evidence. Second, the natural aggregation or flocculation of particles in lake waters is considered, based primarily on conceptual arguments. A description of these processes follows. Both are included in the final lake model.

Vertical Transport

Mortimer (1942) has estimated the mean vertical eddy diffusion coefficients (k_h) in the hypolimnetic waters of several lakes using thermal and chemical data. Analysis of his results indicates that $k_h = 0.0142\ \bar{Z}^{1.49}$ (correlation coefficient = 0.952). Here k_h is expressed in m^2/day and $\bar{Z}$ is in meters. This empirical result is also expressed qualitatively by Mortimer (1969) as follows "The 'average intensity' of turbulent stirring in the hypolimnia of lakes is generally positively correlated with lake dimensions, including depth." These observations for hypolimnetic waters have analogues in the region of the thermocline.

Blanton (1973a) has examined the vertical entrainment of hypolimnetic waters into the epilimnia of stratified lakes, and concluded that "Mean depth seems the most important morphometrical parameter governing the magnitude of entrainment: the greater the mean depth, the greater the entrainment." Blanton (1973b) has calculated vertical diffusion coefficients for

these lakes. Analysis of the data indicates that $k = 0.0016\bar{Z}^{1.58}$ (correlation coefficient = 0.941). Again k has units of m^2/day and $\bar{Z}$ is in meters. Mean depths for these lakes range from 3.2 to 740 m. Blanton does not specify the location of k within the epilimnion-thermocline region of a lake; it is probably characteristic of conditions just above the thermocline.

Blanton (1973b) has also determined the mean static stability (E, sec^{-2}) across the thermoclines of these lakes. E is defined as $(\bar{g}/\rho) \cdot (\partial\rho/\partial Z)$, where $\bar{g}$ is the acceleration of gravity and ρ is the fluid density. E is a measure of the stabilizing buoyant force of the density gradient in a lake. Analysis of Blanton's data indicates that $E = 0.0155\ \bar{Z}^{-0.606}$ (correlation coefficient = 0.922). Deep lakes are thus observed to have lower stabilities than shallow ones, and so may be less able to resist mixing by fluid turbulence.

Ozmidov (1965) has presented evidence indicating that the vertical turbulent exchange coefficient in oceanic waters decreases with increasing density gradient. Considering this result and Blanton's data, an empirical correlation has been sought between k_{th} and $\bar{Z}$. Results are presented in Figure 3; data sources are listed in Table 1. The results indicate that $k_{th} = 0.00682\ \bar{Z}^{1.12}$ (correlation coefficient = 0.924; k_{th} is in m^2/day). Based on these results a mean-depth dependent exchange of phosphorus across the thermocline has been included in the model. The effect of this reaction is to remove PP from the epilimnion to the hypolimnion at a faster rate in deep lakes than in shallow ones.

The vertical exchange coefficient in the thermocline region can be more usefully described by the ratio $k_{th}/\bar{Z}_{th}$, defined here as $\hat{k}_{th}$ (m/day).

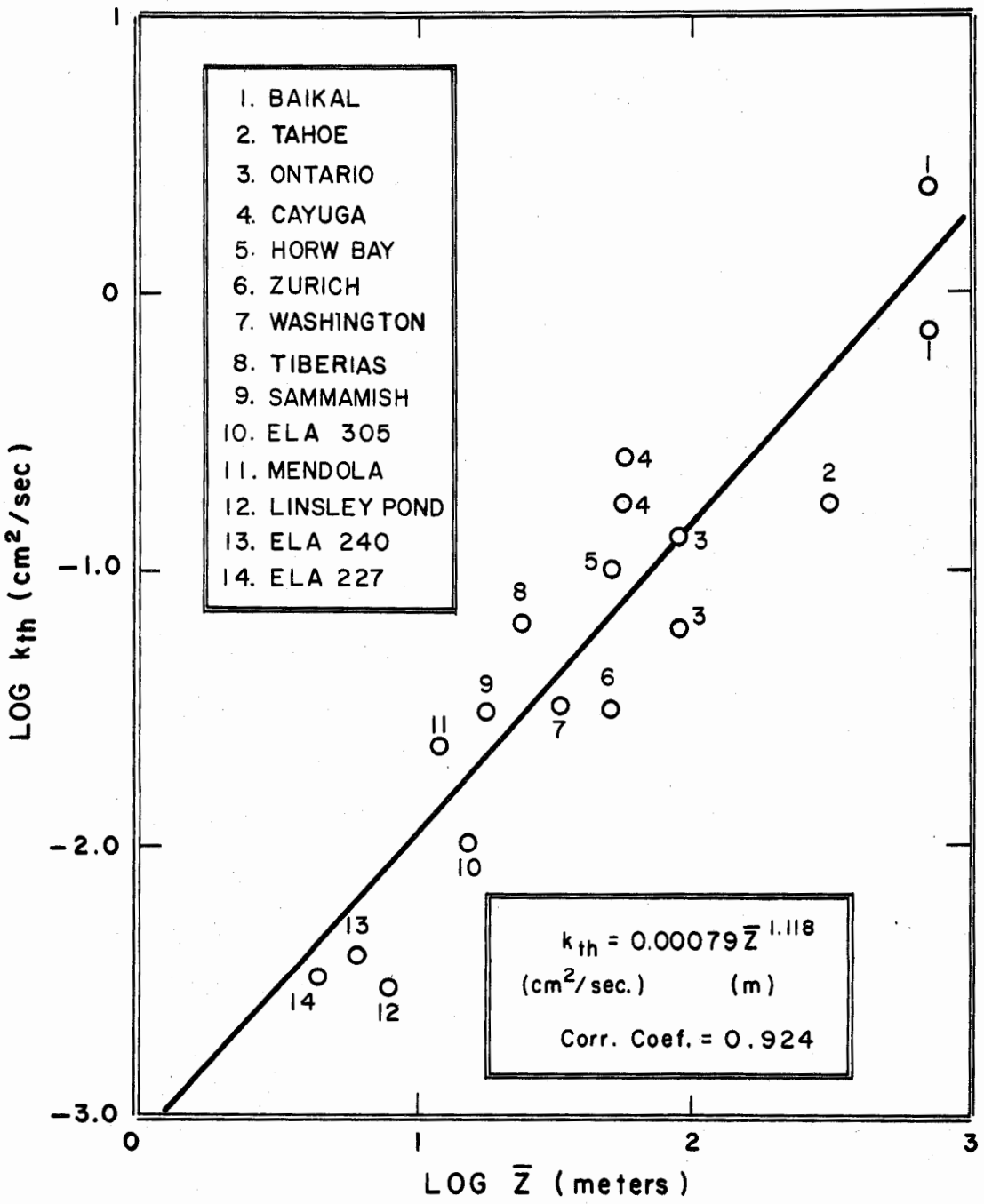

FIGURE 3: CORRELATIONS BETWEEN k_{th} AND $\bar{Z}$

TABLE 1

DATA SOURCES FOR ANALYSIS OF VERTICAL TRANSPORT

Lake	$\bar{Z}$ (m)	Data Source
Baikal	740	Kozhov (1963)[1]
Tahoe	313	Goldman (1974)[1]
Ontario	89	Sweers (1970)[2]
Cayuga	55	Powell and Jassby (1974)
Luzern (Horw Bay)	50	O'Melia (1972a)
Zurich	50	Li (1973)
Washington	33	Edmondson (1974)[1]
Tiberias	24	Lerman and Stiller (1969)
Sammamish	17.7	Bella (1970)
ELA 305	15.1	Schindler (1971)[1]
Mendota	12	Powell and Jassby (1974)
Linsley Pond	~7	Powell and Jassby (1974)
ELA 240	6.1	Schindler (1971)[1]
ELA 227	4.4	Hesslein (1973)[3]

