Work supported by U. S. Army Research Office (Durham) Grant No. DAH C04-70-G0006

EFFICIENT ESTIMATION OF THE MEAN OF AN EXPONENTIAL DISTRIBUTION WHEN AN OUTLIER IS PRESENT

by

Prakash C. Joshi

Department of Biostatistics
University of North Carolina at Chapel Hill

Institute of Statistics Mimeo Series No. 655

January 1970

Efficient Estimation of the Mean of an Exponential Distribution when an Outlier is Present*

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This paper extends some of the results obtained in a recent paper by Kale and Sinha [3] for the exponential distribution. The problem of selecting an efficient estimator of the expected value in the presence of an outlying observation with higher expected value is discussed. An iterative procedure for the estimation of the mean is provided and the method is illustrated by considering an example.

1. INTRODUCTION AND SUMMARY

In a recent paper [3], Kale and Sinha have considered the following problem: Given n independent observations $x_1, x_2, \ldots, x_n$, n-1 of which are from

$$p(x,\sigma) = (1/\sigma)\exp(-x/\sigma),\ x\geq 0,\ \sigma>0$$

and one of which is from $p(x,\sigma/\alpha)$, $0<\alpha\leq 1$, we wish to estimate the parameter σ.

For $0<\alpha<1$, the single observation from $p(x,\sigma/\alpha)$ essentially represents an outlying observation with a higher expected value. If $\alpha=1$ (termed as the homogeneous case by Kale and Sinha), then there is no outlying observation and in this case

$$T_n = \sum_{i=1}^{n} x_i/(n+1) \tag{1.1}$$

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* Supported by U. S. Army Research Office (Durham) Grant No. DAH C04-70-G0006.

is the best linear estimator of σ in the sense that it has the smallest Mean Square Error (MSE) in the class of all linear estimators (see e.g. [4]). To begin with, we do not know the α parameter and the outlying observation. However, in many situations we can reasonably assume that each X_i has an equal chance of being the outlying observation. Under this assumption, Kale and Sinha [3] have suggested an estimator of σ based on the smallest m (<n) order statistics of the form

$$T_m = \left(\sum_{i=1}^{m-1} X_{(i)} + (n-m+1)X_{(m)}\right)/(m+1), \tag{1.2}$$

where $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$ represent the order statistics obtained by rearranging the random variables $X_1, X_2, \ldots, X_n$. They arrived at this estimator by using the fact that the set $\{X_{(m+1)}, X_{(m+2)}, \ldots, X_{(n)}\}$ contains the outlying observation with a high probability. However, they have given no method of determining m^*, the optimum value of m in the sense that the MSE of T_{m^*} is smallest. In this paper, we first derive an exact expression for the MSE of T_m and then tabulate m^* for various values of n and α. Our computations show that even for α as small as ½, m^* is equal to n for all $n \geq 6$. Therefore, we allow the possibility of m being equal to n in equation (1.2). Note that for m=n, equation (1.2) reduces to (1.1).

The use of this table requires the value of α. However, α is not known. So, we first give a simple estimator of α based on intuitive grounds and then provide an iterative procedure for the estimation of σ and α. Finally, we illustrate our method by considering an example.

2. BASIC DISTRIBUTION THEORY RESULTS

In this section we will obtain an exact expression for the MSE of T_m in estimating σ. Let $Y_i = X_i/\sigma$ (i=1,2,...,n). Then, exactly n-1 of these random variables are from

$$p(y,1) = e^{-y}, \qquad y \geq 0,$$

while the remaining one is from $p(y,1/\alpha)$. Now by our assumption each Y_i has an equal chance of being distributed as $p(y,1/\alpha)$. Hence the joint p.d.f. of $Y_1, Y_2, \ldots, Y_n$ is [3]

$$f(y_1, y_2, \ldots, y_n) = (\alpha/n) \sum_{i=1}^{n} \exp[-\sum_{j=1}^{n} y_j + (1-\alpha) y_i].$$

