Mathematical Methods for Sequential Operations Reliability

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1. INTRODUCTION

In many cases the reliability or availability of a system depends on the performance of a number of operations that must be taken in a given sequence. We quote as examples: case (a) the human interaction with abnormal system or accident conditions, when the crew must complete N operations within a finite time window T; case (b) the reliability at time T of a parallel system of N units, only one of which is in operation at a time, i.e the (n+1)-th unit is to go into operation as the n-th fails. The probability $D_m^n(x)$ (or $C_m^n(x)=1-D_m^n(x)$) that the cumulative duration of the m-th,(m+1)-th,...,n-th operations from the sequence does not (or does) exceed x is the basis for the reliability analysis of such systems. In particular, the probability $D_1^N(T)$ of completing the whole sequence of N operations within the time T has two opposite meanings in the above cases. In case (a) $D_1^N(T)$ is the success probability of the man-plant interaction, and the complementary (failure) probability $C_1^N(T)$ is generally required to be several orders of magnitudes below one. In case (b) $D_1^N(T)$ can be termed as the unreliability ($C_1^N(T)$ the reliability) of the system at time T. Now $C_1^N(T)$ should be as close to one as possible. As will be seen below, D_1^N can be evaluated with much greater ease than C_1^N, but in case (a) the latter is the more meaningful quantity and cannot be accurately derived as $1-D_1^N$ if its order of magnitude is low, as required.

The aim of this paper is thus to discuss (first) the basis for deriving independent integral representations of D_1^N and C_1^N according to which (second) various computational methods will be developed: (1) a semi-analytical method based on N-fold series expansion; (2) a numerical method of integration over suitable multidimensional sets; (3) a Monte Carlo simulation approach that can greatly enhance the flexibility. All methods are comparatively tested on Weibull distributions with respect to accuracy, range of application and computational efforts.

2. INTEGRAL REPRESENTATIONS OF D_1^N AND C_1^N (Trombetti,Vestrucci;1989)

We introduce the random vectors (rve.s) $\underline{t}_{m,n}=(t_m,t_{m+1},...,t_n)$, with joint probability density functions (jpd.s) $p_{m,n}(\underline{x})$, $\underline{x}\in\mathbb{R}_+^{n-m+1}$, where the random variable (rva) t_i is the duration of the i-th operation. We also introduce the rva.s $t_m^n=t_m+t_{m+1}+\dots+t_n=S_n(\underline{t})-S_{m-1}(\underline{t})$ where $S_i(\underline{x})$ is the sum of components $1,2,\dots,i$ of $\underline{x}\in\mathbb{R}_+^N$, $i\le N$, and $S_0(\underline{x})=0$. All rva.s are bound to be non-negative and continuously distributed. Hence the cumulative distribution functions (cdf.s) $D_m^n(x)$ and $D_n(x)=D_n^n(x)$, of t_m^n and t_n, approach +0 as x goes to +0. We express the cdf.s and complementary cdf.s by the integral representations (IR.s)

$$D_m^n(T)=\int_{A_{n-m+1}(T)} p_{m,n}(\underline{x})\,d\underline{x}\quad,\quad C_m^n(T)=1-D_m^n(T)=\int_{\bar{A}_{n-m+1}(T)} p_{m,n}(\underline{x})\,d\underline{x}\quad,$$

where the i-th dimensional simplex $A_i(T)\in\mathbb{R}_+^i$ is the polyhedron with i+1 vertices in the origin and the points of coordinate T on the i axes, and $\bar{A}_i(T)$ its complement in $\mathbb{R}_+^i$. To give the second IR a more explicit form we introduce the following partition of $\bar{A}_i$:

$$\bar{A}_i(T)=\bigcup_{j=0}^{i-1} B_i^j(T)\ ,\ B_i^j(T)=\left\{\underline{x}\in\mathbb{R}_+^i:\ S_j(\underline{x})\le T<S_{j+1}(\underline{x})\right\}=$$