1. Temperature profiles obtained from reference. k_{th} calculated in this research.

2. Values calculated for a fixed thermocline have been used to be consistent with other results.

3. The value of k_{th} based on temperature measurements has been used to be consistent other results.

The following relationship has been assumed

$$\hat{k}_{th} = 0.005\overline{Z} \qquad (12)$$

Eq. 12 is based on the empirical results in Fig. 3, together with a rough estimation of the dependence of $\overline{Z}_{th}$ on $\overline{Z}$. Finally, the dependence of $\overline{Z}_e$ on $\overline{Z}$ has been described by $\overline{Z}_e = 1.6\ \overline{Z}^{0.568}$, based on analysis of additional data from Blanton (1973b).

Natural Aggregation

The transport of particles in the air and water environments depends upon particle size. Many aesthetic, health, and general ecological effects also depend upon particle size. Light scattering by smog and deposition of particles in the lung are common examples of aerosol effects. Hahn and Stumm (1970) have presented conceptual arguments for the significance of coagulation in natural waters including lakes. Edzwald et al. (1974) have presented experimental evidence of the occurrence of coagulation in estuaries. Here the possible role of coagulation in removing phosphorus from lakes is outlined in a simplified way.

Contacts may occur between particles by Brownian diffusion (termed perikinetic flocculation), by velocity differences in the water due to fluid motion (termed velocity gradient or orthokinetic flocculation), and by differences in the gravitational settling velocities of the particles themselves. These processes had been examined fruitfully for atmospheric coagulation by Friedlander (1964). The latter two of these will be examined for aquatic coagulation here.

Velocity-gradient flocculation in a CSTR may be described as follows (O'Melia, 1972b):

$$n = n_o \left(1 + \frac{4}{\pi} \phi \, Gt\right)^{-1} \tag{13}$$

where n and n_o are the number concentration of particles at time t and at t=o, respectively, G is the mean velocity gradient, and ϕ is the floc volume fraction, or the volume of solid particles per unit volume of suspension. In arriving at Eq. 13, it is assumed that each contact opportunity produces an aggregate. Stated another way, the particles are assumed to be completely destablized.

Consider the product $(4/\pi)\phi Gt$. For a lake water containing algae so that $[PP]_h = 5\mu g/\ell$, then $\phi \cong 10^{-5}$. This assumes an algal composition of $C_{106} H_{263} O_{110} N_{16} P$ and a water content of 0.9. The mean velocity gradient in lake waters is not known, but a value of 0.5 sec^{-1} will be assumed. Data reported by Okubo (1971) for oceanic waters indicate values of G ranging from 0.012 sec^{-1} to 1.6 sec^{-1}, with most values being 0.1 sec^{-1} or greater. If the time for aggregation in the hypolimnetic waters is taken as $(\bar{Z}_h/g_o)$ and g_o is assumed to be 0.05m/day, then $(4/\pi)\phi Gt) \cong 11 \bar{Z}_h$ ($\bar{Z}_h$ in meters).

The relative aggregation rates by velocity gradients and differential settling may be expressed as follows (Camp, 1946):

$$\frac{N_o}{N_g} = \frac{12 \nu G}{\pi \bar{g} (S-1) (D_2 - D_1)} \tag{14}$$

Here N_o is the orthokinetic flocculation rate, N_g is the sedimentation

flocculation rate, ν is the kinematic viscosity of the water, S is the specific gravity of the particles, and D_1 and D_2 refer to the diameters of two particles. All particles are assumed to have the same specific gravity. For T = 5°C, S = 1.02, D_2 = 100μm, and D_1 = 10μm, then $N_g \cong 6\ N_o$. Flocculation by differential settling can be faster than by velocity gradients.

A detailed flocculation analysis would require knowledge of the size distribution of the particulates, together with chemical information about their stability. For this research it has been assumed that

$$g_h = g_o\ (1 + f\ \overline{Z}_h) \tag{15}$$

for the summer stratification, and

$$g = g_o\ [1 + f(\overline{Z} - \overline{Z}_{eu})] \tag{16}$$

for the winter circulation. The flocculation coefficient has been taken as 0.05 meters $^{-1}$. The preceding analysis of contact rates would suggest a value of f in the order of 50. Hence, the selected value of f has the effect of assuming that only 1 out of every 1000 contacts actually produces an aggregate. No aggregation is assumed in the epilimnion, or in the euphotic zone during the winter circulation.

RESULTS AND DISCUSSION

Model Verification

With the exception of the vertical exchange and flocculation reactions, preliminary values of the reaction coefficients were selected from the literature. Final values were chosen by calibrating the model using available data for Lake Ontario (Thon, 1969; Shiomi and Chawla, 1969; Vollenweider, 1974; Dobson, 1974). Input and morphometric data for Lake Ontario are as follows: L_a = 0.68 gm P/m^2-yr, Q_a = 0.031 m/day, $\bar{Z}$ = 88.5 m. A summary of the coefficients is presented in Table 2. These have been used in generating the predicted stable annual cycle for Lake Ontario (Figure 2).

Verification of this model is provided here in two ways. First, model predictions of the effects of lake depth on eutrophication are compared with Vollenweider's estimates based on observation (1968). Second, predictions of [TP] at the end of the annual winter circulation period are compared with observations made by others on several lakes.

Model predictions are compared with Vollenweider's empirical analysis (1968) in Figure 4. Dashed lines delineate combinations of phosphorus loading and mean depth which Vollenweider considers will produce oligotrophic, mesotrophic, or eutrophic conditions in a lake. Solid curves delineate combinations of L_a and $\bar{Z}$ which are predicted to produce an annual spring [TP] of 20 µg/ℓ. This [TP] is considered here as an indicator of troublesome conditions (see Sawyer, 1947). Other values of [TP] could be assumed; predicted curves would have the same shapes but would be vertically displaced.