If we now let $Y_{(i)} = X_{(i)}/\sigma$, then the joint p.d.f. of $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ is the sum of n components corresponding to n mutually exclusive and exhaustive cases of $Y_{(i)}$ being the outlier. Hence

$$g(y_{(1)}, y_{(2)}, \ldots, y_{(n)}) = (n-1)!\, \alpha \sum_{i=1}^{n} \exp[-\sum_{j=1}^{n} y_{(j)} + (1-\alpha) y_{(i)}], \qquad (2.1)$$

where $0 \leq y_{(1)} \leq y_{(2)} \leq \cdots \leq y_{(n)} < \infty$. Further

$$T_m = \sigma[\sum_{i=1}^{m-1} y_{(i)} + (n-m+1) y_{(m)}]/(m+1).$$

In order to evaluate the MSE of T_m, we now need $E(Y_{(i)})$ and $E(Y_{(i)} Y_{(j)})$. However, equation (2.1) is not very convenient for this purpose. Note that the marginal distributions of the extreme order statistics can be obtained by using some simple probabilistic arguments. Thus, for example, for $Y_{(n)}$ we have

$$F_n(y) = \Pr(Y_{(n)} \leq y)$$

$$= \prod_{i=1}^{n} \Pr(Y_i \leq y)$$

$$= (1-e^{-\alpha y})(1-e^{-y})^{n-1}.$$

From this we can obtain the density, m.g.f. and hence the moments of $Y_{(n)}$.

Similar results are valid for $Y_{(1)}$ also.

We now express T_m in terms of another set of random variables which are easier to handle. To this end, make the transformation

$$z_r = (n-r+1)(y_{(r)}-y_{(r-1)}),\ r=1,2,\ldots,n,$$

where $y_{(o)} = 0$. Then

$$y_{(i)} = \sum_{k=1}^{i} z_k/(n-k+1) \tag{2.2}$$

and

$$T_m/\sigma = \sum_{i=1}^{m} z_i/(m+1). \tag{2.3}$$

Moreover, the joint p.d.f. of $Z_1,Z_2,\ldots,Z_n$ is

$$h(z_1,z_2,\ldots,z_n) = (\alpha/n)\sum_{i=1}^{n} \exp[-\sum_{j=1}^{n} z_j + (1-\alpha)\sum_{k=1}^{i} z_k/(n-k+1)],$$

where $0 \leq z_1,z_2,\ldots,z_n < \infty$. Letting

$$b_r = (n-r+\alpha)/(n-r+1),\ r=1,2,\ldots,n \tag{2.4}$$

we see that

$$\begin{aligned} h(z_1,z_2,\ldots,z_n) = (\alpha/n)[&e^{-b_1z_1-z_2-\ldots-z_n} \\ &+ e^{-b_1z_1-b_2z_2-\ldots-z_n} \\ &+ \ldots.. \\ &+ e^{-b_1z_1-b_2z_2-\ldots-b_nz_n}]. \end{aligned} \tag{2.5}$$

Before proceeding any further, we now evaluate the probability that $Y_{(r)}$ is distributed as $p(y,1/\alpha)$, while the remaining random variables, viz.,

$Y_{(1)},\ldots,Y_{(r-1)},Y_{(r+1)},\ldots,Y_{(n)}$ are distributed as $p(y,1)$. This is precisely the integral of the rth term in the sum given by the R.H.S. of (2.1) or equivalently of (2.5) (see e.g. [3]). Denoting this probability by u_r, we then have

$$u_r = (\alpha/n)\int_0^\infty\int_0^\infty \ldots\int_0^\infty \exp(-\sum_{i=1}^{r} b_i z_i - \sum_{i=r+1}^{n} z_i)dz_1\ldots dz_n$$

$$= \alpha/(nb_1b_2\ldots b_r). \tag{2.6}$$

An alternative method of finding u_r is also given in [3], where it is shown that it can also be expressed as

$$u_r = \alpha\binom{n-1}{r-1}\int_0^\infty (1-e^{-y})^{r-1}\, e^{-(n-r+\alpha)y}dy \tag{2.7}$$

$$= \frac{\alpha\Gamma(n)\Gamma(n-r+\alpha)}{\Gamma(n+\alpha)\Gamma(n-r+1)} . \tag{2.8}$$

Using equation (2.4), the equivalence of (2.6) and (2.8) is easy to prove.