$$=\left\{\underline{x}\in\mathbb{R}_+^i\ :\ \underline{x}_{1,j}\in A_j(T),(S_{j+1}(\underline{x})-T,x_{j+2},x_{j+3},\dots,x_i)\in\mathbb{R}_+^{i-j}\right\}\quad,$$

where x_n is the n-th component of $\underline{x}$ and $\underline{x}_{1,j}=(x_1,x_2,\dots,x_j)$.Then, we recast the IR.s into the more explicit form

$$D_1^N(T)=\int_{A_N(T)} p_{1,N}(\underline{x})\,d\underline{x}\quad, \tag{1}$$

$$C_1^N(T)=Q_{1,1}(T)+\sum_{j=1}^{N-1}\int_{A_j(T)} Q_{1,j+1}(x_1,x_2,\dots x_j,T-S_j(\underline{x}))\,d\underline{x}_{1,j}\quad, \tag{2a}$$

with

$$Q_{1,j}(\underline{x}_{1,j})=\int_{x_j}^{\infty} p_{1,j}(x_1,x_2,\dots,x_{j-1},\xi)\,d\xi\quad.$$

An important special case occurs if the rva.s t_n are all independent, with probability density functions (pdf.s) $p_n(x)$, $x\in\mathbb{R}_+$, cdf.s $P_n(x)$ and complementary cdf.s $Q_n(x)$, such that

$$p_{1,i}(\underline{x}_{1,i})=\prod_{j=1}^{i} p_j(x_j)\quad,\quad P_n(x)=\int_0^x p_n(x_n)\,dx_n\ ,$$

$$Q_n(x)=\int_x^{\infty} p_n(x_n)\,dx_n=1-P_n(x)\quad,\quad C_1^N(T)=Q_1(T)+\sum_{i=1}^{N-1} F_{i+1}(T)\ , \tag{2b}$$

$$F_{i+1}(T) = \int_{A_i(T)} p_{1,i}(\underline{x}_{1,i})Q_{i+1}[T-S_i(\underline{x})]d\underline{x}_{1,i} \quad .$$

Thus the IR of D_1^N involves only one N-simplex integration. When $1-D_1^N$ is inaccurate ($D_1^N \simeq 1$), one can resort to the IR of C_1^N, involving N functions Q_i and N-1 i-simplex integrations, i=1,2,...,N-1.

3. THREE METHODS FOR THE DIRECT EVALUATION OF D_1^N AND C_1^N

To be more definite, some of the discussion below will be referred to independent rva.s t_n with pdf.s of specific interest, i.e. three parameters Weibull distributions (Wd.s) for which $Q_n(x)=1$ if $0<x\le\tau_{on}$ and $Q_n(x)=\exp-[\lambda_n(x-\tau_{on})]^{\beta_n}$, $x>\tau_{on}$. Here $\lambda_n=1/(\eta_n T_{1/2n})$, $\tau_{on}=\gamma_n T_{1/2n}$ where $T_{1/2n}$ is the median time for the n-th operation and in case (a) the values of the parameters β_n,η_n,γ_n can be assigned e.g. according to Hannaman et al.(1985). These values are quoted in Tab.1 for different kind of human actions, following the classification due to Rasmussen (1979).We remark that replacing $x-\tau_{on}$ with x the available time T is replaced by $T_u=T-\sum\tau_{on}>0$ (otherwise $D_1^N=0$, $C_1^N=1$). We turn now to the three methods listed in Sec.1.