TABLE 2

MODEL COEFFICIENTS

Summer Stratification	Winter Circulation
$g_e = 0.1$ m/day	$g = g_o [1 + f (\bar{Z} - \bar{Z}_{eu})]$
$g_h = g_o (1 + f\bar{Z}_h)$	$g_o = 0.05$ m/day
$g_o = 0.05$ m/day	$f = 0.05/m$
$f = 0.05/m$	$\bar{Z}_{eu} = 10m$ for $\bar{Z} > 10m$
$\frac{k_{th}}{\bar{Z}_{th}} = \hat{k}_{th} = 0.005\ \bar{Z}$	$\bar{Z}_{eu} = \bar{Z}$ for $\bar{Z} < 10m$
$\bar{Z}_e = 1.6\ \bar{Z}^{0.568}$	$p_{eu} = 0.06$/day
$p_e = 2.0$/day	$r = 0.03$/day
$r_h = 0.03$/day	

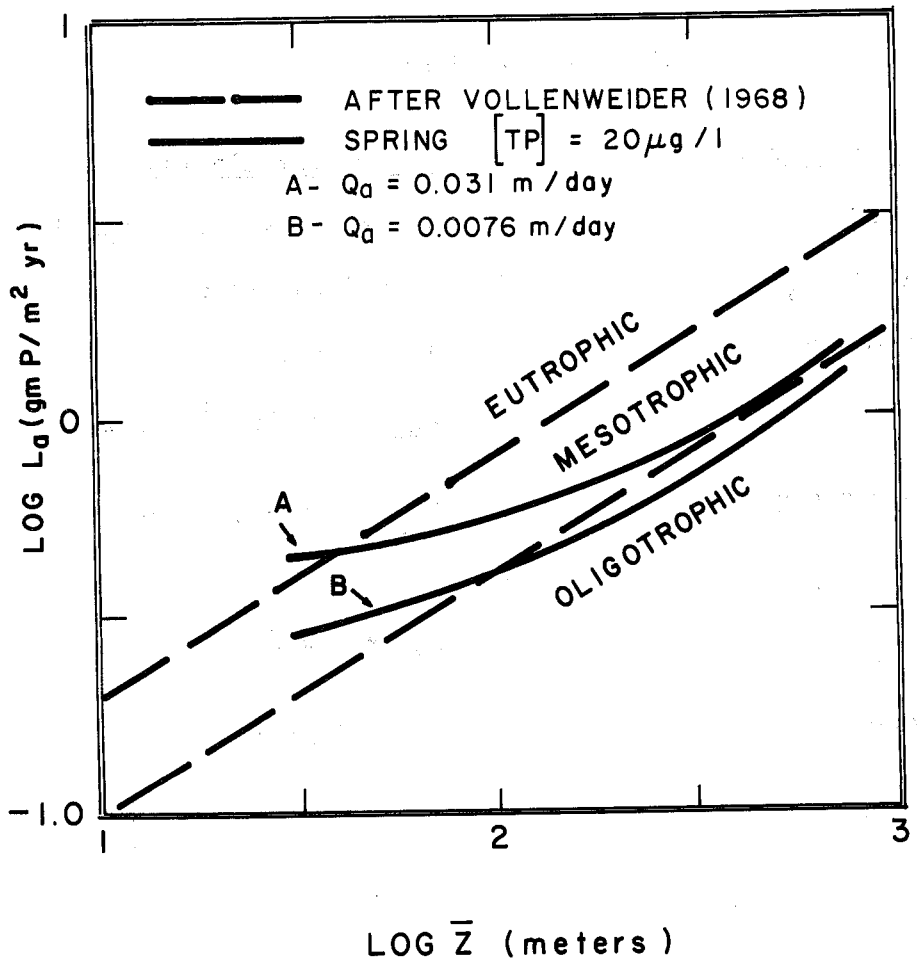

FIG. 4: COMPARISON OF MODEL PREDICTIONS WITH VOLLENWEIDER'S EMPIRICAL RESULTS (1968).

The agreement between calculated and observed curves in Figure 4 is good. This provides some verification for the model, and for the importance of vertical transport in the thermocline region and flocculation as processes affecting water quality in lakes.

The effect of hydraulic loading is illustrated by the model results in Figure 4. Increased hydraulic loading permits increased areal phosphorus loading (L_a). This effect has been noted by Vollenweider (1974) in comparing Lakes Tahoe and Zurich.

The results of model predictions and field observations for several lakes are presented in Figure 5. Predicted annual [TP] are plotted versus observed values. Predictions are made using the coefficients listed in Table 2; i.e., all coefficients are constant except $\hat{k}_{th}$, g_h, and g which are depth-dependent. Data sources for observed [TP] and phosphorus loadings are presented in Table 3.

A lake was selected for presentation in Figure 5 on the following bases: (1) there is no evidence that the lake is not limited by phosphorus, (2) the phosphorus loading has been measured or estimated, (3) hypolimnetic waters must be aerobic, and (4) reliable data for phosphorus concentrations in the lake waters must be available. The agreement between observed and predicted phosphorus concentrations is excellent, and provides strong verification for the model and its assumptions. The analyses of three lakes will be noted. First, the agreement between observed and predicted values for Lake Ontario is good because data from the lake were used in calibrating the model. Second, predicted values for the Sempachersee are high. This can illustrate the dependence of the model on input data, in this case L_a.