We now state a lemma involving the weighted sums of the above mentioned probabilities. This is needed to express the marginal distribution of Z_i in a compact form. The proof of this is given in the Appendix.

Lemma 1. For $m=1,2,\ldots,n$

$$\sum_{r=m}^{n} u_r = (n-m+\alpha)u_m/\alpha \tag{2.9}$$

and

$$\sum_{r=m}^{n} ru_r = \frac{(n-m+\alpha)(n+m\alpha)u_m}{\alpha(\alpha+1)} . \tag{2.10}$$

The sums $\sum_{r=1}^{m-1} u_r$ and $\sum_{r=1}^{m-1} r u_r$ can be obtained by noting that $u_1 = \alpha/(n-1+\alpha)$ and $\sum_{r=1}^{n} u_r = 1$, $\sum_{r=1}^{n} r u_r = (n+\alpha)/(\alpha+1)$.

As mentioned previously, we now obtain the marginal distribution of Z_i in terms of the probability u_i introduced above. Integrating out $Z_1, Z_2, \ldots, Z_{i-1}, Z_{i+1}, \ldots, Z_n$ from (2.5) and using (2.6), we have the marginal p.d.f. of Z_i

$$h(z_i) = (u_1+\ldots+u_{i-1})e^{-z_i} + b_i(u_i+\ldots+u_n)e^{-b_i z_i}$$

$$= e^{-z_i} + [(n-i+\alpha)u_i/\alpha](b_i\, e^{-b_i z_i} - e^{-z_i})$$

on using Lemma 1. Hence

$$E(Z_i) = 1 + \frac{(n-i+\alpha)u_i}{\alpha}\left(\frac{1}{b_i} - 1\right)$$

$$= 1 + \frac{(n-i+\alpha)u_i}{\alpha}\left(\frac{n-i+1}{n-i+\alpha} - 1\right)$$

$$= 1 + \theta u_i, \tag{2.11}$$

where $\theta = (1-\alpha)/\alpha$. Similarly

$$E(Z_i^2) = 2 + 2\theta u_i(2n-2i+1+\alpha)/(n-i+\alpha). \tag{2.12}$$

Next, for $i<j$ the joint p.d.f. of Z_i and Z_j from (2.5) is

$$h(z_i, z_j) = (u_1+\ldots+u_{i-1})e^{-z_i - z_j}$$

$$+ (u_i+\ldots+u_{j-1})b_i\, e^{-b_i z_i - z_j}$$

$$+ (u_j+\ldots+u_n)\, b_i b_j\, e^{-b_i z_i - b_j z_j}.$$

Hence, for $i<j$

$$E(Z_iZ_j) = (u_1+\ldots+u_{i-1}) + (u_i+\ldots+u_{j-1})/b_i$$
$$+ (u_j+\ldots+u_n)/(b_ib_j)$$
$$= 1 + (\frac{1}{b_i} - 1)(u_i+\ldots+u_n)+(\frac{1}{b_ib_j} - \frac{1}{b_i})(u_j+\ldots+u_n)$$
$$= 1 + \theta u_i + \theta u_j(n-i+1)/(n-i+\alpha). \qquad (2.13)$$

Variances and co-variances of $Z_1,Z_2,\ldots,Z_n$ can now be obtained from equations (2.11)-(2.13). Likewise, the means, variances and co-variances of $Y_{(1)},Y_{(2)},\ldots,Y_{(n)}$ can be obtained by using equation (2.2). Thus for example

$$E(Y_{(i)}) = \sum_{k=1}^{i} E(Z_k/(n-k+1)).$$

Now from equation (2.11)

$$E(Z_1/n) = (1+\theta u_1)/n = 1/(n-1+\alpha) = u_1/\alpha$$

and for $k\geq 2$

$$E(\frac{Z_k}{n-k+1}) = \frac{1}{n-k+1} + \frac{(1-\alpha)u_k}{\alpha(n-k+1)} .$$

