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LINEAR FUNCTIONS OF ORDER STATISTICS

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LINEAR FUNCTIONS OF ORDER STATISTICS<sup>1</sup>

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Abstract

Based on an almost sure Wiener process approximation for linear functions of order statistics along with the strong consistency of their variance estimates, a class of sequential tests is considered. These test terminate with probability 1. Asymptotic OC and ASN functions are also studied.

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1. Introduction. Let  $\{X_i, i \geq 1\}$  be a sequence of independent and identically distributed random variables (i.i.d.r.v.) with each  $X_i$  having a continuous distribution function (d.f.)  $F(x)$ ,  $x \in R$ , the real line  $(-\infty, \infty)$ . For every  $n \geq 1$ , the order statistics corresponding to  $X_1, \dots, X_n$  are denoted by  $X_{n,1}, \dots, X_{n,n}$ ; by virtue of the assumed continuity of  $F$ ,  $X_{n,1} < \dots < X_{n,n}$ , with probability 1. Extensive research works have been done on the (optimal) use of linear functions of order statistics for drawing statistical inference on the parameters associated with  $F$ . We may refer to Sarhan and Greenberg (1962) and David (1970) which provide a thorough survey of the basic theory (for finite sample sizes). Consider the sequence

$$(1.1) \quad T_n = n^{-1} \sum_{i=1}^n J(i/(n+1)) X_{n,i}, \quad n \geq 1,$$

where  $J = \{J(u) : u \in I\}$  ( $I = [0, 1]$ ) is a smooth weight function. In the context of asymptotically best estimates of location and scale parameters, it follows from the works of Lloyd, Ogawa, Jung and Blom, among others [viz., Chapter 4 of Sarhan and Greenberg (1962)] that under suitable regularity conditions on  $F$  and  $J$ ,  $T_n$  is an asymptotically most efficient estimator of the parameter

$$(1.2) \quad \mu = \mu(F, J) = \int_{-\infty}^{\infty} x J(F(x)) dF(x).$$

In general, for suitable choice of  $J$ ,  $\mu$  is a functional of the d.f.  $F$  and is usually a measure of some characteristic of  $F$ . Within this framework, we are primarily interested here in sequential tests for  $\mu$  based on  $\{T_n, n \geq 1\}$ . Specifically, we consider

$$(1.3) \quad H_0 : \mu = \mu_0 \quad \text{vs.} \quad H_1 : \mu = \mu_1 = \mu_0 + \Delta, \quad \Delta > 0,$$

where  $\mu_0$  and  $\Delta$  (small) are specified, and we aim to have a test having some prescribed strength  $(\alpha, \beta)$ , where  $0 < \alpha, \beta < 1$ . In our formulation, we assume that under (1.3), the d.f.  $F$  is not completely specified. If, it were so, the Wald (1947) sequential probability ratio test (SPRT) would have been applicable.

Following the Bartlett-Cox motivation of sequential likelihood ratio tests (SLRT) [as further elaborated and rigorously formulated for nonparametric statistics by Sen (1973 a,b) and Sen and Ghosh (1974)], the proposed sequential tests are introduced in Section 2. It is shown in Section 3 that under fairly general conditions (on  $F$  and  $J$ ), These tests terminate with probability 1. For studying the OC and ASN of the proposed procedures, we confine ourselves to the asymptotic situation where we let  $\Delta \rightarrow 0$ . In practice, the derived asymptotic expressions should provide good approximations whenever, in (1.3),  $\Delta$  is taken small. We present these asymptotic results in Sections 4 and 5. The results are based on certain almost sure (a.s.) convergence results on some functionals of the empirical d.f.'s, the proofs of which are mostly sketched in the appendix.

2. Sequential tests based on linear estimates (STBLE). Our procedure is based on an a.s. invariance principle for  $\{T_n - \mu; n \geq 1\}$ . It is well known (viz. [7] and the references cited therein) that under fairly general conditions on  $F$  and  $J$ , as  $n \rightarrow \infty$ ,

$$(2.1) \quad \mathcal{D}(n^{1/2}[T_n - \mu]/\sigma) \rightarrow \mathcal{N}(0,1),$$

where

$$(2.2) \quad \sigma^2 = 2 \iint_{-\infty < x < y < \infty} F(x) [1-F(y)] J(F(x)) J(F(y)) dx dy,$$

and we assume that  $0 < \sigma < \infty$ . By making use of a recent result of Ghosh (1972 a), we strengthen (2.1) to the following a.s. invariance principle : as  $n \rightarrow \infty$ ,