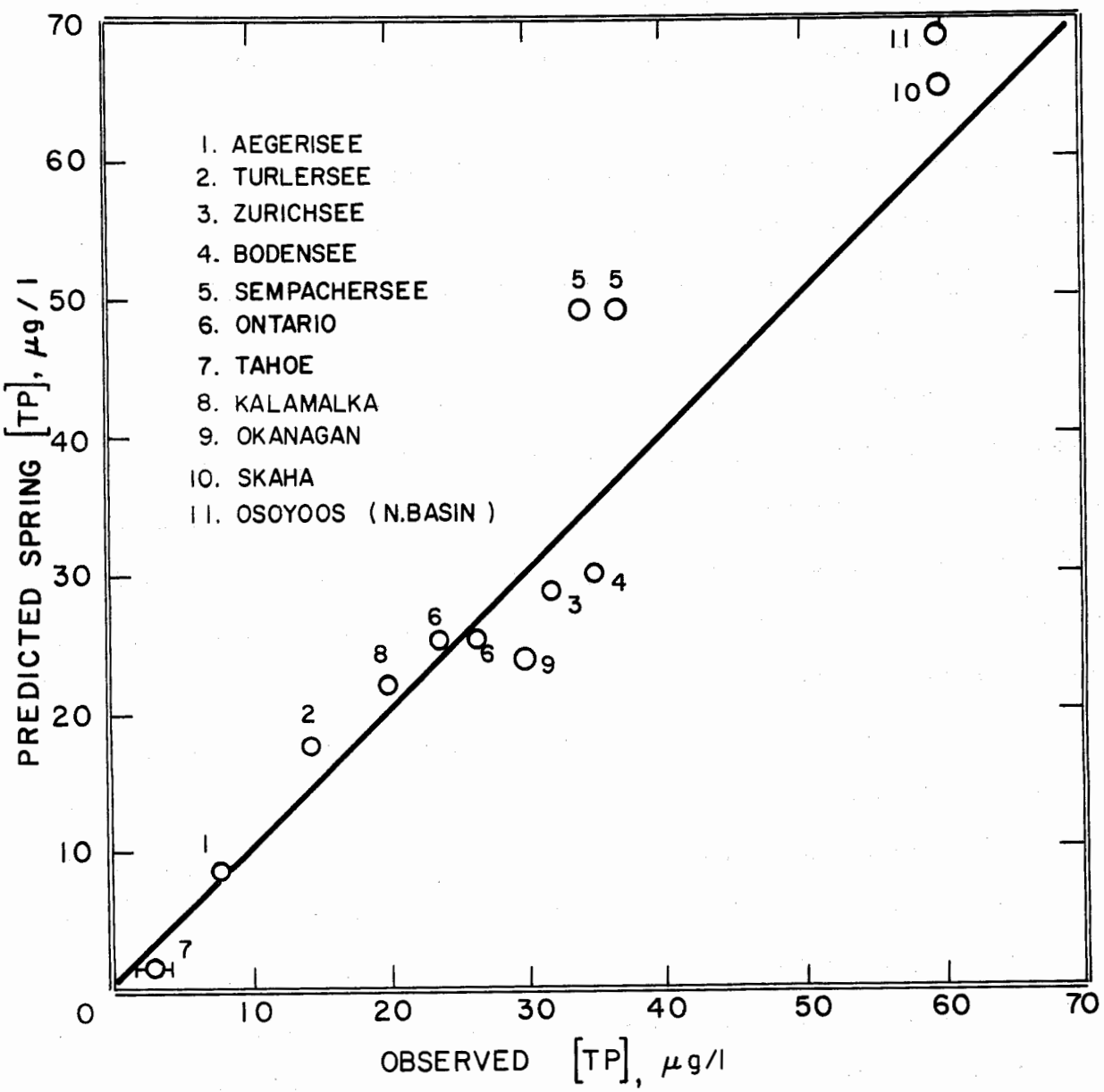

FIGURE 5: COMPARISON OF MODEL PREDICTIONS WITH OBSERVED PHOSPHORUS CONCENTRATION IN SEVERAL LAKES.

TABLE 3

DATA SOURCES FOR FIGURE 5

Lake	Observed Spring [TP] (µg/ℓ)	Reference	Phosphorus Loading (L_a) (gmsP/m^2-yr)	Reference
Aegerisee	7.6	Vollenweider (1974)	0.16	Vollenweider (1974)
Turlersee	14.5	"	0.30	"
Zurichsee	32	"	1.32	"
Bodensee	35	"	1.10	"
Sempachersee	34.2 - 36.7	Perret (1973)	0.77	Imboden (1973)
Ontario	26.4 (1969)	Shiomi and Chawla (1970)	0.68	Vollenweider (1974)
	23.7 (1972)	Dobson (1974)	0.68	"
Tahoe	1.5 - 4.3[2]	Goldman (1974)	0.04	"
Kalamalka	20[1]	Stein and Coulthard (1971)	0.32	Patalas and Salki (1973)
Okanagan	30[1]	"	0.39	"
Skaha	60[1]	"	2.19	"
Osoyoos	60[1]	"	4.20	"

1. [TP] based on yearly average lake conditions.

2. [TP] based on range within lake during winter circulation.

Imboden (1973) estimates a total P loading of 10,700 kg P/yr, of which 8000 kg/yr derive from a population of 7200 inhabitants in the drainage area. All of the P discharges from these inhabitants are assumed to reach the lake. Hence, the high predicted [TP] may be due to an overestimate of L_a. Third, predicted values for Lake Tahoe are low. It is possible that vertical mixing in Lake Tahoe is less intense than that indicated by the k_{th} calculated for the model in this research (compare observed k_{th} for Lake Tahoe with model in Fig. 3). The higher exchange coefficient used in the model would remove PP more rapidly from the epilimnion, and predict more deposition of P than may actually occur. Application of the model to a specific lake and using coefficients specifically determined for that lake should provide even better agreement between model and real system than Figure 5.

General Predictions

The predictions of a stable annual cycle for Lake Ontario presented in Figure 2 indicate the following. First, the model predicts slow changes in phosphorus compartments throughout the year, except at the fall overturn and the spring stagnation. This is due primarily to the use of seasonal averages for the reaction coefficients. Hence, the model in its present form will not simulate rapid temporal changes which may be due to storms and other short-term variations in lake conditions. Second, the summer and winter models approach steady-state (time-invariant) conditions for Lake Ontario. Predictions for other lakes have indicated that this is not always the case. For example, shallow lakes have $[PP]_e$ at the end of the stagnation period

which are less than steady-state values, while deep lakes show $[PP]_e$ in the fall which are above steady-state concentrations.

The model is capable of predicting the temporal response of a lake to sudden changes in land-based nutrient inputs. Verification of these predictions awaits sufficient data at present. For example, the response of Lake Washington to sewage diversion has been demonstrated, but sufficient data on nutrient inputs to verify the model predictions are not available.

The predicted effects of phosphorus loading and mean depth on water quality in lakes are presented in Figure 6 for several hydraulic loadings. These results are calculated for a spring concentration of TP = 20 μg/ℓ in a stable annual cycle. Combinations of L_a and $\bar{Z}$ above and to the left of any curve are predicted to cause eutrophic conditions; combinations which lie in the region below a curve are predicted to permit oligotrophic conditions.

These predictions are presented in other ways in Figures 7 and 8. The effects of phosphorus loading and hydraulic loading are illustrated in Figure 7 for several mean lake depths. The combined effects of L_a, Q_a, and $\bar{Z}$ on water quality are illustrated in Figure 8. A spring concentration of TP = 20 μg/ℓ is again assumed to provide a useful upper limit for oligotrophic waters. These results illustrate that lakes with large depths and high hydraulic loadings can tolerate greater land-based phosphorus loadings. The results indicate that for a lake with given hydrological (Q_a) and morphometrical ($\bar{Z}$) characteristics, the land-based nutrient loading (L_a) should be reduced to a level at or below the three-dimensional surface presented in Figure 8.