From equation (2.6), it is easy to show that for $k\geq 2$

$$u_k-u_{k-1} = (1-\alpha)u_k/(n-k+1). \qquad (2.14)$$

Hence

$$E(\frac{Z_k}{n-k+1}) = \frac{1}{n-k+1} + (u_k-u_{k-1})/\alpha$$

and

$$E(Y_{(i)}) = \frac{u_1}{\alpha} + \sum_{k=2}^{i} \frac{1}{n-k+1} + \frac{1}{\alpha}\sum_{k=2}^{i}(u_k-u_{k-1})$$

$$= \frac{u_i}{\alpha} + \sum_{k=2}^{i} \frac{1}{n-k+1} ,$$

where, by convention, the second term of the R.H.S. is zero for i=1.

Finally, from equation (2.3) we have the MSE of T_m in estimating σ given by

$$\begin{aligned} MSE(T_m)/\sigma^2 &= E\left(\frac{1}{m+1}\sum_{i=1}^{m} Z_i - 1\right)^2 \\ &= \frac{1}{(m+1)^2}\left[\sum_{i=1}^{m} E(Z_i^2) + 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} E(Z_iZ_j) - 2(m+1)\sum_{i=1}^{m} E(Z_i) + (m+1)^2\right]. \end{aligned} \tag{2.15}$$

Using equations (2.11)-(2.13) and Lemma 1 repeatedly, we get

$$MSE(T_m)/\sigma^2 = \frac{1}{m+1} + \frac{2\theta^2}{(m+1)^2}\left[1-(n-m)u_m\left(\frac{1}{\alpha} + \sum_{i=1}^{m}\frac{1}{n-i+\alpha}\right)\right]. \tag{2.16}$$

The details of the proof are given in the Appendix.

Note that for $\alpha=1$

$$MSE(T_m)/\sigma^2 = 1/(m+1)$$

and for m=n

$$MSE(T_n)/\sigma^2 = \frac{1}{n+1} + \frac{2\theta^2}{(n+1)^2} . \tag{2.17}$$

Equation (2.17) has been also obtained by Kale and Sinha [3] by using a different approach. They have also given some special cases of (2.16). The cases considered by them are: (i) any n and m=1, (ii) n=3 and m=2, (iii) n=4 and m=2,3. The result given in (2.16) is valid for all values of m and n. This allows us to find a value of m for which the MSE of T_m is smallest. We will denote this optimum value of m by m*. It is clear that among all the estimators T_m $(m=1,2,\ldots,n)$, T_{m^*} has the maximum efficiency. A numerical evaluation of m* is considered in the next section.

3. OPTIMUM VALUE OF m

The problem of finding m* from equation (2.16) theoretically is quite difficult. However, for given α, the problem can always be solved numerically (at least for small values of n). We need only find the n Mean Square Errors corresponding to n different possible values of m* and then pick the one with the smallest MSE. In Table 1, we tabulate m* for n = 2(1)10(5)20(10)50 obtained by this method. The efficiency of T_{m^*} relative to T_n defined by

$$e_{m^*} = \mathrm{MSE}(T_n)/\mathrm{MSE}(T_{m^*})$$

is also tabulated. The α parameter was increased in steps of 0.05. For $\alpha \geq 0.55$ it was observed that m*=n and these values are not tabulated. Our computations also revealed that the values for the MSE and efficiency as tabled in [3] for the case n=4, m=3 are in error. The correct values for this case are given in Table 2.

In their paper, Kale and Sinha suggest to take m<n. However, in view of Table 1, we allow the possibility of m* being equal to n. In fact for α as low as ½, we recommend the estimator T_n for estimating σ. A theoretical justification for this can be seen from equation (2.16) by comparing the Mean Square Errors for T_{n-1} and T_n for $\alpha=\frac{1}{2}$. Now $\theta=(1-\alpha)/\alpha = 1$,

$$\mathrm{MSE}(T_n)/\sigma^2 = \frac{1}{n+1} + \frac{2}{(n+1)^2}$$

and

$$MSE(T_{n-1})/\sigma^2 = \frac{1}{n} + \frac{2}{n^2}\,[1-u_{n-1}(2+\sum_{i=1}^{n-1}\frac{1}{n-i+\frac{1}{2}})],$$

where

$$u_{n-1} = \frac{\Gamma(n)\Gamma(\frac{1}{2})}{4\ \Gamma(n+\frac{1}{2})}\ .$$

For moderate values of n, the first term of the R.H.S. in both of these expressions will be dominant and hence the $MSE(T_n)$ will be less than the $MSE(T_{n-1})$.