Table 1. Weibull distribution parameters

Type of cognitive processing	β	γ	η
Skill	1.13	0.720	0.388
Rule	1.27	0.148	1.14
Knowledge	0.795	0.389	0.969

<u>Method 1: series expansion method</u>

From power series expansions of $p_n(x_n)$ and $Q[T_u-S_n(\underline{x}_n)]$ followed by termwise integration of the product series, we get

$$F_2=\beta_1 \sum_{i,j=0}^{\infty} \frac{(-1)^{i+j}}{i!\ j!} (\lambda_1 T_u)^{\beta_1(i+1)}(\lambda_2 T_u)^{\beta_2(j+1)} \quad B(\beta_1(i+1),\beta_2 j+1) \ ,$$

$$F_3=\beta_1\beta_2 \sum_{i,j,k=1}^{\infty} \frac{(-1)^{i+j+k}}{i!\ j!\ k!} (\lambda_1 T_u)^{\beta_1(i+1)}(\lambda_2 T_u)^{\beta_2(j+1)}(\lambda_3 T_u)^{\beta_3(k+1)}$$
$$B(\beta_1(i+1),\beta_2 j+1+\beta_3 k+1)\ B(\beta_2(j+1),\beta_3 k+1) \quad ,$$

and so on, in terms of the Beta function B(.,.). The smaller the $\lambda_n T_u$, the better the accuracy and the CPU time. With increasing $\lambda_n T_u$, one would need a too large number of terms to limit the truncation error and double or higher precision to reduce the roundoff error. The latter increases dramatically when a decreasing result for F_n is to be obtained adding positive and negative terms with increasing absolute values.

<u>Method 2: numerical integration</u>

Very efficient methods for the numerical integration over n-simplexes are known (De Doncker,1979) and can be applied to our IR.s of D_1^N and C_1^N in their present form, Eq.s (1) and (2).

Method 3: Montecarlo simulation

Methods 1 and 2 encounter serious difficulties when dealing with parameter uncertainties or rva dependence, and even more when the overall time constraint T is to be supplemented by partial ones referring to single or subgroups of subtasks. The Montecarlo simulation does not present the previous limitations. The basic flow of crew response simulation in case (a) is very simple and a crude or analog Montecarlo approach can easily be developed. However, it is well known that the number of histories required to obtain good estimates is often impractical without biasing. For this reason Variance Reduction Methods are necessary and two of them turn out to be very effective: forcing the success and the failure, and linear extraction (Vestrucci et al., 1989). In the first case the idea is to get at each history "a fraction of success and a fraction of failure". In other words we force the crew to respond to the n-th subtask within the residual available time $T_n = T_u - S_{n-1}(\underline{t})$, assigning a proper weight w_{sn} in order to correct the distortion. The pdf.s used in this approach for the extraction of t_n are $p_n^f(x)=0$ for $x \geq T_n$ and $p_n^f(x)=p_n(x)/P_n(T_n)$ for $x<T_n$. In the n-th subtask we will have a success fraction $w_{sn}=p_n(t_n)/p_n^f(t_n)=P_n(T_n)$ and a failed fraction $1-w_{sn}=Q_n(T_n)$, both dependent on $\underline{t}_{n-1}$ but independent on t_n. The cumulated failed fraction after the n-th subtask is

$$w_{fn} = w_{f,n-1} + (1-w_{sn}) \prod_{1=1}^{n-1} w_{si} \quad , \text{ with } w_{f1}=1-w_{s1}.$$

In the linear extraction technique the basic idea is to favour the sampling of unlikely values of t_n by using uniform distributions. In this frame, the cdf for the linear extraction of t_n is $P_n^{LE}(x)=x/T_n$ (again $x<T_n$) and the weight for the success fraction is given by $w_{sn}^{LE}=p_n(t_n)/p_n^{LE}(t_n)=T_n p_n(t_n)$.

