

ABSTRACT

Williams, Ellison Anne. A Formula for N-Row Macdonald Polynomials. (Dissertation
Directors: Naihuan Jing and Ernie Stitzinger)

We derive a formula for the n-row Macdonald polynomials with the coefficients presented both combinatorically and in terms of very-well-poised hypergeometric series.

A Formula for N-Row Macdonald Polynomials

by

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Submitted in Partial Fulfillment of the Requirements

for the Degree of Doctor of Philosophy in the

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2004

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ACKNOWLEDGEMENTS

I would like to thank Professor Ernest L. Stitzinger and Professor Naihuan Jing for their professional support and encouragement during the course of this work and for their helpful comments and suggestions regarding this manuscript. I am grateful to Prof. Jing for introducing me to Macdonald polynomials and for his suggestion of this problem.

I would also like to thank my husband, Michael Williams, for his comments and suggestions concerning this text.

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1. INTRODUCTION.

Denote the ring of symmetric functions over the field F as Λ_F and let Λ_F^n denote its n^{th} graded space. The space Λ_F^n consists of all symmetric functions of total degree $n \in \mathbb{Z}$, indexed by the partitions $\lambda = (\lambda_1, \dots, \lambda_k)$ for which $\sum_i \lambda_i = n$. There are four Z -bases and one Q -basis of Λ^n . The Q -basis consists of the power sum symmetric functions $p_n = \sum_i x_i^n$, where $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}$, and the four Z -bases are: The monomial symmetric functions $m_\lambda = \sum_{i_1 < \dots < i_k} x_{i_1}^{\lambda_1} \cdots x_{i_k}^{\lambda_k}$, the elementary symmetric functions $e_n = \sum_{i_1 < \dots < i_n} x_{i_1} \cdots x_{i_n}$, where $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_k}$, the complete symmetric functions $h_\lambda = \sum_{i_1 \leq \dots \leq i_k} x_{i_1}^{\lambda_1} \cdots x_{i_k}^{\lambda_k}$, and the Schur functions $s_\lambda(x_1, \dots, x_k) = \det(x_i^{\lambda_j + k - j})_{1 \leq i, j \leq k} / \det(x_i^{k - j})$.

Let $H = Q(q, t)$ be the field of rational functions in q and t . In 1988, Macdonald introduced a new class of two-parameter symmetric functions $P_\lambda(q, t)$, over the ring Λ_H , which generalize several classes of symmetric functions. In particular, taking $q = t$ we obtain the Schur functions, setting $t = 1$ we have the monomial symmetric functions, and letting $q = 0$ gives the Hall-Littlewood functions.

We know from [4] that the (P_λ) are a basis of Λ_H^n . Further, with respect to the scalar product:

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda, \mu} \prod_i i^{m_i} m_i! \prod_{j=1}^{l(\lambda)} \frac{1 - q^{\lambda_j}}{1 - t^{\lambda_j}}$$

we have that

$$\langle P_\lambda, P_\mu \rangle = 0 \quad \text{if} \quad \lambda \neq \mu,$$

where m_i denotes the multiplicity of i in λ and $l(\lambda)$ denote the length of λ . We also know that for each λ , there exists a unique $P_\lambda(q, t)$ such that:

$$P_\lambda = m_\lambda + \sum_{\mu < \lambda} c_{\lambda\mu} m_\mu \quad \text{where} \quad c_{\lambda\mu} \in Q(q, t).$$

Define:

$$Q_\lambda = \frac{P_\lambda}{\langle P_\lambda, P_\lambda \rangle}.$$

Then, the bases (P_λ) and (Q_λ) of Λ_H^n are dual to each other, $\langle Q_\lambda, P_\mu \rangle = \delta_{\lambda,\mu}$, and from [4], for $\lambda = (n)$:

$$Q_{(n)} = \sum_{|\lambda|=n} \prod_i \frac{1}{i^{m_i} m_i!} \prod_{j=1}^{l(\lambda)} \frac{1 - t^{\lambda_j}}{1 - q^{\lambda_j}} p_\lambda.$$

For partitions of the form $\lambda = (\lambda_1, \dots, \lambda_k)$, we will derive a formula for the Macdonald polynomials Q_λ with coefficients presented first combinatorically in section 3 and then as very-well-poised hypergeometric series in section 5. A formula for $Q_{(\lambda_1, \lambda_2)}$ can be found in [2] and following the completion of this work, the author discovered the paper [3] which gives a formula, without proof, for $Q_{(\lambda_1, \lambda_2, \lambda_3)}$.

2. PRELIMINARIES.

Let λ, μ be partitions such that $\mu \subset \lambda$; the diagram of μ being contained in the diagram of λ . For compactness, we will denote the skew diagram $\lambda - \mu$ as $\lambda \setminus \mu$. Moreover, we will view the diagram of a partition with row one being the largest and row n being the smallest (English notation).

The diagram of $\lambda \setminus \mu$ is said to be a *horizontal r -strip* if the number of blocks contained in $\lambda \setminus \mu$ equals r and, of the remaining r blocks, there is at most one in each column of $\lambda \setminus \mu$.

For each block d found in the diagram of λ , let:

$$(1) \quad b_\lambda(d) = \begin{cases} \frac{(1-q^{e(d)}t^{s(d)+1})}{(1-q^{e(d)+1}t^{s(d)})} & \text{if } d \in \lambda \\ 1 & \text{if } d \notin \lambda, \end{cases}$$

where $e(d)$ denotes the number of blocks to the east of d and $s(d)$ denotes the number of blocks to the south of d in the diagram of λ [4].

Let $R_{\lambda \setminus \mu}$ denote the union of the rows and let $C_{\lambda \setminus \mu}$ denote the union of the columns which intersect the diagram of $\lambda \setminus \mu$.

Using a Pieri-type formula [4], for λ, μ such that $\mu \subset \lambda$ is a horizontal r -strip, we have:

$$(2) \quad Q_\mu Q_r = \sum_\lambda \prod_d \frac{b_\mu(d)}{b_\lambda(d)} Q_\lambda$$

for $d \in R_{\lambda \setminus \mu} - C_{\lambda \setminus \mu}$.

Let $r = (r_1, \dots, r_n)$ and $r' = (r_1, \dots, r_{n-1})$ be partitions with the length of row i equal to r_i . In order to derive a formula for $Q_{(r_1, \dots, r_n)}$, we begin with $Q_{(r_1, \dots, r_{n-1})}$.

To utilize (2), we must obtain all partitions $\omega = (w_1, \dots, w_n)$ such that $r' \subset \omega$ and $\omega \setminus r'$ is a horizontal r_n -strip. Denote this set as Ω .

3. CONSTRUCTION OF Ω .

Begin with the desired partition $r' = (r_1, \dots, r_{n-1})$. For all $\omega \in \Omega$, $\omega \setminus r'$ is a horizontal r_n -strip and therefore we must add exactly r_n blocks to $r' = (r_1, \dots, r_{n-1})$. To this end, we strategically enlarge the length of the rows, in some cases creating an n^{th} row, by adding the appropriate number of blocks to row r_i , and below row r_{n-1} , for all i .

Let j_i^0 denote the number of blocks added to row r_i . We may add a maximum of r_n blocks to row r_1 and thus:

$$j_1^0 = 0, \dots, r_n.$$

Consequently, the number of blocks added to all other rows of r' is first dependent on j_1^0 . Second, beginning with row r_1 and adding blocks to r' in order of row succession, we see that the number of blocks which can be added to row r_i is dependent upon the number of blocks added to rows r_k for $1 \leq k < i$. Finally, since we may add at most one block anywhere in r' in order to create any new column which appears in ω , the number of blocks added to row r_m is dependent on the difference between its length and the length of the preceding row, $(r_{m-1} - r_m)$.

Assimilating this, for $m \in \{1, \dots, (n-1)\}$:

$$j_m^0 = \begin{cases} 0, \dots, (r_n - \sum_{s=1}^{m-1} j_s^0) & \text{if } (r_n - \sum_{s=1}^{m-1} j_s^0) \leq (r_{m-1} - r_m) \\ 0, \dots, (r_{m-1} - r_m) & \text{if } (r_n - \sum_{s=1}^{m-1} j_s^0) > (r_{m-1} - r_m). \end{cases}$$

Lastly, for row $w_n \in \omega$, where $0 \leq w_n \leq r_n$, set:

$$w_n = r_n - \sum_{m=1}^{(n-1)} j_m^0.$$

Taking all possible combinations of j_m^0 , beginning with $m = 1$ and ending with $m = (n - 1)$, we generate $\Omega = \{\omega = (w_1, \dots, w_n)\}$.