4. ESTIMATION OF σ AND α

For the estimation of σ we use the estimator T_{m^*}. But m* depends on the parameter α. Now from Table 1 it is clear that m* is same for a wide range of α values. The problem of finding an efficient estimator of α and some tests concerning it will be considered in another paper. Here, we only give a simple estimator of α based on intuitive grounds.

It is easy to show that

$$E(\sum_{i=1}^{n} X_i) = (n-1 + 1/\alpha)\sigma.$$

However, the equation

$$\sum_{i=1}^{n} x_i = (n-1 + 1/\alpha)\hat{\sigma}, \tag{4.1}$$

where $\hat{\sigma}$ is an estimator of σ, leads to an under estimate of α. Thus, for example, in the homogeneous case (α=1) we will use the estimator T_n for estimating σ. But if we put T_n for $\hat{\sigma}$ in (4.1) then we only get $\alpha=\frac{1}{2}$. We therefore suggest to use the equation

$$n\,T_n = (n-1 + 1/\alpha)\hat{\sigma} \tag{4.2}$$

for the estimation of α. This is consistent with the homogeneous case but also involves $\hat{\sigma}$. So we give the following iterative procedure for the estimation of σ and α.

(i) First estimate σ by T_{n-1}, viz.

$$\hat{\sigma} = T_{n-1} = \left(\sum_{i=1}^{n-2} x_{(i)} + 2\, x_{(n-1)}\right)/n.$$

This serves as a good first approximation, since from equation (2.14) $x_{(n)}$ is the outlying observation with highest probability.

(ii) Estimate $\hat{\alpha}$ from equation (4.2).

(iii) Find m* from Table 1 and estimate $\hat{\sigma}$ by T_{m^*}.

(iv) Repeat the steps (ii) and (iii) until a stable value of m* and hence of $\hat{\sigma}$ is obtained.

We now illustrate our procedure by considering the data given by Sukhatme [5] and analyzed for outliers by Carlson [2] and Basu [1].

Example 1. The following ordered observations represent the length of interval, in half minutes, between the successive telephone calls: 1,3,3,15,25,33,39,70.

Here n=8 and the first approximation of $\hat{\sigma}$ is $T_7=19.75$. Equation (4.2) then gives $\hat{\alpha}=0.66$. Hence m*=8 and the revised estimate of σ is $T_8=21.00$.

Next suppose that $x_{(8)}$ is 90 instead of 70; i.e., the data is 1,3,3,15,25, 33,39,90. The value of T_7 remains unchanged but $T_8=23.22$. This gives $\hat{\alpha}=0.42$ and m*=7. So the revised estimate of σ is again $T_7=19.75$.

Note that a larger value of $x_{(8)}$ will lead to a smaller value of $\hat{\alpha}$ and in that case we may use $T_6=20.86$ or even $T_5=20.33$ for estimating σ.

ACKNOWLEDGEMENTS

The author wishes to thank Professor Harry Smith, Jr., the Editor of _Technometrics_ for providing an earlier draft of [3] and to Professor B. K. Kale for his permission to use some of the results contained in [3] prior to their publication.

APPENDIX

We now sketch the proofs of the results mentioned in Section 2.