4. NUMERICAL RESULTS

The three methods have been tested for a sequence of three subtasks composed by a knowledge-based action with $T_{1/2}=5$, a rule-based action having $T_{1/2}=6$ and a skill-based action with $T_{1/2}=9$ time units. Several time window have been considered T=10, 20, 30, 40, 70, 100, 130, in order to span a wide range of probability values. The results are summarized in Tab.2. In the first column the numerical integration results are quoted and they can be considered exact in the sense that increasing the number of knots in the numerical integration, they not change. In addition this method turns out to be the most efficient from the point of view of the computational time except for $T \leq 20$, when the semi-analytical approach is very effective. On the other hand, when $T>20$ the sum of the series becomes very time consuming and for $T>40$ numerical instabilities make the semi-analytical approach unusable. The Montecarlo method yields always good results: with variance reduction techniques 10^4 histories are always enough for practical application, although for the sake of comparison 10^5 histories are been simulated in the extreme case of T=130. Computational times are disadvantageous in the case of Montecarlo in comparison with the previous methods, but always acceptable (few minutes with an AT-PC). The Montecarlo turns out to be easily adaptable to more complicated situations. As a simple example we consider the case in which the time window is known as a rva with a distribution f(T) and not like a fixed value.

Table 2. Results of the application

T	Result	Num. Integr.	Montecarlo	Semi-analytic.
10	F_2	0.185	0.185	0.185
	F_3	0.005	0.005	0.005
	C_1^3	0.9997	0.9997	0.9997
20	F_2	0.303	0.304	0.304
	F_3	0.186	0.186	0.186
	C_1^3	0.643	0.643	0.643
30	F_2	0.0894	0.0851	0.0894
	F_3	0.0915	0.0874	0.0863
	C_1^3	0.216	0.221	0.218
40	F_2	0.0236	0.0238	0.0242
	F_3	0.0245	0.0247	0.0253
	C_1^3	0.0611	0.0616	0.0625
70	F_2	7.17 E-4	7.08 E-4	Numerical instabilities
	F_3	6.47 E-4	6.42 E-4	↓
	C_1^3	1.94 E-3	1.93 E-3	
100	F_2	3.65 E-5	3.54 E-5	
	F_3	3.07 E-5	3.03 E-5	
	C_1^3	1.02 E-4	1.01 E-4	
130	F_2	2.39 E-6	2.41 E-6	
	F_3	1.94 E-6	1.93 E-6	
	C_3^3	6.86 E-6	6.87 E-6	

Table 3. Results with time window uncertainties

	$\langle T\rangle=30$			$\langle T\rangle=100$		
Δ	4.5	9	15	15	30	50
$C_1^3(\langle T\rangle-\Delta)$	0.37	0.59	0.90	4.3E-4	2.2E-3	1.6E-2
C_1^{3*}	0.225	0.26	0.30	1.4E-4	2.2E-4	1.6E-3
$C_1^3(\langle T\rangle+\Delta)$	0.12	0.069	0.033	2.6E-5	6.9E-6	1.2E-6

In this case it is required to compute $C_1^{3*} = \int_0^{\infty} f(T)\, C_1^3(T)\, dT$. Considering as a simple example T uniformly distributed around a mean value $\langle T\rangle$, i.e. $f(T)=1/2\Delta$ for $\langle T\rangle \in [\langle T\rangle-\Delta, \langle T\rangle+\Delta]$, and equal zero otherwise, the Montecarlo method can be applied with trivial modifications. The results are presented in Tab.3 for $\langle T\rangle=30$ and 100 and $\Delta/\langle T\rangle=15$, 30 and 50% .As expected, when Δ increases, the values of C_1^{3*} differ more and more from the values of C_1^3 (see Tab.3), and the order of magnitude becomes closer to the bound $C_1^3(\langle T\rangle-\Delta)$.

CONCLUSIONS

The probabilities, $D_1^N(T)$ and $C_1^N(T)$, that the cumulative duration of N sequential actions does not or does exceed a given time window T can represent (according to the type of system operation) important quantities as success and failure probabilities, system unavailability or reliability. They have first been expressed by the integral representations, Eq.s(1,2), then evaluated by various computational methods. Problems arising when $C_1^N(T)$ has an extremely small order of magnitude have been solved successfully by one or more methods. The Montecarlo method has proven also capable of dealing with a further degree of randomness, when T itself is to be treated as a random variable.

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