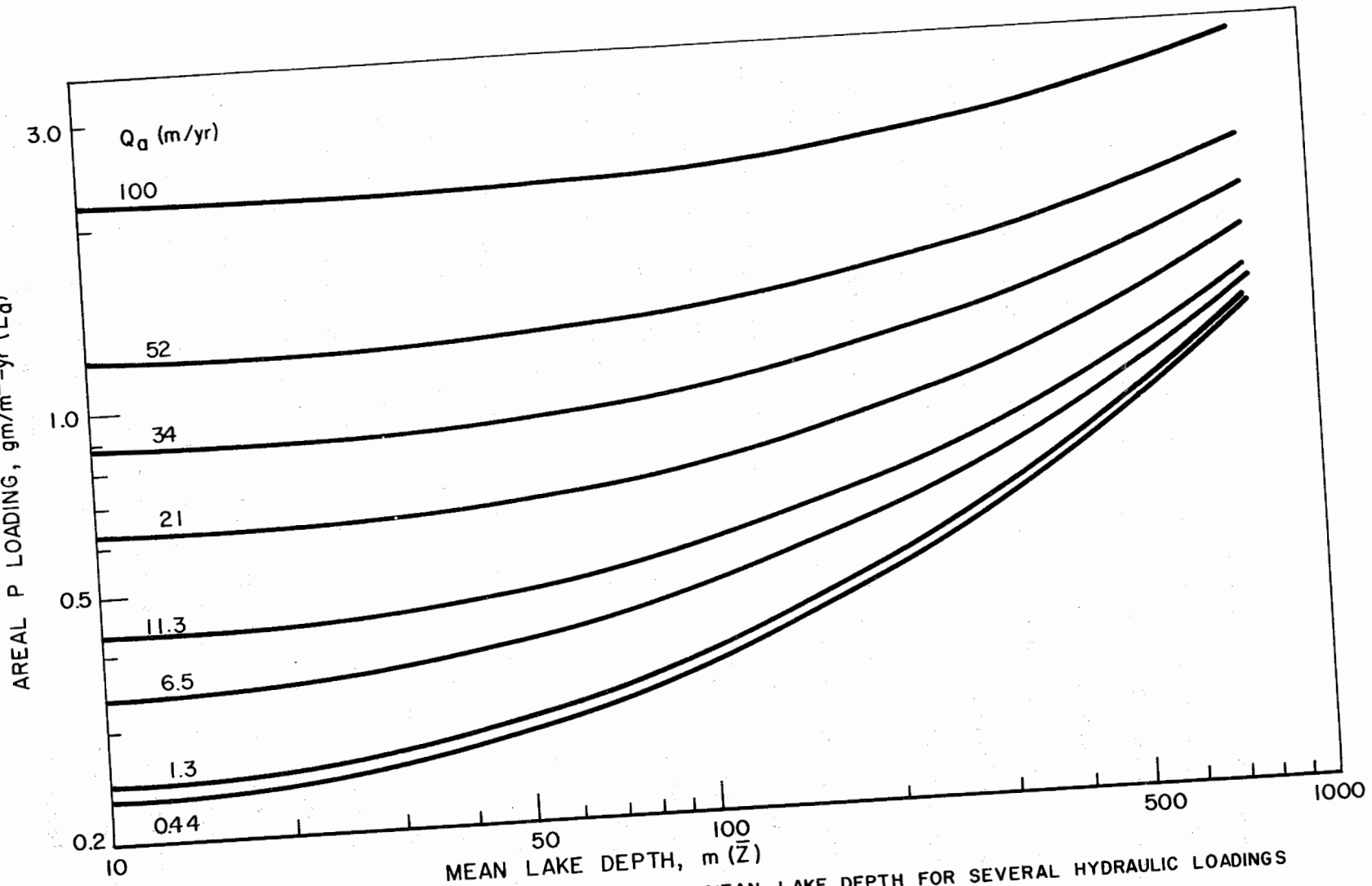

FIGURE 6: PERMISSIBLE P LOADING AS A FUNCTION OF MEAN LAKE DEPTH FOR SEVERAL HYDRAULIC LOADINGS ([TP] = 20 μg/l)

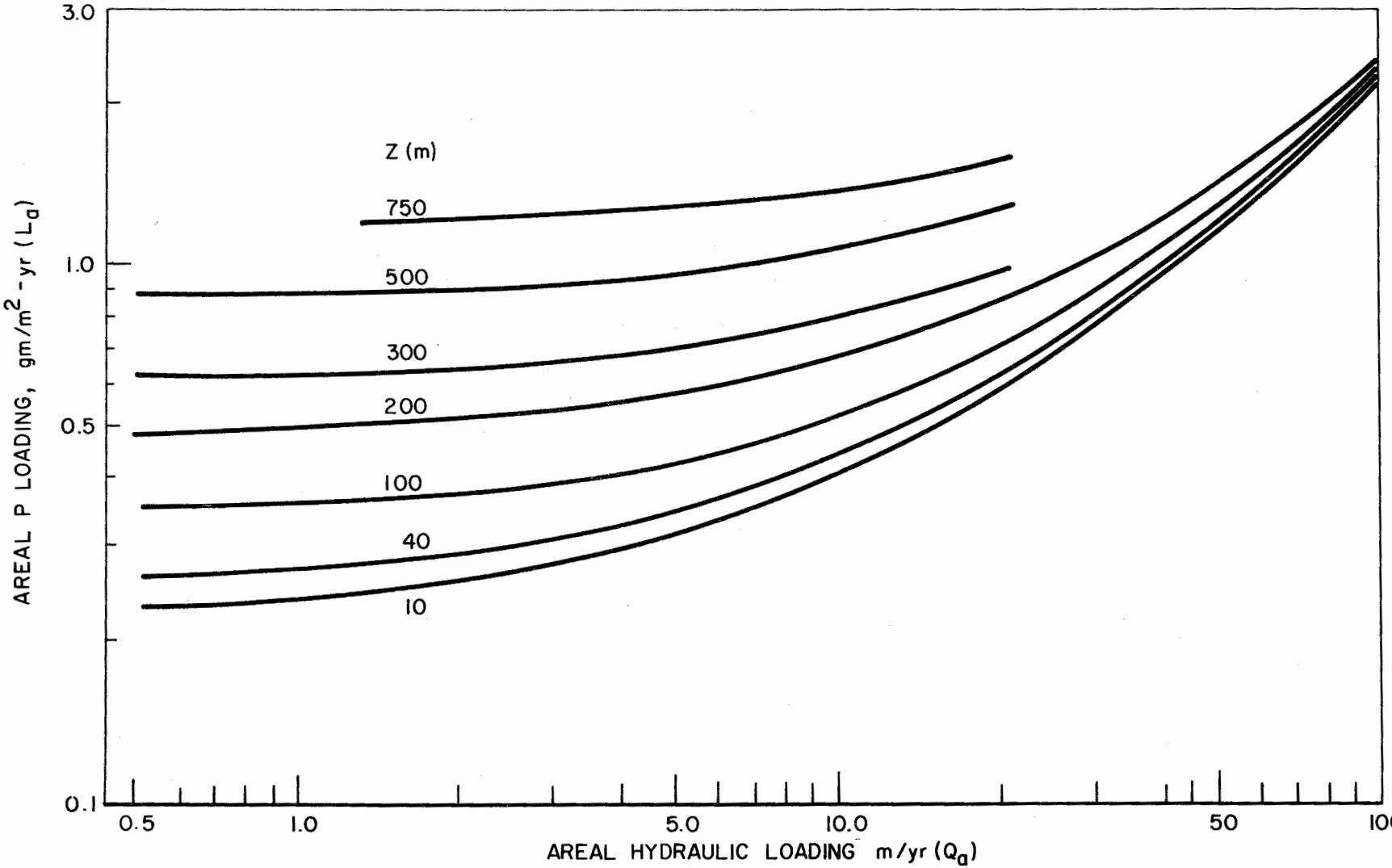

FIGURE 7: PERMISSIBLE P LOADING AS A FUNCTION OF HYDRAULIC LOADING FOR SEVERAL MEAN LAKE DEPTHS ([TP] = 20μg/l)