(i) Proof of (2.9). We have

$$\sum_{r=m}^{n} u_r = u_n + u_{n-1} + \ldots + u_m$$

$$= \frac{\Gamma(n)}{\Gamma(n+\alpha)} \left\{ \frac{\alpha\Gamma(\alpha)}{\Gamma(1)} + \frac{\alpha\Gamma(\alpha+1)}{\Gamma(2)} + \ldots + \frac{\alpha\Gamma(n-m+\alpha)}{\Gamma(n-m+1)} \right\}$$

on using equation (2.8). Adding the terms successively, we get

$$\sum_{r=m}^{n} u_r = \frac{\Gamma(n)\Gamma(n-m+\alpha+1)}{\Gamma(n+\alpha)\Gamma(n-m+1)}$$

$$= (n-m+\alpha)u_m/\alpha \ .$$

It should be noted that equation (2.9) is valid for any $\alpha>0$ and with the help of (2.7) it can be rewritten as

$$\sum_{r=m}^{n} \binom{n-1}{r-1} \int_0^\infty (1-e^{-y})^{r-1} e^{-(n-r+\alpha)y} dy = \frac{\Gamma(n)\Gamma(n-m+\alpha+1)}{\alpha\Gamma(n+\alpha)\Gamma(n-m+1)} \ . \tag{A.1}$$

(ii) Proof of (2.10). To prove (2.10), it is convenient to use equation (2.7). Now consider the sum

$$\sum_{r=m}^{n} (n-r)u_r = \alpha \sum_{r=m}^{n} \int_0^\infty (n-r)\binom{n-1}{r-1}(1-e^{-y})^{r-1} e^{-(n-r+\alpha)y} dy.$$

Putting $n-1=N$ and $\alpha+1=\beta$ we see that

$$\sum_{r=m}^{n}(n-r)u_r = (\beta-1)N\sum_{r=m}^{N}\binom{N-1}{r-1}\int_0^{\infty}(1-e^{-y})^{r-1}\,e^{-(N-r+\beta)y}\,dy$$

$$= \frac{(\beta-1)N\Gamma(N)\Gamma(N-m+\beta+1)}{\beta\Gamma(N+\beta)\Gamma(N-m+1)} \qquad \text{by (A.1)}$$

$$= \frac{\alpha\Gamma(n)\Gamma(n-m+\alpha+1)}{(\alpha+1)\Gamma(n+\alpha)\Gamma(n-m)}$$

$$= (n-m+\alpha)(n-m)u_m/(\alpha+1).$$

The result now follows on using equation (2.9).

It is clear that the same method can be applied to evaluate similar weighted sums of u_r. Thus, for example, we can first evaluate $\sum_{r=m}^{n}(n-r)(n-r-1)u_r$ and then $\sum_{r=m}^{n}r^2u_r$. However, we do not need them for our present work.

(iii) Proof of (2.16). Let

$$S_m = (m+1)^2\text{MSE}(T_m)/\sigma^2 .$$

Then from equation (2.15) we can write

$$S_m = \sum_{i=1}^{m}E(Z_i^2-2Z_i)+2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}E(Z_iZ_j) - 2m\sum_{i=1}^{m}E(Z_i)+(m+1)^2.$$

From equations (2.11)-(2.13) we therefore have

$$S_m = 2\theta\sum_{i=1}^{m}\frac{n-i+1}{n-i+\alpha}u_i + 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m}(1+\theta u_i + \frac{n-i+1}{n-i+\alpha}\theta u_j)$$

$$-2m\sum_{i=1}^{m}(1+\theta u_i) + (m+1)^2$$

$$= 2\theta\sum_{i=1}^{m}\frac{n-i+1}{n-i+\alpha}u_i + m(m-1) + 2\theta\sum_{i=1}^{m-1}(m-i)u_i$$

$$+ 2\theta \sum_{i=1}^{m-1} \frac{n-i+1}{n-i+\alpha} \sum_{j=i+1}^{m} u_j - 2m^2 - 2m\theta \sum_{i=1}^{m} u_i + (m+1)^2 .$$

After a rearrangement of terms, this can be written as

$$S_m = (m+1) + \frac{2\theta(n-m+1)u_m}{n-m+\alpha} - 2\theta \sum_{i=1}^{m-1} i\, u_i - 2m\theta u_m$$

$$+ 2\theta \sum_{i=1}^{m-1} \frac{n-i+1}{n-i+\alpha} \sum_{j=i}^{m} u_j . \qquad \text{(A.2)}$$