$$(2.3) \quad n(T_n - \mu) = \sigma W(n) + O([n \log \log n]^{1/4} [\log n]^{1/2}) \text{ a.s.},$$

where  $W = \{W(t) : 0 \leq t < \infty\}$  is a standard Brownian motion on  $R^+ = [0, \infty)$ ; see Theorem A.1 in the appendix. Also, in general,  $\sigma^2$ , defined by (2.2), is not known [even under (1.3)]. To estimate it, we define, for every  $n \geq 1$ , the empirical d.f.  $F_n$  by

$$(2.4) \quad F_n(x) = n^{-1} \sum_{i=1}^n u(x - X_i), \quad x \in R,$$

where  $u(t)$  is 1 or 0 according as  $t$  is  $\geq$  or  $< 0$ . Let then

$$(2.5) \quad \sigma_n^2 = 2 \iint_{-\infty < x < y < \infty} J(F_n(x)) J(F_n(y)) F_n(x) [1 - F_n(y)] dx dy.$$

We prove in Theorem A.2 that

$$(2.6) \quad \sigma_n^2 / \sigma^2 \rightarrow 1 \text{ a.s., as } n \rightarrow \infty.$$

By looking at the Bartlett-Cox [2] SLRT, one observes that their theory is based on a Wiener-process approximation for the sequence of likelihood ratios along with the a.s. convergence of the maximum likelihood estimators; analogues of these for linear functions of order statistics are provided in (2.3) and (2.6). As such, by the same motivation as in [2, 9, 10, 11], we may proceed as follows.

Let  $(\alpha, \beta)$  be the desired strength of the test. We take  $0 < \alpha, \beta < \frac{1}{2}$ , and consider two positive numbers  $B (\geq \beta / (1 - \alpha))$  and  $A (\leq (1 - \beta) / \alpha)$ , so that  $0 < B < 1 < A < \infty$ . Let then

$$(2.7) \quad b = \log B \text{ and } a = \log A, \text{ so that } -\infty < b < 0 < a < \infty.$$

Also for  $\Delta (> 0)$ , defined by (1.3), consider an initial sample of size  $n_0 = n_0(\Delta)$ , where we assume that for small  $\Delta$ ,  $n_0(\Delta)$  is moderately large. A more precise statement will be introduced in Section 4. Then, starting with the initial sample of size  $n_0(\Delta)$ , we continue drawing observations, one by one, so long as

$$(2.8) \quad b \sigma_n^2 < n \Delta \left\{ T_n - \frac{1}{2}(\mu_0 + \mu_1) \right\} < a \sigma_n^2, \quad n \geq n_0(\Delta).$$

If  $N (= N(\Delta))$  is the smallest positive integer ( $\geq n_0(\Delta)$ ) for which (2.8) is violated, we accept  $H_0$  or  $H_1$  according as  $N \Delta \left\{ T_N - \frac{1}{2}(\mu_0 + \mu_1) \right\}$  is  $\leq b \sigma_N^2$  or  $\geq a \sigma_N^2$ .  $N(\Delta)$  is therefore the stopping variable; if the process does not terminate, we let  $N(\Delta) = \infty$ .

Note that, in structure, our procedure is similar to the one in Sen (1973 a) where, instead of  $(T_n, \sigma_n^2)$ , sequences of U-statistics and their estimated variances were used. However, in mathematical manipulations, we need here an altogether different approach.

3. Termination of the STBLE. We assume that

$$(3.1) \quad J(u) \text{ and } J'(u) \text{ exist and are continuous in } u \in I,$$

$$(3.2) \quad J'(u) \text{ is of bounded variation in } u \in I,$$

$$(3.3) \quad v_r = \int_{-\infty}^{\infty} |x|^r dF(x) < \infty,$$

for some  $r (\geq 2)$ , to be specified later on.