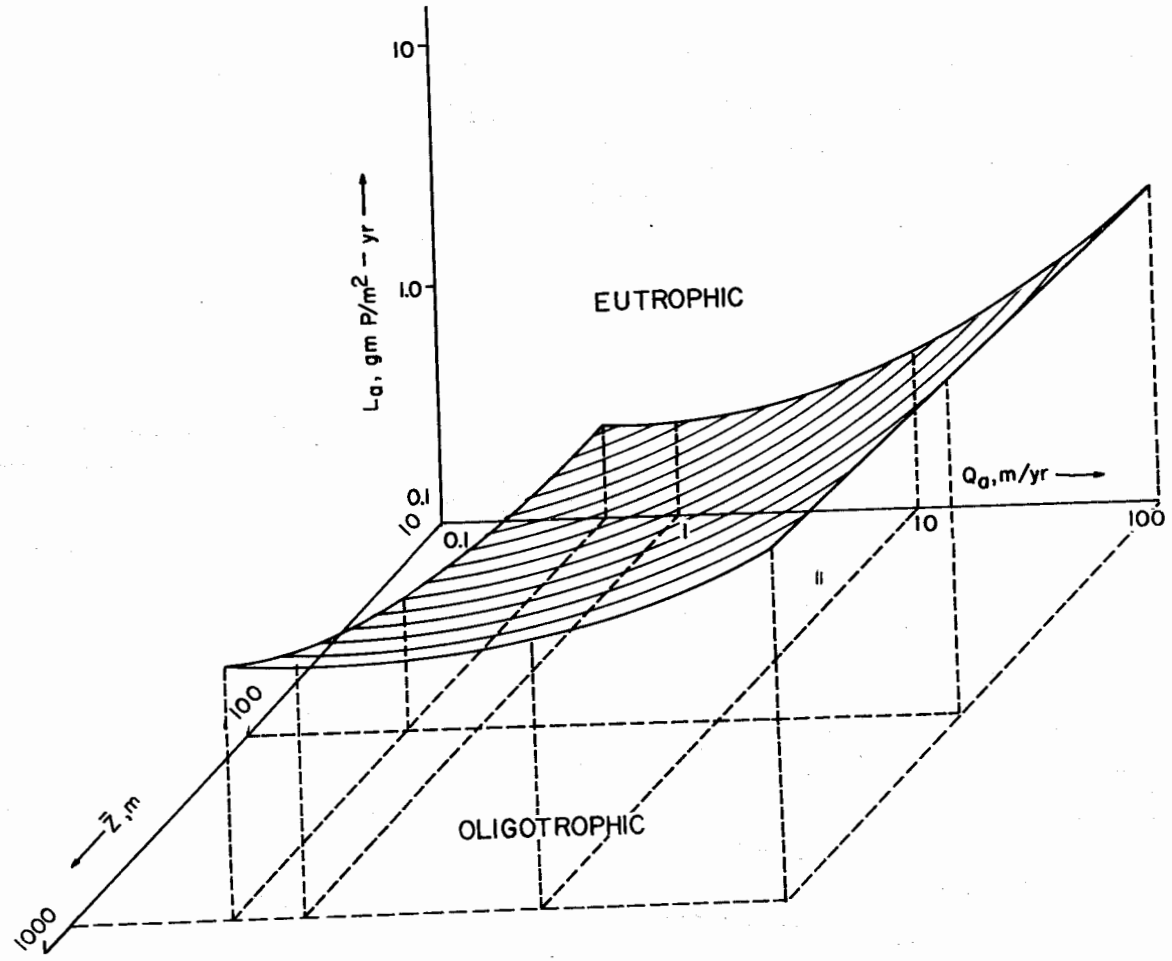

FIGURE 8: INTERRELATIONSHIP AMONG AREAL P LOADING (L_a), AREAL HYDRAULIC LOADING (Q_a), AND MEAN LAKE DEPTH ($\bar{z}$), [TP] = 20 μg/l.

These predictions are based on the coefficients listed in Table 2. Where available evidence indicates that other values may be appropriate, these can be used in the model. It must also be emphasized that lakes which develop anoxic hypolimnetic waters cannot be described by the model in its present form.

Predictions for North Carolina Lakes and Reservoirs

Many lakes in North Carolina are man-made impoundments, constructed for power generation, flood control, water supply, recreation and other purposes. Measurements or estimates of nutrient inputs have not been made for most of these waters. As a result, predictions have been made of the phosphorus retention capacity of these lakes. This is the fraction or percent of the phosphorus entering a lake which is retained in the lake sediments. A high retention capacity indicates that most of the land-based nutrient loading to a lake reaches and is retained in the lake sediments. High retention factors suggest that oligotrophic conditions will occur. However, a prediction of water quality ([TP]) requires information about nutrient loading (L_a).

Predictions of P retention factors for several lakes in North Carolina are presented in Table 4. Hyco and Belews Reservoirs are used for cooling. Lakes Norman and Wylie are used for hydropower generation and cooling. Lakes James and Waterlee are used for hydropower generation. Kerr and Fontana reservoirs are used for hydropower generation, flood control, and recreation.

Several and perhaps all of these reservoirs have a discharge from the lake bottom. During the summer stratification, hypolimnetic waters are

TABLE 4

PREDICTIONS OF PHOSPHORUS RETENTION IN NORTH CAROLINA RESERVOIRS
(Preliminary)

Reservoir	$\bar{Z}$(m)	Q_a(m/yr)	Hydraulic Detention Time (days)	P Retention (%)*
Hyco	6.3	7.7	300	61.2
Belews	15.3	5.6	1,000	67.9
Lake James	14.0	22	228	35.6
Lake Norman	10.3	18	206	39.9
Lake Wylie	6.9	79	32	9.9
Lake Waterlee	6.9	93	27	8.1
Fontana	41.1	82	183	16
Kerr	9.2	49	150	14.4

*Assumes hypolimnetic discharge to downstream during stratification period.

released downstream. This requires some changes in the basic input-output model. First, the discharge of nutrients from the epilimnion to downstream ($Q\{[OP]_e + [PP]_e\}$) is set equal to zero. Second, a downward advective transport of phosphorus across the thermocline is added to allow water entering the lake to reach the lake for subsequent discharge. The vertical advective flux of nutrients is given by $Q_a A_e\{[OP]_e + [PP]_e\}$. Finally, the advective discharge of P from the hypolmnion is given by $Q\{[OP]_h + [PP]_h\}$. The result of these modifications is to increase the input of P to the lake sediments and reduce P in the lake waters. The effect is most pronounced for deep reservoirs with low hydraulic loadings. Reservoirs with bottom discharges are predicted to have somewhat greater assililation capacities for nutrients than lakes with epilimnetic withdrawals.