Now applying Lemma 1 to $\sum_{j=i}^{m} u_j$, it is easy to show that

$$\sum_{j=i}^{m} u_j = (n-i+\alpha)u_i/\alpha - (n-m)u_m/\alpha .$$

Substituting this in equation (A.2) we have

$$S_m = (m+1) + \frac{2\theta(n-m+1)u_m}{n-m+\alpha} - 2\theta \sum_{i=1}^{m-1} i\, u_i - 2m\theta u_m$$

$$+ \frac{2\theta}{\alpha} \sum_{i=1}^{m-1} (n-i+1)u_i - \frac{2\theta}{\alpha} (n-m)u_m \sum_{i=1}^{m-1} (1 + \frac{1-\alpha}{n-i+\alpha}) .$$

Applying Lemma 1 to $\sum_{i=1}^{m-1} i\, u_i$ and $\sum_{i=1}^{m-1} (n-i+1)u_i$ and simplifying, we get

$$S_m = (m+1) + 2\theta^2 - 2\theta^2 (n-m)u_m [\frac{1}{\alpha} + \sum_{i=1}^{m} \frac{1}{n-i+\alpha}] .$$

Recalling the definition of S_m, the result follows.

Table 1. m* and the efficiency e_{m*} of T_{m*}

α	0.05		0.10		0.15		0.20		0.25	
n	m*	e_{m*}	m*	e_{m*}	m*	e_{m*}	m*	e_{m*}	m*	e_{m*}
2	1	88.59	1	21.96	1	9.66	1	5.38	1	3.43
3	1	74.71	2	17.58	2	7.88	2	4.49	2	2.93
4	2	67.10	2	15.97	2	6.83	3	3.95	3	2.63
5	3	60.42	3	14.51	3	6.29	4	3.57	4	2.41
6	4	54.84	4	13.27	4	5.81	4	3.29	5	2.25
7	5	50.19	5	12.22	5	5.40	5	3.10	6	2.12
8	6	46.28	6	11.33	6	5.05	6	2.93	7	2.01
9	7	42.96	7	10.57	7	4.75	7	2.79	8	1.93
10	8	40.10	8	9.91	8	4.49	8	2.66	9	1.85
15	12	30.73	13	7.65	13	3.59	13	2.22	13	1.62
20	17	25.06	17	6.36	18	3.06	18	1.96	18	1.49
30	27	18.45	27	4.87	27	2.46	28	1.68	28	1.34
40	36	14.77	37	4.03	37	2.14	38	1.52	38	1.25
50	46	12.39	47	3.49	47	1.93	48	1.42	48	1.20

α	0.30		0.35		0.40		0.45		0.50	
n	m*	e_{m*}	m*	e_{m*}	m*	e_{m*}	m*	e_{m*}	m*	e_{m*}
2	1	2.39	1	1.79	1	1.41	1	1.16	1	1.00
3	2	2.11	2	1.62	2	1.32	2	1.13	2	1.00
4	3	1.93	3	1.52	3	1.27	3	1.11	3	1.00
5	4	1.80	4	1.45	4	1.23	4	1.09	4	1.00
6	5	1.70	5	1.39	5	1.20	5	1.08	6	1.00
7	6	1.63	6	1.35	6	1.17	6	1.07	7	1.00
8	7	1.56	7	1.31	7	1.16	7	1.06	8	1.00
9	8	1.51	8	1.28	8	1.14	8	1.05	9	1.00
10	9	1.47	9	1.26	9	1.13	9	1.05	10	1.00
15	14	1.33	14	1.17	14	1.08	14	1.03	15	1.00
20	19	1.25	19	1.13	19	1.06	19	1.02	20	1.00
30	29	1.17	29	1.08	29	1.04	29	1.01	30	1.00
40	39	1.12	39	1.06	39	1.03	39	1.01	40	1.00
50	49	1.10	49	1.05	49	1.02	49	1.00	50	1.00

Table 2. MSE and efficiency of T_3 for n=4

α	MSE	Efficiency
0.1	0.4427	15.09
0.2	0.3749	3.95
0.3	0.3296	1.93
0.4	0.2993	1.27
0.5	0.2792	1.00
0.6	0.2661	0.89
0.7	0.2579	0.83
0.8	0.2531	0.81
0.9	0.2507	0.80
1.0	0.2500	0.80

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