Example 3.1. Let $r' = (5, 4, 3)$ and $r_4 = 2$. Then, $j_1^0 = \{0, 1, 2\}$, $j_2^0 = \{0, 1\}$, and $j_3^0 = \{0, 1\}$. Thus:

$\Omega = \{(5, 4, 3, 2), (5, 4, 4, 1), (5, 5, 3, 1), (6, 4, 3, 1), (5, 5, 4), (6, 4, 4), (6, 5, 3), (7, 4, 3)\}$. \square

4. CONSTRUCTION OF THE COEFFICIENTS.

For the partition $\lambda = (\lambda_1, \dots, \lambda_n)$, define:

$$v_i = \{p \text{ where } p = \max\{1, \dots, n\} \text{ such that } \lambda_p \geq i\}.$$

Proposition 4.1. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition with row i , λ_i . For each row λ_m and each block $d_k \in \lambda_m$, $1 \leq k \leq \lambda_m$, numbered left to right,

$$s(d_k) = (v_k - m).$$

Proof. Let λ' denote the conjugate partition of λ , the partition whose diagram is the transpose of the diagram of λ . Let d_k be the k^{th} block of row λ_m . Then, d_k is the m^{th} block of the k^{th} row of λ' . Since row λ_m of λ is equal to column m of λ' , it follows that the number of blocks in row λ'_m is equal to p where:

$$p = \max\{1, \dots, n\} \quad \text{such that} \quad \lambda_p \geq m.$$

This implies that the number of blocks to the east of d_k in the diagram of λ' is equal to $p - m$. Taking the transpose, it follows that the number of blocks south of d_k in the diagram of λ equals $p - m$. ■

Remark 4.1. As demonstrated by the proof of Proposition 4.1, for $\lambda = (\lambda_1, \dots, \lambda_n)$, $v_i \equiv \lambda'_i$. As we shall see, the desirability of using v_i rather than the conjugate partition is evidenced by the considerable facility that it gives to the implementation of the principal formula. For example, when $\lambda = (6, 4, 2, 2)$ and $i = 3$, we may simply observe from λ that $v_i = 2$. □

Proposition 4.2. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition with row i , λ_i . For each row λ_m , let $\{d_k\}$, $1 \leq k \leq \lambda_m$, denote the set of blocks which compose λ_m , numbered left to right. Then:

$$b_{\lambda_m} = b_{\lambda}(\{d_k\}) = \prod_{i=0}^{\lambda_m-1} \frac{(1 - q^{\lambda_m-(i+1)} t^{v_{(i+1)}-(m-1)})}{(1 - q^{\lambda_m-i} t^{v_{(i+1)}-m})}$$

Proof. For the product limits $\{i = 0, \dots, (\lambda_m - 1)\}$, let block k , d_k , for $1 \leq k \leq \lambda_m$, correspond to $i = (k - 1)$. Then, $\{i = 0, \dots, (\lambda_m - 1)\}$ corresponds directly to $\{d_k\}$, where $1 \leq k \leq \lambda_m$.

It is easily seen that the number of blocks east of block d_k in the diagram of λ is equal to $(\lambda_m - k)$. By Proposition 4.1, we know that the number of blocks south of d_k in the diagram of λ is equal to $(v_k - m)$. Therefore:

$$b_{\lambda}(d_k) = \frac{(1 - q^{\lambda_m-k} t^{v_{(k)}-(m-1)})}{(1 - q^{\lambda_m-k+1} t^{v_{(k)}-m})}.$$

Shifting block b_k to block b_i where $i = (k - 1)$, we have:

$$b_{\lambda}(d_k) = b_{\lambda}(d_i) = \frac{(1 - q^{\lambda_m-(i+1)} t^{v_{(i+1)}-(m-1)})}{(1 - q^{\lambda_m-i} t^{v_{(i+1)}-m})}.$$

Taking the product over $\{i = 0, \dots, (\lambda_m - 1)\}$ to encompass $\{d_k\}$, $1 \leq k \leq \lambda_m$, yields the desired result. ■

Let $\omega \in \Omega$. In order to utilize the Pieri-type formula, we must construct the required coefficient for each $Q_{(\omega)}$:

$$\prod_d \frac{b_{r'}(d)}{b_{\omega}(d)} \quad d \in R_{\omega \setminus r'} - C_{\omega \setminus r'}.$$

For $m \in \{1, \dots, (n - 1)\}$, we compute these coefficients according to row, r_m , and according to the number of blocks added to r_m , j_m^0 , where $\omega_m = (r_m + j_m^0)$.

Proposition 4.3. Let $\omega \in \Omega$ and $r' = (r_1, \dots, r_{n-1})$.

For row $w_m = (r_m + j_m^0)$, $1 \leq m \leq (n-1)$, such that $d \in w_m$ and $d \in R_{\omega \setminus r'} - C_{\omega \setminus r'}$,

$$(3) \quad \prod_d \frac{b_{r_m}(d)}{b_{w_m}(d)} = T_{j_m^0, 0} T_{j_m^0, j_k^0, 0} \quad m < k \leq (n-1)$$

where

$$(4) \quad T_{j_m^0, 0} = \begin{cases} \prod_{i=r_n - \sum_{s=1}^{n-1} j_s^0}^{r_m-1} \frac{(1-q^{r_m-(i+1)}t^{v(i+1)-(m-1)})(1-q^{r_m+j_m^0-i}t^{v(i+1)-m})}{(1-q^{r_m-i}t^{v(i+1)-m})(1-q^{r_m+j_m^0-(i+1)}t^{v(i+1)-(m-1)})} & j_m^0 \neq 0 \\ 1 & j_m^0 = 0 \end{cases}$$

and

$$(5) \quad T_{j_m^0, j_k^0, 0} = \begin{cases} \prod_{i=r_k}^{r_k+j_k^0-1} \frac{(1-q^{r_m-i}t^{k-m-1})(1-q^{r_m+j_m^0-(i+1)}t^{k-m})}{(1-q^{r_m-(i+1)}t^{k-m})(1-q^{r_m+j_m^0-i}t^{k-m-1})} & j_m^0 \neq 0 \text{ and } j_k^0 \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

restricting $v_{(i+1)}$ to r' ; $v_{(i+1)} = \{p \text{ where } p = \max\{1, \dots, (n-1)\}$ such that $r_p \geq (i+1)\}$.

Proof. First, note that within this proof, we use the expressions b'_{r_m} and b'_{w_m} to identify “incomplete stages” in the development of b_{r_m} and b_{w_m} .

For row r_m , we include only the $d \in r_m$ for which $d \in R_{\omega \setminus r'} - C_{\omega \setminus r'}$. Since row $w_n = (r_n - \sum_{m=1}^{n-1} j_m^0)$, it follows that, for each row of ω and hence of r' , the blocks $\{d_1, \dots, d_{(r_n - \sum_{m=1}^{n-1} j_m^0)}\}$ are not included in $R_{\omega \setminus r'} - C_{\omega \setminus r'}$. Therefore, by Proposition 4.2:

$$(6) \quad b'_{r_m} = \prod_{i=r_n - \sum_{m=1}^{n-1} j_m^0}^{r_m-1} \frac{(1 - q^{r_m-(i+1)}t^{v(i+1)-(m-1)})}{(1 - q^{r_m-i}t^{v(i+1)-m})}.$$

Similarly, for row $w_m = (r_m + j_m^0)$:

$$(7) \quad b'_{w_m} = \prod_{i=r_n - \sum_{m=1}^{n-1} j_m^0}^{r_m-1} \frac{(1 - q^{r_m+j_m^0-(i+1)}t^{v(i+1)-(m-1)})}{(1 - q^{r_m+j_m^0-i}t^{v(i+1)-m})}$$

where we restrict $v_{(i+1)}$ to r' in order to exclude any blocks $d \notin r'$ which lie beneath row r_m .

If $j_m^0 = 0$, we do not include blocks from r_m in (3), and thus from (6) and (7):

$$\frac{b'_{r_m}}{b'_{w_m}} = T_{j_m^0, 0} = \begin{cases} \prod_{i=r_n-\sum_{s=1}^{n-1} j_s^0}^{r_m-1} \frac{(1-q^{r_m-(i+1)}t^{v(i+1)-(m-1)})(1-q^{r_m+j_m^0-i}t^{v(i+1)-m})}{(1-q^{r_m-i}t^{v(i+1)-m})(1-q^{r_m+j_m^0-(i+1)}t^{v(i+1)-(m-1)})} & j_m^0 \neq 0 \\ 1 & j_m^0 = 0. \end{cases}$$

However, we do not include any $d \in w_m$ such that $d \in C_{\omega \setminus r'}$; therefore, we must get rid of any terms corresponding to these blocks in the quotient (4). These blocks, which generate the unwanted terms, directly correspond to the j_k^0 , for $(m+1) \leq k \leq (n-1)$, which were added below them on row r_k to create row w_k . In order for these terms to appear in (4), we must have $j_m^0 \neq 0$ and $j_k^0 \neq 0$. Therefore, to correct this in order to include only the $d \in w_m$ such that $d \in R_{\omega \setminus r'} - C_{\omega \setminus r'}$, we multiply (4) by:

$$T_{j_m^0, j_k^0, 0} = \begin{cases} \prod_{i=r_k}^{r_k+j_k^0-1} \frac{(1-q^{r_m-i}t^{k-m-1})(1-q^{r_m+j_m^0-(i+1)}t^{k-m})}{(1-q^{r_m-(i+1)}t^{k-m})(1-q^{r_m+j_m^0-i}t^{k-m-1})} & j_m^0 \neq 0 \text{ and } j_k^0 \neq 0 \\ 1 & \text{otherwise,} \end{cases}$$

yeilding the desired result. ■

5. CONSTRUCTION OF THE PRINCIPAL FORMULA.

Let $r = (r_1, \dots, r_n)$ be a partition. For $l \in \{1, \dots, (r_n - 1)\}$, let

$$j_1^l = 0, \dots, (r_n - \sum_{s=1}^{n-1} \sum_{c=0}^{l-1} j_s^c).$$

And, for $m \in \{2, \dots, (n - 1)\}$, let

$$j_m^l = \begin{cases} 0, \dots, (r_n - \sum_{s=1}^{n-1} \sum_{c=0}^{l-1} j_s^c - \sum_{s=1}^{m-1} j_s^l) \\ \\ \text{if } (r_n - \sum_{s=1}^{n-1} \sum_{c=0}^{l-1} j_s^c - \sum_{s=1}^{m-1} j_s^l) \leq (r_{m-1} - r_m + \sum_{c=0}^{l-1} (j_{m-1}^c - j_m^c)) \\ \\ 0, \dots, (r_{m-1} - r_m + \sum_{c=0}^{l-1} (j_{m-1}^c - j_m^c)) \\ \\ \text{if } (r_n - \sum_{s=1}^{n-1} \sum_{c=0}^{l-1} j_s^c - \sum_{s=1}^{m-1} j_s^l) > (r_{m-1} - r_m + \sum_{c=0}^{l-1} (j_{m-1}^c - j_m^c)). \end{cases}$$