Theorem 3.1. Under (3.1), (3.2) and (3.3) with  $r = 2$ , the STBLE terminates with probability one, i.e., for every given  $\mu_0$ ,  $\Delta$  and  $\mu$ ,  
 $P\{N(\Delta) < \infty \mid \mu\} = 1.$

Proof. For every  $n \geq n_0(\Delta)$ , by (2.8),

$$(3.4) \quad P\{N(\Delta) > n \mid \mu\} \leq P\{n^{-1} b \sigma_n^2 < \Delta [T_n - \frac{1}{2}(\mu_0 + \mu_1)] < n^{-1} a \sigma_n^2 \mid \mu\}.$$

If  $\mu \neq \frac{1}{2}(\mu_0 + \mu_1)$ , we note that by Theorem 1 of Ghosh (1972 b), under conditions less restrictive than (3.1) - (3.3),  $T_n \rightarrow \mu$ , a.s., as  $n \rightarrow \infty$  and by Theorem A.2, under (3.1) - (3.3),  $\sigma_n^2 \rightarrow \sigma^2$  a.s., as  $n \rightarrow \infty$ , so that the right hand side (rhs) of (3.4) converges to 0 as  $n \rightarrow \infty$ . If  $\mu = \frac{1}{2}(\mu_0 + \mu_1)$ , we rewrite the rhs of (3.4) as

$$(3.5) \quad P\{n^{-1/2} b \sigma_n / \Delta < n^{1/2} [T_n - \mu] / \sigma_n < n^{-1/2} a \sigma_n / \Delta \mid \mu\},$$

where by the results of Moore (1968),  $n^{1/2} [T_n - \mu] / \sigma$  is asymptotically normal with zero mean and unit variance, and by Theorem A.2,  $\sigma_n / \sigma \rightarrow 1$  a.s., as  $n \rightarrow \infty$ . Consequently, (3.5) converges to 0 as  $n \rightarrow \infty$ . Q.E.D.

4. OC function of the STABLE. We shall provide here an asymptotic expression for the OC function, where we let  $\Delta \rightarrow 0$ . Note that for fixed  $\mu_0$  and  $\mu (\neq \mu_0)$ , the OC function will converge to 1 or 0 [according as  $\mu < \mu_0$  or  $> \mu_0$ ] when  $\Delta \rightarrow 0$ . Hence, to avoid the limiting non-degeneracy, we let

$$(4.1) \quad \mu = \mu_0 + \varphi \Delta, \quad \varphi \in I^* = \{t : |t| \leq K\},$$

where  $K (< \infty)$  is some positive number ( $> 1$ ). Also, we assume that

$$(4.2) \quad \lim_{\Delta \rightarrow 0} n_0(\Delta) = \infty \quad \text{but} \quad \lim_{\Delta \rightarrow 0} \Delta^2 n_0(\Delta) = 0.$$

For motivation of (4.1) and (4.2), we again refer to Sen (1973 a, b) and

Sen and Ghosh (1974). Finally, let  $L_{J,F}(\varphi, \Delta)$  be the OC (probability of acceptance of  $H_0$  when  $J, F$  and  $\varphi$  are given,  $\Delta$  is specified by (1.3) and  $u$ , by (4.1)) of the STBLE.

Theorem 4.1. If, in addition to (3.1) and (3.2)  $J''(u)$  is bounded for  $u \in I$  and (3.3) holds for some  $r > 2$ , then under (4.1) and (4.2),

$$(4.3) \quad \lim L_{J,F}(\varphi, \Delta) = P(\varphi) = \begin{cases} (A^{1-2\varphi-1})/(A^{1-2\varphi-B}A^{1-2\varphi}), & \varphi \neq 1/2 \\ a/(a-b), & \varphi = 1/2. \end{cases}$$

Thus, asymptotically (as  $\Delta \rightarrow 0$ ), the OC function does not depend on  $F$  and  $J$  and the STBLE is asymptotically consistent.

Proof. Let  $E_1 = E_1(\Delta) = \{(m\Delta(T_m - \frac{1}{2}[\mu_0 + \mu_1])) \leq b\sigma_m^2 \text{ for some } n \geq n_0(\Delta) \text{ before being } \geq a\sigma_m^2 \text{ for a smaller } m\}$ ,  $E_2 = E_2(\Delta, \eta) = \{|\sigma_m^2/\sigma^2 - 1| \leq \eta, \forall m \geq n_0(\Delta)\}$ ,  $\eta > 0$ , and let  $E_j^c$ ,  $j=1,2$  denote the complementary events. Then, by definition, for every  $\eta > 0$ ,

$$(4.4) \quad L_{J,F}(\varphi, \Delta) = P_\varphi\{E_1(\Delta)\} = P_\varphi\{E_1(\Delta)E_2(\Delta)\} + P_\varphi\{E_1(\Delta)E_2^c(\Delta)\},$$

where by Theorem A.2 and (4.2),  $P_\varphi\{E_1(\Delta)E_2^c(\Delta)\} \leq P_\varphi\{E_2^c(\Delta)\} \rightarrow 0$  as  $\Delta \rightarrow 0$ .