Model predictions (Table 4) indicate that the retention of P in lake sediments will be low for several reservoirs in North Carolina. This is due to their shallow depth and high hydraulic loading. For a reservoir such as Lake Wylie with a hydraulic detention of 32 days, the almost river-like nature of the system prevents phosphorus from being retained in the sediments (predicted retention = 9.9%). For Belews Reservoir with a greater depth and a detention time of 1000 days, considerable phosphorus retention (68%) is predicted.

These estimates should be viewed with caution because (1) the model has not been verified for a reservoir-type system having a hypolimnetic discharge, (2) the possible occurrence of anoxic hypolimnia has not been evaluated, (3) stratification has been assumed, but may not always occur in cooling reservoirs and (4) waters in central and western North Carolina

typically contain high concentrations of natural turbidity (e.g., clays) which could increase natural aggregation and improve P retention.

If predictions of water quality in North Carolina's lakes and reservoirs are to be made, two things must be done. First, measurements or estimates of nutrient inputs to these waters must be made. Measurements are preferred, difficult, and expensive. Lacking this, nutrient inputs may be estimated using land use data, as demonstrated by Vollenweider (1968). These estimates use land development, population density, industrial activities, existing wastewater treatment facilities, fertilizer applications, and animal populations as independent variables from which the inputs of nutrients are estimated in terms of their location, type, magnitude, temporal variation, and possible future changes.

Second, the model developed in this research should be used to make water quality predictions. These predictions will be most reliable for lakes with oxic hypolimnetic waters, stable nutrient loadings, and epilimnetic discharges. Predictions for reservoir systems with hypolimnetic discharges can be made, but additional verification of the the model should be sought. Similarly, prediction of the time required to reach a new stable level of water quality after a change in nutrient inputs can also be made, but further verification of these model predictions should be attempted. Finally, the model should not be used for anoxic systems. Additional research can add this dimension to the utility of this modeling approach.

Copies of a computer program to facilitate numerical solution of the model may be obtained from C. R. O'Melia at the University of North Carolina.

REFERENCES

Bella, David R., (1970), "Dissolved Oxygen Variation in Stratified Lakes", Proc. ASCE, Jour. SED, Vol. 96, No. SA5, 1129-1146.

Blanton, J. O. (1973a), "Vertical Entrainment into the Epilimnia of Stratified Lakes", Limnology and Oceanography, Vol. 18, 697-704.

Blanton, J. O. (1973b), "Rates of Vertical Entrainment in Stratified Lakes", International Association of Hydrological Sciences, Publication No. 109, 301-305.

Camp, Thomas R. (1946), "Sedimentation and the Design of Settling Tanks", Trans ASCE, Vol. III, 895-958.

Dobson, H. (1974), Personal communication.

Edmondson, W. T. (1974), "Preliminary Report for the North American Project-Lake Washington", Workshop on Measuring and Monitoring of Eutrophication, Canada Centre for Inland Waters, Burlington, Ontario.

Edzwald, J. K., Upchurch, J. B., and O'Melia, C. R. (1974), "Coagulation in Estuaries", Environmental Science and Technology, Vol. 8, 58-63.

Friedlander, Shelton K., 1964, "The Similarity Theory of the Particle Size Distribution of the Atmospheric Aerosol", Proceedings of the 1st National Conference on Aerosols, Czechoslovak Academy of Sciences, pp. 115-130.

Goldman, C. R. (1974), "Preliminary Report for the North American Project - Lake Tahoe, California - Nevada", Workshop on Measurement and Monitoring of Eutrophication, Canada Centre for Inland Waters, Burlington, Ontario.

Hahn, Hermann H., and Stumm, Werner (1970), "The Role of Coagulation in Natural Waters", American Journal of Science, Vol. 268, 354-368.

Hesslein, R., and Quay, P., (1973) "Vertical Eddy Diffusion Studies in the Thermocline of a Small Stratified Lake", Jour. Fisheries Research Board of Canada, Vol. 30, 1491-1500.

Imboden, D. M. (1973), "Limnologische Transport-und Nährstoffmodelle", Schweizerische Zeitschrift fur Hydrologie, Vol. 35, 29-68.

Imboden, D. M. (1974), "Phosphorus Model of Lake Eutrophication", Limnology and Oceanography, Vol. 19, 297-304.

Jewell, William J., and McCarty, Perry L. (1971), "Aerobic Decomposition of Algae", Environmental Science and Technology, Vol. 5, 1023-1031.

Kozhov, M. (1963), Lake Baikal and Its Life, Monographiae Biologicae Vol. XI, W. Junk Publishers, The Hague, p. 41.

Lerman, A., and Stiller, M. (1969), "Vertical Eddy Diffusion in Lake Tiberias", Verh. International Verein. Limnol., Vol.17, 323-333.

Li, Yuan-Hui (1973), "Vertical Eddy Diffusion Coefficient in Lake Zurich", Schweizerische Zeitschrift für Hydrologie, Vol.35, 1-7.

Lorenzen, Marc, and Mitchell, Ralph (1973), "Theoretical Effects of Artificial Destratification on Algal Production in Impoundments", Environmental Science and Technology, Vol.7, 939-944.

Mortimer, Clifford H. (1942), "The Exchange of Dissolved Substances between Mud and Water in Lakes, III and IV", Journal Ecol., Vol.30, 147-201.

Mortimer, Clifford H. (1969), "Physical Factors with Bearing on Eutrophication in Lakes in General and in Large Lakes in Particular", in Eutrophication: Causes, Consequences, Correctives, National Academy of Sciences, Washington, D.C., 340-368.

O'Connor, D. J., and Mueller, J. A. (1970), "A Water Quality Model of Chlorides in Great Lakes", Proc. ASCE, Journal SED, Vol.96, No. SA4, 955-975.

Okubo, A. (1971), "Horizontal and Vertical Mixing in the Sea", in Impingement of Man on the Oceans, edited by D. W. Hood, John Wiley and Sons, New York.

O'Melia, C. R. (1972a), "An Approach to the Modeling of Lakes," Schweizerische Zeitschrift für Hydrologie, Vol. 34, 1-33.

O'Melia C. R. (1972b), "Coagulation and Flocculation", Chapter 2 in Physico-chemical Processes for Water Quality Control, pp. 61-109, Wiley-Interscience, New York.