Remark 5.1. The purpose of the j_i^l , $1 \leq i \leq (n - 1)$, is to allow us to systematically reduce row $w_n = (r_n - \sum_{i=1}^{n-1} j_i^0)$ to zero. To this end, we first must compute the j_i^0 in order to generate Ω . Then, restricting the j_i^l such that $\sum_{i=1}^{n-1} j_i^l \neq 0$, $0 < \sum_{i=1}^{n-1} j_i^l \leq (r_n - \sum_{s=1}^{n-1} \sum_{c=0}^{l-1} j_s^c)$, we are able to achieve the desired result. Further, restrict the j_i^0 such that $\sum_{i=1}^{n-1} j_i^0 \neq 0$, excluding the original partition $r = (r_1, \dots, r_n)$ from this reduction. We must now add at least one block to the partition r^l , and to any subsequently created partitions (i.e., $\omega \in \Omega$); thus, the maximum number of times that we may need to repeat this reduction process is r_n , yielding $(r_n - \sum_{i=1}^{n-1} \sum_{l=0}^{r_n-1} j_i^l) = 0$. \square

Example 5.1. Building upon Example 3.1, for the partition $\omega = (6, 4, 3, 1)$, $j_1^0 = 1$, $j_2^0 = 0$, and $j_3^0 = 0$, we desire to reduce row w_4 to zero. Using the restriction given in Remark 5.1, we have $j_1^1 = \{0, 1\}$, $j_2^1 = \{0, 1\}$, $j_3^1 = \{0, 1\}$ where $0 < \sum_{m=1}^3 j_m^1 \leq 1$. Taking all possible combinations of the j_m^1 dictated by the $j_1^0 = 1$, $j_2^0 = 0$, and $j_3^0 = 0$, $1 \leq m \leq 3$, and applying them to ω , yields the set of three-row partitions $\{(7, 4, 3), (6, 5, 3), (6, 4, 4)\}$. \square

Given j_m^l , $1 \leq m \leq (n - 1)$, (4) becomes:

$$(8) \quad T_{j_m^l, 0} = \begin{cases} \prod_{i=(r_n - \sum_{s=1}^{l-1} \sum_{c=0}^l j_s^c)}^{(r_m + \sum_{c=0}^{l-1} j_m^c - 1)} \frac{(1-q^{r_m + \sum_{c=0}^{l-1} j_m^c - (i+1)} t^{v(i+1) - (m-1)}) (1-q^{r_m + \sum_{c=0}^l j_m^c - i} t^{v(i+1) - m})}{(1-q^{r_m + \sum_{c=0}^{l-1} j_m^c - i} t^{v(i+1) - m}) (1-q^{r_m + \sum_{c=0}^l j_m^c - (i+1)} t^{v(i+1) - (m-1)})} & j_m^l \neq 0 \\ 1 & j_m^l = 0 \end{cases}$$

and, (5) becomes:

$$(9) \quad T_{j_m^l, j_k^l, 0} = \begin{cases} \prod_{i=r_k + \sum_{c=0}^{l-1} j_k^c}^{r_k + \sum_{c=0}^l j_k^c - 1} \frac{(1-q^{r_m + \sum_{c=0}^{l-1} j_m^c - i} t^{k-m-1}) (1-q^{r_m + \sum_{c=0}^l j_m^c - (i+1)} t^{k-m})}{(1-q^{r_m + \sum_{c=0}^{l-1} j_m^c - (i+1)} t^{k-m}) (1-q^{r_m + \sum_{c=0}^l j_m^c - i} t^{k-m-1})} & j_m^l \neq 0 \text{ and } j_k^l \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

where for $l = 0$, $\sum_{c=0}^{l-1} j_m^c = 0$ and where $v_{(i+1)}$ is restricted to the partition $(r_1 + \sum_{c=0}^{l-1} j_1^c, \dots, r_{n-1} + \sum_{c=0}^{l-1} j_{n-1}^c)$; $v_{(i+1)} = \{p \text{ where } p = \max\{1, \dots, (n-1)\} : (r_p + \sum_{c=0}^{l-1} j_p^c) \geq (i+1)\}$.

Theorem. For the partition $r = (r_1, \dots, r_n)$, $m \in \{1, \dots, (n-1)\}$, and $l \in \{1, \dots, (r_n - 1)\}$, we have

$$\begin{aligned}
Q_{(r_1, \dots, r_n)} &= Q_{(r_1, \dots, r_{n-1})} Q_{(r_n)} \\
&+ \sum_{j_m^0, j_m^l} \sum_{i=1}^{r_n-1} (-1)^i \prod_{l=0}^{i-1} \left(\left(\prod_{m=1}^{n-1} T_{j_m^l, 0} \right) \left(\prod_{k=m+1}^{n-1} T_{j_m^l, j_k^l, 0} \right) \right) \cdot \\
&\quad Q_{(r_1 + \sum_{l=0}^{i-1} j_1^l, \dots, r_{n-1} + \sum_{l=0}^{i-1} j_{n-1}^l)} Q_{(r_n - \sum_{m=1}^{n-1} \sum_{l=0}^{i-1} j_m^l)} \\
&+ \sum_{\substack{j_m^0, j_m^l \\ \sum_{m=1}^{n-1} j_m^{n-1} \neq 0}} (-1)^{r_n} \prod_{l=0}^{r_n-1} \left(\left(\prod_{m=1}^{n-1} T_{j_m^l, 0} \right) \left(\prod_{k=m+1}^{n-1} T_{j_m^l, j_k^l, 0} \right) \right) \cdot \\
&\quad Q_{(r_1 + \sum_{l=0}^{r_n-1} j_1^l, \dots, r_{n-1} + \sum_{l=0}^{r_n-1} j_{n-1}^l)}
\end{aligned}$$

where $\sum_{j_m^0, j_m^l}$ signifies to sum over all possible combinations of the values of j_1^0 and j_m^0 such that $0 < \sum_{m=2}^{n-1} (j_1^0 + j_m^0) \leq r_n$ and all possible combinations of the values j_m^l dictated by the j_m^0 such that $\sum_{m=1}^{n-1} j_m^l \neq 0$.

Proof. We want to show that:

$$\begin{aligned}
&Q_{(r_1, \dots, r_n)} - Q_{(r_1, \dots, r_{n-1})} Q_{(r_n)} \\
&- \sum_{j_1^0, j_m^0} \sum_{i=1}^{r_n-1} (-1)^i \prod_{l=0}^{i-1} \left(\left(\prod_{m=1}^{n-1} T_{j_m^l, 0} \right) \left(\prod_{k=m+1}^{n-1} T_{j_m^l, j_k^l, 0} \right) \right) \cdot \\
(10) \quad &\quad Q_{(r_1 + \sum_{l=0}^{i-1} j_1^l, \dots, r_{n-1} + \sum_{l=0}^{i-1} j_{n-1}^l)} Q_{(r_n - \sum_{m=1}^{n-1} \sum_{l=0}^{i-1} j_m^l)} \\
&- \sum_{\substack{j_1^0, j_m^0 \\ \sum_{m=1}^{n-1} j_m^{n-1} \neq 0}} (-1)^{r_n} \prod_{l=0}^{r_n-1} \left(\left(\prod_{m=1}^{n-1} T_{j_m^l, 0} \right) \left(\prod_{k=m+1}^{n-1} T_{j_m^l, j_k^l, 0} \right) \right) \cdot \\
&\quad Q_{(r_1 + \sum_{l=0}^{r_n-1} j_1^l, \dots, r_{n-1} + \sum_{l=0}^{r_n-1} j_{n-1}^l)} = 0.
\end{aligned}$$

Using (2), for $\sum_{m=1}^{n-1} j_m^0 \neq 0$, we have:

$$\begin{aligned}
(11) \quad &Q_{(r_1, \dots, r_{n-1})} Q_{(r_n)} = Q_{(r_1, \dots, r_n)} \\
&+ \sum_{j_1^0, j_m^0} \left(\prod_{m=1}^{n-1} T_{j_m^0, 0} \right) \left(\prod_{k=m+1}^{n-1} T_{j_k^0, j_k^0, 0} \right) Q_{(r_1 + j_1^0, \dots, r_{n-1} + j_{n-1}^0, r_n - \sum_{m=1}^{n-1} j_m^0)}.
\end{aligned}$$

Substituting (11) into (10), we need only to show:

$$\begin{aligned}
& \sum_{j_1^0, j_m^0} \left(\prod_{m=1}^{n-1} T_{j_m^0, 0} \right) \left(\prod_{k=m+1}^{n-1} T_{j_k^0, j_k^0, 0} \right) \cdot \\
& \quad \mathcal{Q}_{(r_1+j_1^0, \dots, r_{n-1}+j_{n-1}^0, r_n-\sum_{m=1}^{n-1} j_m^0)} \\
& + \sum_{j_1^0, j_m^0} \sum_{i=1}^{r_n-1} (-1)^i \prod_{l=0}^{i-1} \left(\left(\prod_{m=1}^{n-1} T_{j_m^l, 0} \right) \left(\prod_{k=m+1}^{n-1} T_{j_m^l, j_k^l, 0} \right) \right) \cdot \\
(12) \quad & \quad \mathcal{Q}_{(r_1+\sum_{l=0}^{i-1} j_1^l, \dots, r_{n-1}+\sum_{l=0}^{i-1} j_{n-1}^l)} \mathcal{Q}_{(r_n-\sum_{m=1}^{n-1} \sum_{l=0}^{i-1} j_m^l)} \\
& = \sum_{\substack{j_1^0, j_m^0 \\ \sum_{m=1}^{n-1} j_m^{n-1} \neq 0}} (-1)^{r_n-1} \prod_{l=0}^{r_n-1} \left(\left(\prod_{m=1}^{n-1} T_{j_m^l, 0} \right) \left(\prod_{k=m+1}^{n-1} T_{j_m^l, j_k^l, 0} \right) \right) \cdot \\
& \quad \mathcal{Q}_{(r_1+\sum_{l=0}^{r_n-1} j_1^l, \dots, r_{n-1}+\sum_{l=0}^{r_n-1} j_{n-1}^l)} \cdot
\end{aligned}$$