We also denote by  $N^{(ij)}(\Delta)$  the stopping variable, defined similarly as  $N(\Delta)$ , but  $b\sigma_n^2$  and  $a\sigma_n^2$  being replaced by  $b\sigma^2(1+(-1)^i\eta)$  and

$a\sigma^2(1+(-1)^j\eta)$ , respectively, and let  $L_{J,F}^{(ij)}(\varphi, \Delta)$  be the corresponding

OC function, for  $i, j=1,2$ . Then, from the above, we conclude that for

every  $\eta > 0$  and  $\epsilon > 0$ , there exists as  $\Delta_0(> 0)$ , such that for  $0 < \Delta \leq \Delta_0$ ,

$$(4.5) \quad L_{J,F}^{(21)}(\varphi, \Delta) - \epsilon \leq L_{J,F}(\varphi, \Delta) \leq L_{J,F}^{(12)}(\varphi, \Delta) + \epsilon, \forall \varphi \in I^*.$$

Since  $\epsilon > 0$  and  $\eta > 0$  are arbitrary, to prove the theorem, it suffices to show that

$$(4.6) \quad \lim_{\eta \rightarrow 0} \left\{ \lim_{\Delta \rightarrow 0} L_{J,F}^{(i,j)}(\varphi, \Delta) \right\} = P(\varphi), \text{ for } i, j = 1, 2.$$

For every real  $d$  and  $c_1 < 0 < c_2$ , let us denote by

$$(4.7) \quad P(c_1, c_2, d) = \begin{cases} (e^{2dc_2} - 1)/(e^{2dc_2} - e^{2dc_1}), & d \neq 0, \\ c_2/(c_2 - c_1) & d = 0. \end{cases}$$

Then, from Theorem 4.3 of Anderson (1960), we have for every  $\gamma > 0$ ,

$$\begin{aligned} & \lim_{T \rightarrow \infty} P\{W(t), 0 \leq t \leq T, \text{ first crosses the line } \gamma^{-1} c_1 + \gamma dt \\ & \text{ before crossing the line } \gamma^{-1} c_2 + \gamma dt\} \\ & = P(c_1, c_2, d). \end{aligned}$$

If we now let  $n^*(\Delta) = \max\{k : \Delta^2 k \leq -\log \Delta\}$ , then as  $\Delta \rightarrow 0$ ,  $n^*(\Delta) \rightarrow \infty$ ,  $\Delta^2 n^*(\Delta) \rightarrow \infty$  but  $\Delta ([n^*(\Delta)]^{1/4} [\log n^*(\Delta)]^{3/4}) \rightarrow 0$ , so that by our Theorem A.1, we have for  $\mu = \mu_0 + \varphi \Delta$ ,

$$(4.9) \quad \sup_{n_0(\Delta) \leq n \leq n^*(\Delta)} |n \Delta (T_n - \frac{1}{2} [\mu_0 + \mu_1]) - n \Delta^2 (\varphi - \frac{1}{2}) - \Delta(W(n))| \rightarrow 0, \text{ with probability 1, as } \Delta \rightarrow 0,$$

and by virtue of Lemma 4.2 of Strassen (1967), we obtain that as  $\Delta \rightarrow 0$ ,  $\{\Delta W(n) : N_0(\Delta) \leq n \leq n^*(\Delta)\}$  behaves like a continuous Brownian motion on  $(0, \infty)$ . Hence, (4.6) follows from (4.7) - (4.9) along with the fact that  $c_1 = \Delta^{-1} b(1 + (-1)^i \eta)$ ,  $c_2 = \Delta^{-1} a(1 + (-1)^j \eta)$ ,  $i, j = 1, 2$ ,  $d = (\frac{1}{2} - \varphi)$  and  $\eta (> 0)$  is arbitrary. Q. E. D.

Note that, if we let

$$(4.10) \quad B = \beta/(1 - \alpha) \quad \text{and} \quad A = (1 - \beta)/\alpha,$$

then from (4.3), (2.7) and (4.10), we obtain that

$$(4.11) \quad \lim_{\Delta \rightarrow 0} L_{J,F}(0, \Delta) = 1 - \alpha, \quad \lim_{\Delta \rightarrow 0} L_{J,F}(1, \Delta) = \beta,$$

so that asymptotically, the STBLE has the prescribed strength  $(\alpha, \beta)$  i.e., it is asymptotically consistent

5. ASN function of the STBLE. We require here more stringent conditions, as we face here the moment convergent problem, which is stronger than the convergence in distribution problem of Section 3.