Ozmidov. R. V. (1965), "On the Turbulent Exchange in a Stably Stratified Ocean", Izvestiya, Bull. Acad. Sci. USSR, Atmospheric and Ocean Physics Series, Vol. 1, 853-860, translated by American Geophysical Union, Washington,

Patalas, K., and Salki, A. (1973), "Crustacean Plankton and the Eutrophication of Lakes in the Okanagan Valley, B. C.", Jour. Fisheries Res. Bd. of Canada, Vol. 30, 519-542.

Perret, Peter, (1973), "Ubersuchungen über die Bodenfauna des Litorals im Eutrophierenden Sempachersee", Schweizerische Zeitschrift Für Hydrologie, Vol. 35, No. 1, 69-113.

Powell, T., and Jassby, A. (1974) "The Estimation of Vertical Eddy Diffusivities below the Thermocline in Lakes", Water Resources Research, Vol. 10, 191-198.

Rawson, D. S. (1965) "Morphometry as a Dominate Factor in the Productivity of Large Lakes", Verh. Int. Verein. Limnol., Vol. 12, 164-175.

Sakamoto, Mitsuru (1966), "Primary Production by Phytoplankton Community in Some Japanese Lakes and its Dependence on Lake Depth", Arch. Hydrobiol., Vol. 62, 1-28.

Sawyer, Clair N. (1947), "Fertilization of Lakes by Agricultural and Urban Drainage", Jour. New England Water Works Assoc., Vol. 61, 109-127.

Schindler, D. W. (1971), "Light, Temperature, and Oxygen Regimes of Selected Lakes in the Experimental Lakes Area, Northwestern Ontario", Jour. Fisheries Research Board of Canada, Vol. 28, 157-169.

Shiomi, M. T., and Chawla, V. K. (1970), "Nutrients in Lake Ontario", Proc. 13th Conf. Great Lakes Res., International Association Great Lakes Res., 715-732.

Snodgrass, William J. (1974), "A Predictive Phosphorus Model for Lakes - Development and Testing", Ph.D. Dissertation, University of North Carolina, Chapel Hill, North Carolina.

Stein, J. R., and Coulthard, T. L. (1971), "A Report on the Okanagan Water Investigation 1969-70", Water Investigations Branch, British Columbia Water Resources Service, Victoria, B. C. Canada, 176 pp.

Stumm, Werner, and Morgan, James J. (1962), "Stream Pollution by Algal Nutrients", Trans. 12th Conference Sanitary Engineering, University of Kansas, 16-26.

Sweers, H. E. (1970), "Vertical Diffusivity Coefficient in a Thermocline", Limnology and Oceanography, Vol. 15, 273-280.

Thomas, Eugene A. (1969), "The Process of Eutrophication in Central European Lakes", in Eutrophication, Causes, Consequences, Correctives, National Academy of Sciences, Washington, D. C., 29-49.

Thon, J. (1969), "Empirical Relationship for Total Phosphorus System in Lower Great Lakes; Proc. 12th Conference on Great Lakes Res., Inter. Assoc. of Great Lakes Res., 768-787.

Vollenweider, R. A. (1968), "Scientific Fundamentals of the Eutrophication of Lakes and Flowing Waters, with Particular Reference to Nitrogen and Phosphorus as Factors in Eutrophication", OECD Technical Report DAS/CS1/68.27, Paris, pp. 159.

Vollenweider, R. A. (1969), "Möglichkeiten und Grenzen elementarer Modelle der Stoffbilanz von Seen", Arch. Hydrobiol., Vol. 66, 1-36.

Vollenweider, R. A. (1974), "Input - Output Models", Schweizerische Zeitschrift fur Hydrologie, (in press).

LIST OF PATENTS AND PUBLICATIONS

Patents

No applications for a patent has been or will be made.

Publications

Snodgrass, William J., "A Predictive Phosphorus Model for Lakes-Development and Testing", Ph.D. Dissertation, Department of Environmental Sciences and Engineering, University of North Carolina at Chapel Hill, 1974.

GLOSSARY

A	surface area of a lake (L^2)
A_s	surface area of the sediment-water interface (L^2)
A_{th}	horizontal cross-sectional area of a lake at the thermocline (L^2)
D	diameter of a particle (L)
E	static stability (T^{-2})
f	flocculation coefficient (L^{-1})
G	mean velocity gradient (T^{-1})
g	sedimentation coefficient of entire lake (L/T)
g_h	sedimentation coefficient in the hypolimnion (L/T)
g_e	sedimentation coefficient in the epilimnion (L/T)
g_o	sedimentation coefficient in the absence of flocculation (L/T)
$\overline{g}$	gravity acceleration (L/T^2)
k	vertical transport coefficient (L^2/T)
k_h	vertical transport coefficient in the hypolimnion (L^2/T)
k_{th}	vertical transport coefficient in the thermocline (L^2/T)
$\hat{k}_{th}$	vertical exchange coefficient in the thermocline region, $= k_{th}/\overline{Z}_{th}$ (L/T)
L_a	areal phosphorus loading (M/L^2-T)
n	number concentration of particles in a system (L^{-3})
n_o	initial number concentration of particles in a system (L^{-3})
N_o	orthokinetic flocculation rate ($L^{-3}T^{-1}$)

N_g	sedimentation flocculation rate ($L^{-3}T^{-1}$)
p_e	production rate coefficient in the epilimnion (T^{-1})
p_{eu}	production rate coefficient in the euphotic zone, circulation model (T^{-1})
Q	volumetric rate of discharge from a lake (L^3/T)
Q_j	volumetric rate of inflow to a lake from source j (L^3/T)
Q_a	areal rate of total water inflow or discharge (L/t)
r	decomposition rate coefficient for entire lake (T^{-1})
r_h	decomposition rate coefficient for the hypolimnion (T^{-1})
S	specific gravity of a particle
t	time (T)
V	volume of entire lake (L^3)
V_e	volume of the epilimnion (L^3)
V_{eu}	volume of the euphotic zone in circulation model (L^3)
V_h	volume of the hypolimnion (L^3)
$\overline{Z}$	mean lake depth (L)
$\overline{Z}_e$	mean depth of the epilimnion (L)
$\overline{Z}_{eu}$	mean depth of the euphotic zone, circulation model (L)
$\overline{Z}_h$	mean depth of the hypolimnion (L)
$\overline{Z}_{th}$	mean depth of thermocline region (L)
ϕ	floc volume fraction
υ	kinematic viscosity of water (L^2/T)
ρ	density of water (M/L^3)

σ	volumetric sedimentation coefficient (T^{-1})
[OP]	concentration of orthophosphate (M/L^3)
[PP]	concentration of particulate phosphorus (M/L^3)
[TP]	concentration of total phosphorus (M/L^3)
$[\bar{\ }]$	steady-state concentration (M/L^3)