Note that each Pieri-formula expansion (via (2) and (11)) of

$$(13) \quad \sum_{j_1^0, j_m^0} \left(\prod_{m=1}^{n-1} T_{j_m^0, 0} \right) \left(\prod_{k=m+1}^{n-1} T_{j_k^0, j_k^0, 0} \right) \mathcal{Q}_{(r_1+j_1^0, \dots, r_{n-1}+j_{n-1}^0, r_n-\sum_{m=1}^{n-1} j_m^0)}$$

yields two terms. To show our desired result, we will show that with each successive “numbered” Pieri-formula expansion of (13), the term which contains the product of two Macdonald polynomials cancels with the correspondingly numbered term i in

$$\begin{aligned}
(14) \quad & \sum_{j_1^0, j_m^0} \sum_{i=1}^{r_n-1} (-1)^i \prod_{l=0}^{i-1} \left(\left(\prod_{m=1}^{n-1} T_{j_m^l, 0} \right) \left(\prod_{k=m+1}^{n-1} T_{j_m^l, j_k^l, 0} \right) \right) \cdot \\
& \quad \mathcal{Q}_{(r_1+\sum_{l=0}^{i-1} j_1^l, \dots, r_{n-1}+\sum_{l=0}^{i-1} j_{n-1}^l)} \mathcal{Q}_{(r_n-\sum_{m=1}^{n-1} \sum_{l=0}^{i-1} j_m^l)}
\end{aligned}$$

and that, in the final Pieri expansion of (13), we will be left only with

$$\begin{aligned}
(15) \quad & \sum_{\substack{j_1^0, j_m^0 \\ \sum_{m=1}^{n-1} j_m^{n-1} \neq 0}} (-1)^{r_n-1} \prod_{l=0}^{r_n-1} \left(\left(\prod_{m=1}^{n-1} T_{j_m^l, 0} \right) \left(\prod_{k=m+1}^{n-1} T_{j_m^l, j_k^l, 0} \right) \right) \cdot \\
& \quad \mathcal{Q}_{(r_1+\sum_{l=0}^{r_n-1} j_1^l, \dots, r_{n-1}+\sum_{l=0}^{r_n-1} j_{n-1}^l)} \cdot
\end{aligned}$$

To this end, we will induct on i .

Consider $i = 1$. Completing the first Pieri expansion of (13), and computing $i = 1$ from (14), (12) becomes:

$$\begin{aligned}
& \sum_{j_1^0, j_m^0} \left(\prod_{m=1}^{n-1} T_{j_m^0, 0} \right) \left(\prod_{k=m+1}^{n-1} T_{j_k^0, j_k^0, 0} \right) \mathcal{Q}_{(r_1+j_1^0, \dots, r_{n-1}+j_{n-1}^0)} \mathcal{Q}_{(r_n-\sum_{m=1}^{n-1} j_m^0)} \\
& - \sum_{j_1^0, j_m^0} \prod_{l=0}^1 \left(\left(\prod_{m=1}^{n-1} T_{j_m^l, 0} \right) \left(\prod_{k=m+1}^{n-1} T_{j_m^l, j_k^l, 0} \right) \right) \cdot \\
& \qquad \qquad \qquad \mathcal{Q}_{(r_1+\sum_{l=0}^1 j_1^l, \dots, r_{n-1}+\sum_{l=0}^1 j_{n-1}^l, r_n-\sum_{m=1}^{n-1} \sum_{l=0}^1 j_m^l)} \\
& - \sum_{j_1^0, j_m^0} \left(\left(\prod_{m=1}^{n-1} T_{j_m^0, 0} \right) \left(\prod_{k=m+1}^{n-1} T_{j_m^0, j_k^0, 0} \right) \right) \mathcal{Q}_{(r_1+j_1^0, \dots, r_{n-1}+j_{n-1}^0)} \mathcal{Q}_{(r_n-\sum_{m=1}^{n-1} j_m^0)} \\
& + \sum_{j_1^0, j_m^0} \sum_{i=2}^{r_n-1} (-1)^i \prod_{l=0}^{i-1} \left(\left(\prod_{m=1}^{n-1} T_{j_m^l, 0} \right) \left(\prod_{k=m+1}^{n-1} T_{j_m^l, j_k^l, 0} \right) \right) \cdot \\
& \qquad \qquad \qquad \mathcal{Q}_{(r_1+\sum_{l=0}^{i-1} j_1^l, \dots, r_{n-1}+\sum_{l=0}^{i-1} j_{n-1}^l)} \mathcal{Q}_{(r_n-\sum_{m=1}^{n-1} \sum_{l=0}^{i-1} j_m^l)} \\
& = - \sum_{j_1^0, j_m^0} \prod_{l=0}^1 \left(\left(\prod_{m=1}^{n-1} T_{j_m^l, 0} \right) \left(\prod_{k=m+1}^{n-1} T_{j_m^l, j_k^l, 0} \right) \right) \cdot \\
& \qquad \qquad \qquad \mathcal{Q}_{(r_1+\sum_{l=0}^1 j_1^l, \dots, r_{n-1}+\sum_{l=0}^1 j_{n-1}^l, r_n-\sum_{m=1}^{n-1} \sum_{l=0}^1 j_m^l)} \\
& + \sum_{j_1^0, j_m^0} \sum_{i=2}^{r_n-1} (-1)^i \prod_{l=0}^{i-1} \left(\left(\prod_{m=1}^{n-1} T_{j_m^l, 0} \right) \left(\prod_{k=m+1}^{n-1} T_{j_m^l, j_k^l, 0} \right) \right) \cdot \\
& \qquad \qquad \qquad \mathcal{Q}_{(r_1+\sum_{l=0}^{i-1} j_1^l, \dots, r_{n-1}+\sum_{l=0}^{i-1} j_{n-1}^l)} \mathcal{Q}_{(r_n-\sum_{m=1}^{n-1} \sum_{l=0}^{i-1} j_m^l)}.
\end{aligned}$$

Therefore, in the first Pieri expansion of (13), the term containing the product of two Macdonald polynomials cancelled with the correspondingly numbered $i = 1$ in (14), as desired.

Assume that with each further “numbered” Pieri expansion of (13), up to $(r_n - 2)$, the term containing a product of two Macdonald polynomials cancels with the correspondingly numbered i in (14). We will show the result for $(r_n - 1)$.

For $i = (r_n - 2)$, we have that (12) equals:

$$\begin{aligned}
& \sum_{j_1^0, j_m^0} (-1)^{r_n-3} \prod_{l=0}^{r_n-3} \left(\left(\prod_{m=1}^{n-1} T_{j_m^l, 0} \right) \left(\prod_{k=m+1}^{n-1} T_{j_m^l, j_k^l, 0} \right) \right) \cdot \\
& \quad \mathcal{Q}_{(r_1 + \sum_{l=0}^{r_n-3} j_1^l, \dots, r_{n-1} + \sum_{l=0}^{r_n-3} j_{n-1}^l)} \mathcal{Q}_{(r_n - \sum_{m=1}^{n-1} \sum_{l=0}^{r_n-3} j_m^l)} \\
& + \sum_{j_1^0, j_m^0} (-1)^{r_n-2} \prod_{l=0}^{r_n-2} \left(\left(\prod_{m=1}^{n-1} T_{j_m^l, 0} \right) \left(\prod_{k=m+1}^{n-1} T_{j_m^l, j_k^l, 0} \right) \right) \cdot \\
& \quad \mathcal{Q}_{(r_1 + \sum_{l=0}^{r_n-2} j_1^l, \dots, r_{n-1} + \sum_{l=0}^{r_n-2} j_{n-1}^l, r_n - \sum_{m=1}^{n-1} \sum_{l=0}^{r_n-2} j_m^l)} \\
& + \sum_{j_1^0, j_m^0} (-1)^{r_n-2} \prod_{l=0}^{r_n-3} \left(\left(\prod_{m=1}^{n-1} T_{j_m^l, 0} \right) \left(\prod_{k=m+1}^{n-1} T_{j_m^l, j_k^l, 0} \right) \right) \cdot \\
& \quad \mathcal{Q}_{(r_1 + \sum_{l=0}^{r_n-3} j_1^l, \dots, r_{n-1} + \sum_{l=0}^{r_n-3} j_{n-1}^l)} \mathcal{Q}_{(r_n - \sum_{m=1}^{n-1} \sum_{l=0}^{r_n-3} j_m^l)} \\
& + \sum_{j_1^0, j_m^0} (-1)^{r_n-1} \prod_{l=0}^{r_n-2} \left(\left(\prod_{m=1}^{n-1} T_{j_m^l, 0} \right) \left(\prod_{k=m+1}^{n-1} T_{j_m^l, j_k^l, 0} \right) \right) \cdot \\
& \quad \mathcal{Q}_{(r_1 + \sum_{l=0}^{r_n-2} j_1^l, \dots, r_{n-1} + \sum_{l=0}^{r_n-2} j_{n-1}^l)} \mathcal{Q}_{(r_n - \sum_{m=1}^{n-1} \sum_{l=0}^{r_n-2} j_m^l)} \\
& = \sum_{j_1^0, j_m^0} (-1)^{r_n-2} \prod_{l=0}^{r_n-2} \left(\left(\prod_{m=1}^{n-1} T_{j_m^l, 0} \right) \left(\prod_{k=m+1}^{n-1} T_{j_m^l, j_k^l, 0} \right) \right) \cdot \\
& \quad \mathcal{Q}_{(r_1 + \sum_{l=0}^{r_n-2} j_1^l, \dots, r_{n-1} + \sum_{l=0}^{r_n-2} j_{n-1}^l, r_n - \sum_{m=1}^{n-1} \sum_{l=0}^{r_n-2} j_m^l)} \\
& + \sum_{j_1^0, j_m^0} (-1)^{r_n-1} \prod_{l=0}^{r_n-2} \left(\left(\prod_{m=1}^{n-1} T_{j_m^l, 0} \right) \left(\prod_{k=m+1}^{n-1} T_{j_m^l, j_k^l, 0} \right) \right) \cdot \\
& \quad \mathcal{Q}_{(r_1 + \sum_{l=0}^{r_n-2} j_1^l, \dots, r_{n-1} + \sum_{l=0}^{r_n-2} j_{n-1}^l)} \mathcal{Q}_{(r_n - \sum_{m=1}^{n-1} \sum_{l=0}^{r_n-2} j_m^l)} \cdot
\end{aligned}$$