Theorem 5.1. Suppose that (3.1), (3.2), (4.1), (4.2) hold, (3.3) holds for some  $r > 4$  and  $J''(u)$  is continuous in  $u \in I$ . Then

$$(5.1) \quad \lim_{\Delta \rightarrow \infty} \{ \Delta^2 E_{\varphi} [N(\Delta)] \} = \Psi(\varphi, \sigma^2), \quad \forall \varphi \in I^*,$$

where  $E_{\varphi}$  stands for the expectation under  $\mu = \mu_0 + \varphi \Delta$  and

$$(5.2) \quad \Psi(\varphi, \sigma^2) = \begin{cases} [aP(\varphi) + b\{1 - P(\varphi)\}] \sigma^2 (\varphi - \frac{1}{2})^{-1}, & \varphi \neq 1/2, \\ - (a-b) P'(\frac{1}{2}) \sigma^2, & \varphi = 1/2, \end{cases}$$

with  $P(\varphi)$  defined by (4.3) and  $P'(\varphi) = (\partial/\partial\varphi)P(\varphi)$ .

Proof. We consider first the case of  $\varphi \neq 1/2$ . Define

$$(5.3) \quad n_1(\Delta) = [\epsilon \Delta^{-2}] \quad \text{and} \quad n_2(\Delta) = [K \Delta^{-2} (-\log \Delta)],$$

where  $K (< \infty)$  is a positive constant and  $\epsilon (> 0)$  is arbitrary. Then,

$$(5.4) \quad \Delta^2 E_{\varphi} [N(\Delta)] = \Delta^2 \left[ \sum_{n \leq n_1} + \sum_{n_1 < n \leq n_2} + \sum_{n > n_2} n P_{\varphi} \{N(\Delta) = n\} \right],$$

where the first term on the rhs of (5.4) is  $< \epsilon$ . At this stage, we make use of the following results proved in Theorem A.3 in the appendix. For every  $n > 0$ , under the hypothesis of Theorem 5.1, there exist positive numbers  $K_1, K_2$  and an integer  $n_0$ , such that for  $n \geq n_0$ , for some  $\delta > 0$ ,

$$(5.5) \quad P\{|\sigma_n^2/\sigma^2 - 1| > \eta\} \leq K_2 n^{-1-\delta},$$

$$(5.6) \quad P\{|T_n - \mu - \bar{Z}_n| > K_1 n^{-1} (\log n)^2\} \leq K_2 n^{-1-\delta},$$

where  $\bar{Z}_n = n^{-1} \sum_{i=1}^n Z_i$ ,  $n \geq 1$ , and

$$(5.7) \quad Z_i = - \int \{u(x - X_i) - F(x)\} J(F(x)) dx, \quad i \geq 1.$$

Suppose, in (2.8), we replace  $b\sigma_n^2$ ,  $a\sigma_n^2$  by  $b\sigma^2(1+(-1)^i\eta)$ ,  $a\sigma^2(1+(-1)^i\eta)$  and  $[T_n - \frac{1}{2}(\mu_0 + \mu_1)]$  by  $[\bar{Z}_n + (\varphi - \frac{1}{2})\Delta]$ , respectively, and denote the corresponding stopping variable by  $\tilde{N}^{(ii)}(\Delta)$ , for  $i = 1, 2$ . By making use of the fact that

$$(5.8) \quad \sum_{n_1 < n \leq n_2} n P_{\varphi} \{N(\Delta) = n\} = [n_1(\Delta) + 1] P_{\varphi} \{N(\Delta) > n_1(\Delta)\} + \sum_{n_1 < n \leq n_2} P_{\varphi} \{N(\Delta) > n\} - [n_2(\Delta)] P_{\varphi} \{N(\Delta) > n_2(\Delta)\},$$

and similarly for  $\tilde{N}^{(ii)}(\Delta)$ ,  $i = 1, 2$ , we obtain on using (5.3), (5.5), (5.6) and a few standard steps that as  $\Delta \rightarrow 0$ ,

$$(5.9) \quad \Delta^2 \sum_{n_1 < n \leq n_2} n P_{\varphi} \{\tilde{N}^{(11)}(\Delta) = n\} - O([n_1(\Delta)]^{-\delta} (-\log \Delta)) \\ \leq \Delta^2 \sum_{n_1 < n \leq n_2} n P_{\varphi} \{N(\Delta) = n\} \leq \Delta^2 \sum_{n_1 < n \leq n_2} n P_{\varphi} \{\tilde{N}^{(22)}(\Delta) = n\} \\ + O([n_1(\Delta)]^{-\delta} (-\log \Delta)),$$

where by (3.5), as  $\Delta \rightarrow 0$ , for every (fixed)  $\epsilon > 0$ ,

$$(5.10) \quad [n_1(\Delta)]^{-\delta} (-\log \Delta) = \epsilon^{-\delta} \Delta^{2\delta} (-\log \Delta) \rightarrow 0.$$