Performing the Pieri expansion on (13) a final time, (12) is equal to:

$$\begin{aligned}
& \sum_{j_1^0, j_m^0} (-1)^{r_n-2} \prod_{l=0}^{r_n-2} \left(\left(\prod_{m=1}^{n-1} T_{j_m^l, 0} \right) \left(\prod_{k=m+1}^{n-1} T_{j_m^l, j_k^l, 0} \right) \right) \cdot \\
& \quad Q_{(r_1 + \sum_{l=0}^{r_n-2} j_1^l, \dots, r_{n-1} + \sum_{l=0}^{r_n-2} j_{n-1}^l)} Q_{(r_n - \sum_{m=1}^{n-1} \sum_{l=0}^{r_n-2} j_m^l)} \\
& + \sum_{j_1^0, j_m^0} (-1)^{r_n-1} \prod_{l=0}^{r_n-1} \left(\left(\prod_{m=1}^{n-1} T_{j_m^l, 0} \right) \left(\prod_{k=m+1}^{n-1} T_{j_m^l, j_k^l, 0} \right) \right) \cdot \\
& \quad Q_{(r_1 + \sum_{l=0}^{r_n-1} j_1^l, \dots, r_{n-1} + \sum_{l=0}^{r_n-1} j_{n-1}^l)} \\
& + \sum_{j_1^0, j_m^0} (-1)^{r_n-1} \prod_{l=0}^{r_n-2} \left(\left(\prod_{m=1}^{n-1} T_{j_m^l, 0} \right) \left(\prod_{k=m+1}^{n-1} T_{j_m^l, j_k^l, 0} \right) \right) \cdot \\
& \quad Q_{(r_1 + \sum_{l=0}^{r_n-2} j_1^l, \dots, r_{n-1} + \sum_{l=0}^{r_n-2} j_{n-1}^l)} Q_{(r_n - \sum_{m=1}^{n-1} \sum_{l=0}^{r_n-2} j_m^l)} \\
& = \sum_{\substack{j_1^0, j_m^0 \\ \sum_{m=1}^{n-1} j_m^{r_n-1} \neq 0}} (-1)^{r_n-1} \prod_{l=0}^{r_n-1} \left(\left(\prod_{m=1}^{n-1} T_{j_m^l, 0} \right) \left(\prod_{k=m+1}^{n-1} T_{j_m^l, j_k^l, 0} \right) \right) \cdot \\
& \quad Q_{(r_1 + \sum_{l=0}^{r_n-1} j_1^l, \dots, r_{n-1} + \sum_{l=0}^{r_n-1} j_{n-1}^l)}.
\end{aligned}$$

Since, by definition of j_m^l , we must have

$$r_n - \sum_{m=1}^{n-1} \sum_{l=0}^{r_n-1} j_m^l = 0. \quad \blacksquare$$

Example 5.2. We shall build upon Example 3.1 to calculate the expansion of the Macdonald polynomial $Q_{(5,4,3,2)}$.

Given the nature of the partition $r = (5, 4, 3, 2)$, it is fitting and instructive to use the following notation:

$$\begin{aligned}
\prod_{m=1}^{n-1} T_{j_m^l, 0} &\equiv T_{(j_1^l, \dots, j_{(n-1)}^l)}^l \\
\prod_{k=m+1}^{n-1} T_{j_m^l, j_k^l, 0} &\equiv T_{(0, \dots, j_{m+1}^l, \dots, j_{(n-1)}^l)}^{ll} \quad j_m^l, j_{m+i}^l \neq 0, \quad 1 \leq i \leq (n-m-1)
\end{aligned}$$

For $2 \leq m \leq 3$, we have:

$$j_1^0 = \{0, 1, 2\} \quad j_1^l = \{0, \dots, (2 - \sum_{s=1}^3 \sum_{c=0}^{l-1} j_s^c)\},$$

$$j_m^0 = \begin{cases} 0, \dots, (2 - \sum_{s=1}^{m-1} j_s^0) & \text{if } (2 - \sum_{s=1}^{m-1} j_s^0) \leq (r_{m-1} - r_m) \\ 0, \dots, (r_{m-1} - r_m) & \text{if } (2 - \sum_{s=1}^{m-1} j_s^0) > (r_{m-1} - r_m), \end{cases}$$

$$j_m^l = \begin{cases} 0, \dots, (2 - \sum_{s=1}^3 \sum_{c=0}^{l-1} j_s^c - \sum_{s=1}^{m-1} j_s^l) \\ \text{if } (2 - \sum_{s=1}^3 \sum_{c=0}^{l-1} j_s^c - \sum_{s=1}^{m-1} j_s^l) \leq (r_{m-1} - r_m + \sum_{c=0}^{l-1} (j_{m-1}^c - j_m^c)) \\ 0, \dots, (r_{m-1} - r_m + \sum_{c=0}^{l-1} (j_{m-1}^c - j_m^c)) \\ \text{if } (2 - \sum_{s=1}^3 \sum_{c=0}^{l-1} j_s^c - \sum_{s=1}^{m-1} j_s^l) > (r_{m-1} - r_m + \sum_{c=0}^{l-1} (j_{m-1}^c - j_m^c)). \end{cases}$$

We begin with j_m^0 since these values determine the subsequent choices for j_m^l . We have:

$$\begin{aligned} j_1^0 &= \{0, 1, 2\} & j_1^1 &= \{0, 1\} \\ j_2^0 &= \{0, 1\} & j_2^1 &= \{0, 1\} \\ j_3^0 &= \{0, 1\} & j_3^1 &= \{0, 1\}. \end{aligned}$$

Taking all possible combinations of the $j_m^l, l = \{0, 1\}$, such that $0 < \sum_{m=1}^3 j_m^0 \leq 2$ and $0 < \sum_{m=1}^3 j_m^1 \leq 1$, we have:

$$\begin{aligned} Q_{(5,4,3,2)} &= Q_{(5,4,3)} Q_{(2)} - \\ &\sum_{j_1^0, j_m^0} \left(\prod_{m=1}^3 T_{j_m^0, 0} \right) \left(\prod_{k=m+1}^3 T_{j_m^0, j_k^0, 0} \right) Q_{(5+j_1^0, 4+j_2^0, 3+j_3^0)} Q_{(2-\sum_{m=1}^3 j_m^0)} \\ &+ \sum_{j_1^0, j_m^0} \prod_{l=0}^1 \left(\left(\prod_{m=1}^3 T_{j_m^l, 0} \right) \left(\prod_{k=m+1}^3 T_{j_m^l, j_k^l, 0} \right) \right) Q_{(5+\sum_{l=0}^1 j_1^l, 4+\sum_{l=0}^1 j_2^l, 3+\sum_{l=0}^1 j_3^l)} \\ &= Q_{(5,4,3)} Q_{(2)} + \left(T_{(1,0,0)}^0 T_{(1,0,0)}^1 - T_{(2,0,0)}^0 \right) Q_{(7,4,3)} \\ &+ \left(T_{(1,0,0)}^0 T_{(0,1,0)}^1 + T_{(0,1,0)}^0 T_{(1,0,0)}^1 - T_{(1,1,0)}^0 T_{(0,1,0)}^{00} \right) Q_{(6,5,3)} \end{aligned}$$

$$\begin{aligned}
& + \left(T_{(1,0,0)}^0 T_{(0,0,1)}^1 + T_{(0,0,1)}^0 T_{(1,0,0)}^1 - T_{(1,0,1)}^0 T_{(0,0,1)}^{00} \right) Q_{(6,4,4)} \\
& + \left(T_{(0,1,0)}^0 T_{(0,0,1)}^1 + T_{(0,0,1)}^0 T_{(0,1,0)}^1 - T_{(0,1,1)}^0 T_{(0,0,1)}^{00} \right) Q_{(5,5,4)} \\
& - T_{(1,0,0)}^0 Q_{(6,4,3)} Q_{(1)} - T_{(0,1,0)}^0 Q_{(5,5,3)} Q_{(1)} - T_{(0,0,1)}^0 Q_{(5,4,4)} Q_{(1)}. \quad \square
\end{aligned}$$

6. VERY-WELL-POISED HYPERGEOMETRIC SERIES.

We may express the coefficients $T_{j_m}^l$, for $m \in \{1, \dots, (n-1)\}$, and $l \in \{0, \dots, r_{n-1}\}$, in terms of very-well-poised hypergeometric series.

We will use the following notations.

$$\begin{aligned} (a; q)_0 &= 1 \\ (a; q)_n &= \prod_{i=0}^{n-1} (1 - aq^i) \\ (a; q)_\infty &= \prod_{i=0}^{\infty} (1 - aq^i) \\ (a; q)_n &= \frac{(a; q)_\infty}{(aq^n; q)_\infty} \\ (a_1, \dots, a_m; q)_n &= (a_1; q)_n \cdots (a_m; q)_n \\ (a_1, \dots, a_m; q)_\infty &= (a_1; q)_\infty \cdots (a_m; q)_\infty \end{aligned}$$

The *basic hypergeometric series* $\phi_{r+1,r}$ is of the form:

$$\left[\begin{array}{cccc} a_1, & a_2 & , \dots, & a_{r+1} \\ b_1, & b_2 & , \dots, & b_r \quad q, z \end{array} \right]$$

where

$$\phi_{r+1,r} = \sum_{n=0}^{\infty} \frac{(a_1, \dots, a_{r+1}; q)_n}{(b_1, \dots, b_r; q)_n (q; q)_n} z^n.$$

We say that $\phi_{r+1,r}$ is *very-well-poised*, denoted

$$W_{r+1,r} = (a_1; a_4, \dots, a_{r+1}; q, z),$$

if

$$a_1q = a_2b_1 = \dots = a_{r+1}b_r \quad \text{and} \quad a_2 = qa_1^{\frac{1}{2}}, a_3 = -qa_1^{\frac{1}{2}}.$$

From [1], we have:

$$(16) \quad W_{6,5}(a; b, c, d; q, \frac{aq}{bcd}) = \frac{(aq, \frac{aq}{bc}, \frac{aq}{bd}, \frac{aq}{cd}; q)_\infty}{(\frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, bcd; q)_\infty}.$$

The Coefficients $T_{j_m^0,0}$ and $T_{j_m^0, j_k^0, 0}$.

We first compute the coefficient $T_{j_m^0}$ in terms of very-well-poised hypergeometric series.

Beginning with row r_m , we identify the ‘‘indentions’’ which lie ‘‘under’’ it in the diagram of r' . In other words, we identify all rows r_i , $m < i \leq (n-1)$, such that $r_i < r_{i+1}$.

For $k_1 \in \{1, \dots, (n-m)\}$, let r_{n-k_1} be the largest indexed row of r' such that $r_{n-k_1} \leq r_m$.

Proposition 6.1. Suppose $r_{n-k_1} = r_m$. It follows that $k_1 = (n-m)$ and $r_m \equiv r_{n-1}$. Then:

$$T_{j_m^0,0} = \begin{cases} W_{6,5}(q^{r_m-r_n+\sum_{s=1}^{n-1} j_s^0 t^{n-m-1}}; qt^{-1}, q^{-j_m^0}, q^{r_{n-1}-r_n+\sum_{s=1}^{n-1} j_s^0}; q, q^{j_m^0} t^{n-m}) & j_m^0 \neq 0 \\ 1 & j_m^0 = 0. \end{cases}$$

Proof. Let $r_{n-k_1} = r_m$; $k_1 = (n-m)$ and $r_m \equiv r_{n-1}$. There are two cases: $j_m = 0$ and $j_m^0 \neq 0$.

For $j_m^0 = 0$, we have that $T_{j_m^0,0} = 1$ by Proposition 4.3.

For $j_m^0 \neq 0$, using properties of hypergeometric series and (16), (4) becomes:

$$\begin{aligned}
T_{j_m^0, 0} &= \prod_{i=r_n-\sum_{s=1}^{n-1} j_s^0}^{r_{n-1}-1} \frac{(1-q^{r_m-(i+1)}t^{n-m})(1-q^{r_m+j_m^0-i}t^{n-m-1})}{(1-q^{r_m-i}t^{n-m-1})(1-q^{r_m+j_m^0-(i+1)}t^{n-m})} \\
&= \frac{(q^{r_m-r_{n-1}}t^{n-m}, q^{r_m-r_{n-1}+j_m^0+1}t^{n-m-1}; q)_{r_{n-1}-r_n+\sum_{s=1}^{n-1} j_s^0}}{(q^{r_m-r_{n-1}+1}t^{n-m-1}, q^{r_m-r_{n-1}+j_m^0}t^{n-m}; q)_{r_{n-1}-r_n+\sum_{s=1}^{n-1} j_s^0}} \\
&= \frac{P}{Q} \\
&= W_{6,5}(q^{r_m-r_n+\sum_{s=1}^{n-1} j_s^0}t^{n-m-1}; qt^{-1}, q^{-j_m^0}, q^{r_{n-1}-r_n+\sum_{s=1}^{n-1} j_s^0}; q, q^{r_m-r_{n-1}+j_m^0}t^{n-m}) \\
&= W_{6,5}(q^{r_m-r_n+\sum_{s=1}^{n-1} j_s^0}t^{n-m-1}; qt^{-1}, q^{-j_m^0}, q^{r_{n-1}-r_n+\sum_{s=1}^{n-1} j_s^0}; q, q^{j_m^0}t^{n-m}).
\end{aligned}$$

Where:

$$\begin{aligned}
P &= (q^{r_m-r_{n-1}}t^{n-m}, q^{r_m+j_m^0-r_{n-1}+1}t^{n-m-1}, q^{r_m-r_n+\sum_{s=1}^{n-1} j_s^0+1}t^{n-m-1}, \\
&\quad q^{r_m+j_m^0-r_n+\sum_{s=1}^{n-1} j_s^0}t^{n-m}; q)_\infty \\
Q &= (q^{r_m-r_n+\sum_{s=1}^{n-1} j_s^0}t^{n-m}, q^{r_m+j_m^0-r_n+\sum_{s=1}^{n-1} j_s^0+1}t^{n-m-1}, \\
&\quad q^{r_m-r_{n-1}+1}t^{n-m-1}, q^{r_m+j_m^0-r_{n-1}}t^{n-m}; q)_\infty \blacksquare
\end{aligned}$$

Remark 6.1. Let $r_m = r_{n-k_1}$. It follows that for all $m < k \leq (n-1)$, we have $j_k^0 = 0$, and thus $T_{j_m^0, j_k^0, 0} = 1$. \square

Consider the case when $r_{n-k_1} \neq r_m$. Let r_{n-k_2} be the largest indexed row of $r' = (r_1, \dots, r_{n-1})$ such that $r_{n-k_1} \neq r_{n-k_2}$. If $r_{n-k_2} \neq r_m$, let r_{n-k_3} be the largest indexed row of r' for which $r_{n-k_2} \neq r_{n-k_3}$, ect. Continue this process until one reaches row r_{n-k_f} for which $r_{n-k_f} = r_m$, creating the chain:

$$r_{n-k_1} < r_{n-k_2} < \dots < r_{n-k_f} = r_m$$

for $k_i \in \{1, \dots, (n-m)\}$, $1 \leq i \leq f$.

Proposition 6.2. Suppose $r_{n-k_1} \neq r_m$. Then, (4) becomes:

$$T_{j_m^0, 0} = \begin{cases} W_{6,5}(q^{r_m-r_n+\sum_{s=1}^{n-1} j_s^0} t^{n-m-k_1}; \\ qt^{-1}, q^{-j_m^0}, q^{r_{n-k_1}-r_n+\sum_{s=1}^{n-1} j_s^0}; q, q^{r_m-r_{n-k_1}+j_m^0} t^{n-m-k_1+1}). \\ \prod_{h=2}^{f-1} W_{6,5}(q^{r_m-r_{n-k(h+1)}+j_m^0} t^{n-k(h+1)-m}; \\ qt^{-1}, q^{j_m^0}, q^{r_{n-k_h}-r_{n-k(h+1)}}; q, q^{r_m-r_{n-k_h}} t^{n-m-k(h+1)+1}) & j_m^0 \neq 0 \\ 1 & j_m^0 = 0. \end{cases}$$

Proof. Using our chain, we are able to identify the indentions under row r_m in the diagram of r' ; thus, we obtain all intervals of blocks $d \in r_m$ on which $s(d)$ changes. Using properties of hypergeometric series and (16), we decompose (4) as follows:

$$\begin{aligned} T_{j_m^0, 0} &= \prod_{i=r_n-\sum_{s=1}^{n-1} j_s^0}^{r_{n-k_1}-1} j_s^0 \frac{(1 - q^{r_m-(i+1)} t^{n-m-k_1+1})(1 - q^{r_m+j_m^0-i} t^{n-m-k_1})}{(1 - q^{r_m-i} t^{n-m-k_1})(1 - q^{r_m+j_m^0-(i+1)} t^{n-m-k_1+1})}. \\ &\prod_{h=2}^{f-1} \left(\prod_{i=r_{n-k_h}}^{r_{n-k(h+1)}-1} \frac{(1 - q^{r_m-(i+1)} t^{n-m-k(h+1)+1})(1 - q^{r_m+j_m^0-i} t^{n-m-k(h+1)})}{(1 - q^{r_m-i} t^{n-m-k(h+1)})(1 - q^{r_m+j_m^0-(i+1)} t^{n-m-k(h+1)+1})} \right) \\ &= \frac{(q^{r_m-r_{n-k_1}} t^{n-m-k_1+1}, q^{r_m+j_m^0-r_{n-k_1}+1} t^{n-m-k_1}; q)_{r_{n-k_1}-r_n+\sum_{s=1}^{n-1} j_s^0}}{(q^{r_m+j_m^0-r_{n-k_1}} t^{n-m-k_1+1}, q^{r_m-r_{n-k_1}+1} t^{n-m-k_1}; q)_{r_{n-k_1}-r_n+\sum_{s=1}^{n-1} j_s^0}}. \\ &\prod_{h=2}^{f-1} \frac{(q^{r_m-r_{n-k(h+1)}} t^{n-m-k(h+1)+1}, q^{r_m+j_m^0-r_{n-k(h+1)}+1} t^{n-m-k(h+1)}; q)_{r_{n-k(h+1)}-r_{n-k_h}}}{(q^{r_m-r_{n-k(h+1)}+1} t^{n-m-k(h+1)}, q^{r_m-r_{n-k(h+1)}} t^{n-m-k(h+1)+1}; q)_{r_{n-k(h+1)}-r_{n-k_h}}} \\ &= \frac{S}{C}. \\ &\prod_{h=2}^{f-1} \frac{P}{Q} \\ &= W_{6,5}(q^{r_m-r_n+\sum_{s=1}^{n-1} j_s^0} t^{n-m-k_1}; qt^{-1}, q^{-j_m^0}, q^{r_{n-k_1}-r_n+\sum_{s=1}^{n-1} j_s^0}; q, q^{r_m+j_m^0-r_{n-k_1}} t^{n-m-k_1+1}). \\ &\prod_{h=2}^{f-1} W_{6,5}(q^{r_m-r_{n-k(h+1)}+j_m^0} t^{n-k(h+1)-m}; qt^{-1}, q^{j_m^0}, q^{r_{n-k_h}-r_{n-k(h+1)}}; q, q^{r_m-r_{n-k_h}} t^{n-m-k(h+1)+1}). \end{aligned}$$

Where: $S = (q^{r_m-r_{n-k_1}} t^{n-m-k_1+1}, q^{r_m+j_m^0-r_{n-k_1}+1} t^{n-m-k_1},$

$$\begin{aligned}
& q^{r_m - r_n + j_m^0 + \sum_{s=1}^{n-1} j_s^0} t^{n-m-k_1+1}, q^{r_m - r_n + \sum_{s=1}^{n-1} j_s^0 + 1} t^{n-m-k_1}; q)_\infty \\
C &= (q^{r_m + j_m^0 - r_{n-k_1}} t^{n-m-k_1+1}, q^{r_m + j_m^0 - r_{n-k_1}} t^{n-m-k_1}, \\
& q^{r_m - r_n + \sum_{s=1}^{n-1} j_s^0} t^{n-m-k_1+1}, q^{r_m - r_n + \sum_{s=1}^{n-1} j_s^0 + j_m^0 + 1} t^{n-m-k_1}; q)_\infty \\
P &= (q^{r_m - r_{n-k(h+1)}} t^{n-m-k(h+1)+1}, q^{r_m + j_m^0 - r_{n-k(h+1)} + 1} t^{n-m-k(h+1)}, \\
& q^{r_m - r_{n-k_h} + 1} t^{n-m-k(h+1)}, q^{r_m - r_{n-k_h} + j_m^0} t^{n-m-k(h+1)+1}; q)_\infty \\
Q &= (q^{r_m - r_{n-k(h+1)} + 1} t^{n-m-k(h+1)}, q^{r_m + j_m^0 - r_{n-k(h+1)} + 1} t^{n-m-k(h+1)+1}, \\
& q^{r_m - r_{n-k_h} + 1} t^{n-m-k(h+1)+1}, q^{r_m - r_{n-k_h} + j_m^0 + 1} t^{n-m-k(h+1)+1}; q)_\infty. \blacksquare
\end{aligned}$$

Proposition 6.3. For $r_{n-k_1} \neq r_m$, the coefficient $T_{j_m^0, j_k^0, 0}$ becomes:

$$T_{j_m^0, j_k^0, 0} = W_{6,5}(q^{r_m - r_k - j_k^0} t^{k-m-1}; q^{-j_k^0}, qt^{-1}, q^{-j_m^0}; q, q^{r_m - r_k + j_m^0} t^{k-m}).$$

Proof. Using properties of hypergeometric series and (16), we have:

$$\begin{aligned}
T_{j_m^0, j_k^0, 0} &= \prod_{i=r_k}^{r_k + j_k^0 - 1} \frac{(1 - q^{r_m - i} t^{k-m-1})(1 - q^{r_m + j_m^0 - (i+1)} t^{k-m})}{(1 - q^{r_m - (i+1)} t^{k-m})(1 - q^{r_m + j_m^0 - i} t^{k-m-1})} \\
&= \frac{(q^{r_m - r_k - j_k^0 - 1} t^{k-m-1}, q^{r_m - r_k + j_m^0 - j_k^0} t^{k-m}; q)_{j_k^0}}{(q^{r_m - r_k - j_k^0} t^{k-m}, q^{r_m - r_k + j_m^0 - j_k^0 + 1} t^{k-m-1}; q)_{j_k^0}} \\
&= \frac{(q^{r_m - r_k - j_k^0 + 1} t^{k-m-1}, q^{r_m - r_k + j_m^0 - j_k^0} t^{k-m}, q^{r_m - r_k} t^{k-m}, q^{r_m - r_k + j_m^0 + 1} t^{k-m-1}; q)_\infty}{(q^{r_m - r_k - j_k^0} t^{k-m}, q^{r_m - r_k + j_m^0 - j_k^0 + 1} t^{k-m-1}, q^{r_m - r_k + 1} t^{k-m-1}, q^{r_m - r_k + j_m^0} t^{k-m}; q)_\infty} \\
&= W_{6,5}(q^{r_m - r_k - j_k^0} t^{k-m-1}; q^{-j_k^0}, qt^{-1}, q^{-j_m^0}; q, q^{r_m - r_k + j_m^0} t^{k-m}). \blacksquare
\end{aligned}$$

The Coefficients $\mathbf{T}_{j_m^l, 0}$ and $\mathbf{T}_{j_m^l, j_k^l, 0}$.

For $l = \{1, \dots, (r_n - 1)\}$, we compute the coefficients $T_{j_m^l, 0}$ and $T_{j_m^l, j_k^l, 0}$ in terms of very-well-poised hypergeometric series. To this end, fix l and carry out the procedure previously described to obtain the chain:

$$(r_{n-k_1} + \sum_{c=0}^{l-1} j_{n-k_1}^c) < \dots < (r_{n-k_f} + \sum_{c=0}^{l-1} j_{n-k_f}^c) = (r_m + \sum_{c=0}^{l-1} j_m^c).$$

Notation 6.1. Set:

$$\begin{aligned} \sum_{c=0}^{l-1} j_m^c &\equiv J_m \\ \sum_{s=1}^{n-1} \sum_{c=0}^l j_s^c &\equiv J_s \\ \sum_{c=0}^{l-1} j_{n-k_i}^c &\equiv J_{n-k_i} \end{aligned}$$

Proposition 6.4. Suppose that $(r_{n-k_1} + J_{n-k_1}) = (r_m + J_m)$. It follows that $(r_m + J_m) \equiv (r_{n-1} + J_{n-1})$ and (8) becomes:

$$T_{j_m^l, 0} = \begin{cases} W_{6,5}(q^{r_m-r_n+J_m+J_s} t^{n-m-k_1}; qt^{-1}, q^{-j_m^l}, q^{r_{n-k_1}+J_{n-k_1}-r_n} J_s; \\ \quad q, q^{r_m-r_{n-k_1}-J_{n-k_1}+J_m+j_m^l} t^{n-m-k_1})_{\bullet} \\ \prod_{h=2}^{f-1} W_{6,5}(q^{r_m-r_{n-k_i}-J_{n-k_i}+J_m} t^{n-k_i-m+1}; qt^{-1}, q^{-j_m^l}, \\ \quad q^{r_{n-k_i+1}-r_{n-k_i}-J_{n-k_i}+J_{n-k_i+1}}; q, q^{r_m-r_{n-k_i+1}-J_{n-k_i+1}+J_m+j_m^l} t^{n-m-k_i}) & j_m^l \neq 0 \\ 1 & j_m^l = 0. \end{cases}$$

Proof. Let $(r_{n-k_1} + J_{n-k_1}) = (r_m + J_m) \Rightarrow (r_m + J_m) \equiv (r_{n-1} + J_{n-1})$. There are two cases: $j_m^l = 0$ and $j_m^l \neq 0$.

For $j_m^l = 0$, by Proposition 4.3, we have that $T_{j_m^l, 0} = 1$.

For $j_m^l \neq 0$, using properties of hypergeometric series, we have:

$$\begin{aligned} T_{j_m^l, 0} &= \prod_{i=r_n-J_s}^{r_{n-1}+J_{n-1}-1} \frac{(1 - q^{r_m+J_m-(i+1)} t^{n-m})(1 - q^{r_m+J_m+j_m^l-i} t^{n-m-1})}{(1 - q^{r_m+J_m-i} t^{n-m-1})(1 - q^{r_m+J_m+j_m^l-(i+1)} t^{n-m})} \\ &= \frac{(q^{r_m+J_m-r_{n-1}-J_{n-1}} t^{n-m}, q^{r_m+J_m+j_m^l-r_{n-1}-J_{n-1}+1} t^{n-m-1}; q)_{r_{n-1}+J_{n-1}-r_n+J_s}}{(q^{r_m+J_m-J_{n-1}-r_{n-1}+1} t^{n-m-1}, q^{r_m-r_{n-1}-J_{n-1}+J_m+j_m^l} t^{n-m}; q)_{r_{n-1}-r_n+J_s+J_{n-1}}} \\ &= \frac{P}{Q} \\ &= W_{6,5}(q^{r_m-r_n+J_s+J_m} t^{n-m-1}; qt^{-1}, q^{-j_m^l}, q^{r_{n-1}-r_n+J_s+J_{n-1}}; q, q^{r_m-r_{n-1}+J_m-J_{n-1}+j_m^l} t^{n-m}). \end{aligned}$$

Where:

$$\begin{aligned}
P &= (q^{r_m-r_{n-1}+J_m-J_{n-1}}t^{n-m}, q^{r_m-r_{n-1}+J_m-J_{n-1}+j_m^l+1}t^{n-m-1}, \\
&\quad q^{r_m-r_n+J_s+J_m+1}t^{n-m-1}, q^{r_m-r_n+J_s+J_m+j_m^l}t^{n-m}; q)_\infty \\
Q &= (q^{r_m-r_n+J_s+J_m}t^{n-m}, q^{r_m-r_n+J_s+J_m+j_m^l+1}t^{n-m-1}, \\
&\quad q^{r_m-r_{n-1}+J_m-J_{n-1}}t^{n-m+1}, q^{r_m-r_{n-1}+J_m-J_{n-1}+j_m^l}t^{n-m}; q)_\infty. \blacksquare
\end{aligned}$$

Remark 6.2. Since $(r_m + J_m) \equiv (r_{n-1} + J_{n-1}) \Rightarrow j_k^l = 0$ for all $m < k \leq (n-1)$, we have that $T_{j_m^l, j_k^l, 0} = 1$. \square

Proposition 6.5. Suppose that $(r_{n-k_1} + J_{n-k_1}) \neq (r_m + J_m)$. Obtaining the chain

$$(r_{n-k_1} + J_{n-k_1}) < \dots < (r_{n-k_f} + J_{n-k_f}) = (r_m + J_m),$$

we have that:

$$T_{j_m^l, 0} = \begin{cases} W_{6,5}(q^{r_m-r_n+J_s+J_m}t^{n-m-k_1}; qt^{-1}, q^{-j_m^l}, q^{r_{n-k_1}-r_n+J_{n-k_1}+J_s}; \\ \quad q, q^{r_m-r_{n-k_1}-J_{n-k_1}+J_m+j_m^l}t^{n-m-k_1+1}). \\ \prod_{h=2}^{f-1} W_{6,5}(q^{r_m-r_{n-k(h+1)}+J_m+j_m^l-J_{n-k(h+1)}}t^{n-k(h+1)-m}; qt^{-1}, q^{j_m^l}, \\ \quad q^{r_{n-k_h}-r_{n-k(h+1)}+J_{n-k_h}-J_{n-k(h+1)}}; q, q^{r_m-r_{n-k_h}-J_{n-k_h}+J_m}t^{n-m-k(h+1)}) & j_m^l \neq 0 \\ 1 & j_m^l = 0. \end{cases}$$

Proof. For $(r_{n-k_1} + J_{n-k_1}) \neq (r_m + J_m)$, using properties of hypergeometric series, (8) becomes:

$$\begin{aligned}
T_{j_m^l, 0} &= \prod_{i=r_n-J_s}^{r_{n-k_1}+J_{n-k_1}-1} \frac{(1 - q^{r_m+J_m-(i+1)}t^{n-m-k_1+1})(1 - q^{r_m+J_m+j_m^l-i}t^{n-m-k_1})}{(1 - q^{r_m+J_m-i}t^{n-m-k_1})(1 - q^{r_m+J_m+j_m^l-(i+1)}t^{n-m-k_1+1})} \\
&\prod_{h=2}^{f-1} \left(\prod_{i=r_{n-k_h}+J_{n-k_h}}^{r_{n-k(h+1)}+J_{n-k(h+1)}-1} \frac{(1 - q^{r_m+J_m-(i+1)}t^{n-m-k(h+1)+1})(1 - q^{r_m+J_m+j_m^l-i}t^{n-m-k(h+1)})}{(1 - q^{r_m+J_m-i}t^{n-m-k(h+1)})(1 - q^{r_m+J_m+j_m^l-(i+1)}t^{n-m-k(h+1)+1})} \right) \\
&= \frac{(q^{r_m-r_{n-k_1}+J_m-J_{n-k_1}}t^{n-m-k_1+1}, q^{r_m-r_{n-k_1}+J_m-J_{n-k_1}+j_m^l+1}t^{n-m-k_1}; q)_{r_{n-k_1}-r_n+J_s+J_{n-k_1}}}{(q^{r_m-r_{n-k_1}+J_m-J_{n-k_1}+j_m^l}t^{n-m-k_1+1}, q^{r_m-r_{n-k_1}+J_m-J_{n-k_1}}t^{n-m-k_1}; q)_{r_{n-k_1}-r_n+J_{n-k_1}+J_s}} \\
&\quad \prod_{h=2}^{f-1} \frac{P_1}{Q_1}
\end{aligned}$$

$$\begin{aligned}
&= \frac{P_2}{Q_2} \cdot \prod_{h=2}^{f-1} \frac{P_3}{Q_3} \\
&= W_{6,5}(q^{r_m-r_n+J_m+J_s} t^{n-m-k_1}; qt^{-1}, q^{-j_m^l}, q^{r_{n-k_1}+J_{n-k_1}-r_n+J_s}; \\
&\quad q, q^{r_m-r_{n-k_1}-J_{n-k_1}+J_m+j_m^l} t^{n-m-k_1+1}). \\
&\quad \prod_{i=2}^{f-1} W_{6,5}(q^{r_m-r_{n-k(h+1)}+J_m+j_m^l-J_{n-k(h+1)}} t^{n-k(h+1)-m}; \\
&\quad qt^{-1}, q^{j_m^l}, q^{r_{n-k_h}-r_{n-k(h+1)}+J_{n-k_h}-J_{n-k(h+1)}}; q, q^{r_m-r_{n-k_h}-J_{n-k_h}+J_m} t^{n-m-k(h+1)}).
\end{aligned}$$

Where:

$$\begin{aligned}
P_1 &= (q^{r_m-r_{n-k(h+1)}+J_m-J_{n-k(h+1)}} t^{n-m-k(h+1)+1}, \\
&\quad q^{r_m-r_{n-k(h+1)}+J_m-J_{n-k(h+1)}+j_m^l+1} t^{n-m-k(h+1)}; q)_{r_{n-k(h+1)}-r_{n-k_h}+J_{n-k(h+1)}-J_{n-k_h}} \\
Q_1 &= (q^{r_m-r_{n-k(h+1)}+J_m-J_{n-k(h+1)}+1} t^{n-m-k(h+1)}, \\
&\quad q^{r_m-r_{n-k(h+1)}+J_m-J_{n-k(h+1)}+j_m^l} t^{n-m-k(h+1)+1}; q)_{r_{n-k(h+1)}+J_{n-k(h+1)}-r_{n-k_h}-J_{n-k_h}} \\
P_2 &= (q^{r_m+J_m-r_{n-k_1}-J_{n-k_1}} t^{n-m-k_1+1}, q^{r_m+J_m+j_m^l-r_{n-k_1}-J_{n-k_1}+1} t^{n-m-k_1}, \\
&\quad q^{r_m-r_n+J_s+J_m+j_m^l} t^{n-m-k_1+1}, q^{r_m-r_n+J_s+J_m+1} t^{n-m-k_1}; q)_{\infty} \\
Q_2 &= (q^{r_m-r_{n-k_1}+J_m-J_{n-k_1}+j_m^l} t^{n-m-k_1+1}, q^{r_m-r_{n-k_1}+J_m-J_{n-k_1}+j_m^l} t^{n-m-k_1}, \\
&\quad q^{r_m-r_n+J_s+J_m} t^{n-m-k_1+1}, q^{r_m-r_n+J_s+J_m+j_m^l+1} t^{n-m-k_1}; q)_{\infty} \\
P_3 &= (q^{r_m-r_{n-k(h+1)}+J_m-J_{n-k(h+1)}} t^{n-m-k(h+1)+1}, \\
&\quad q^{r_m-r_{n-k(h+1)}+J_m-J_{n-k(h+1)}+j_m^l+1} t^{n-m-k(h+1)}, q^{r_m-r_{n-k_h}+J_{n-k_h}+J_m+1} t^{n-m-k(h+1)}, \\
&\quad q^{r_m-r_{n-k_h}-J_{n-k_h}+J_m+j_m^l} t^{n-m-k(h+1)+1}; q)_{\infty} \\
Q_3 &= (q^{r_m-r_{n-k(h+1)}-J_{n-k(h+1)}+J_m+1} t^{n-m-k(h+1)}, \\
&\quad q^{r_m+J_m+j_m^l-r_{n-k(h+1)}-J_{n-k(h+1)}} t^{n-m-k(h+1)+1}, \\
&\quad q^{r_m-r_{n-k_h}+J_m-J_{n-k_h}} t^{n-m-k(h+1)+1}, q^{r_m-r_{n-k_h}-J_{n-k_h}+J_m+j_m^l+1} t^{n-m-k(h+1)}; q)_{\infty}. \blacksquare
\end{aligned}$$

Proposition 6.6. For $T_{j_m^l, j_k^l, 0}$, $m < k \leq (n-1)$, we have:

$$T_{j_m^l, j_k^l, 0} = \begin{cases} W_{6,5}(q^{r_m-r_k-J_k+J_m-j_k^l} t^{k-m-1}; \\ q^{-j_k^l}, qt^{-1}, q^{-j_m^l}; q, q^{r_m-r_k+J_m+J_k+j_m^l} t^{k-m}) & j_m^l \neq 0 \text{ and } j_k^l \neq 0 \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Using properties of hypergeometric series, (9) becomes:

$$\begin{aligned} T_{j_m^l, j_k^l, 0} &= \prod_{i=r_k+J_k}^{r_k+J_k+j_k^l-1} \frac{(1-q^{r_m+J_m-i} t^{k-m-1})(1-q^{r_m+J_m+j_m^l-(i+1)} t^{k-m})}{(1-q^{r_m+J_m-(i+1)} t^{k-m})(1-q^{r_m+J_m+j_m^l-i} t^{k-m-1})} \\ &= \frac{(q^{r_m-r_k+J_m-J_k-j_k^l+1} t^{k-m-1}, q^{r_m-r_k+J_m-J_k+j_m^l-j_k^l} t^{k-m}; q)_{j_k^l}}{(q^{r_m-r_k-J_k+J_m-j_k^l} t^{k-m}, q^{r_m-r_k+J_m-J_k+j_m^l-j_k^l+1} t^{k-m-1}; q)_{j_k^l}} \\ &= \frac{P}{Q} \\ &= W_{6,5}(q^{r_m-r_k-J_k+J_m-j_k^l} t^{k-m-1}; q^{-j_k^l}, qt^{-1}, q^{-j_m^l}; q, q^{r_m-r_k+J_m+J_k+j_m^l} t^{k-m}). \end{aligned}$$

Where:

$$\begin{aligned} P &= (q^{r_m-r_k-J_k+J_m-j_k^l+1} t^{k-m-1}, q^{r_m-r_k+J_m-J_k+j_m^l-j_k^l} t^{k-m}, \\ &\quad q^{r_m-r_k-J_k+J_m} t^{k-m}, q^{r_m-r_k+J_m-J_k+j_m^l+1} t^{k-m-1}; q)_\infty \\ Q &= (q^{r_m-r_k-J_k+J_m-j_k^l} t^{k-m}, q^{r_m-r_k+J_m-J_k+j_m^l-j_k^l+1} t^{k-m-1}, \\ &\quad q^{r_m-r_k-J_k+J_m+1} t^{k-m-1}, q^{r_m-r_k+J_m-J_k+j_m^l} t^{k-m}; q)_\infty. \blacksquare \end{aligned}$$

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