Further, for  $n > n_2(\Delta)$ , we note that by (5.5) and (5.6),

$$(5.11) \quad P_{\varphi} \{N(\Delta) > n\} \leq P_{\varphi} \{b\sigma_n^2 < n\Delta [T_n - \frac{1}{2}(\mu_0 + \mu_1)] < a\sigma_n^2\} \\ \leq P_{\varphi} \{b\sigma^2(1+\eta) + R_n < n\Delta [\bar{Z}_n + (\varphi - \frac{1}{2})\Delta] < a\sigma^2(1+\eta) + R_n\} + O(n^{-1-\delta}),$$

where  $R_n = n\Delta [T_n - \mu - \bar{Z}_n]$ , so that by (5.6)

$$(5.12) \quad P\{|R_n| > K_1 \Delta (\log n)^2\} \leq K_2 n^{-1-\delta}, \text{ for } n \geq n_0.$$

[In the sequel, we choose  $\Delta_0 (> 0)$  so small that for  $0 < \Delta \leq \Delta_0$ ,  $n_1(\Delta) \geq n_0$ .]

Since  $\varphi \neq 1/2$ , rewriting the rhs of (5.11) as

$$(5.13) \quad P_{\varphi} \left\{ \frac{b\sigma^2(1+\eta)}{\Delta\sqrt{n}} + \frac{R_n}{\Delta\sqrt{n}} - \sqrt{n}\Delta\left(\varphi - \frac{1}{2}\right) < \sqrt{n}\bar{Z}_n < \frac{a\sigma^2(1+\eta)}{\Delta\sqrt{n}} + \frac{R_n}{\Delta\sqrt{n}} - \sqrt{n}\Delta\left(\varphi - \frac{1}{2}\right) \right\},$$

we observe that (i)  $b\sigma^2(1+\eta)/\Delta\sqrt{n} \leq b\sigma^2(1+\eta)/\Delta\sqrt{n_2(\Delta)} \rightarrow 0$  as  $\Delta \rightarrow 0$ ,  $a\sigma^2(1+\eta)/\Delta\sqrt{n} \rightarrow 0$  as  $\Delta \rightarrow 0$ , (ii) by (5.12),  $R_n/\Delta\sqrt{n} = O(n^{-1/2}(\log n)^2) = o(1)$  (as  $\Delta \rightarrow 0$ ) with a probability  $\geq 1 - K_2 n^{-1-\delta}$ , (iii)  $\sqrt{n}\Delta|\varphi - \frac{1}{2}| = \sqrt{n/n_2(\Delta)} [K^{1/2}(-\log \Delta)^{1/2}] |\varphi - \frac{1}{2}| \rightarrow \infty$ , and (iv)  $\bar{Z}_n$  is an average over  $n$  i.i.d.r.v., where by the hypothesis of the theorem,  $E|Z_1|^r < \infty$  for some  $r > 4$ , so that by using the Markov inequality along with (i) - (iii), we claim that for  $\Delta (> 0)$  small, for all  $n > n_2(\Delta)$

$$(5.14) \quad P_{\varphi} \{N(\Delta) > n\} = O(n^{-1-\delta}), \text{ where } \delta > 0.$$

Consequently,  $\Delta^2 \sum_{n > n_2} P_{\varphi} \{N(\Delta) > n\} \rightarrow 0$  as  $\Delta \rightarrow 0$ , i.e.,  $\Delta^2 \sum_{n > n_2} n P_{\varphi} \{N(\Delta) = n\} \rightarrow 0$  as  $\Delta \rightarrow 0$ . A similar treatment holds for both the stopping variables  $\tilde{N}^{(ii)}(\Delta)$ ,  $i=1,2$ . Hence, it suffices to show that

$$(5.15) \quad \lim_{\eta \rightarrow 0} \left\{ \lim_{\Delta \rightarrow 0} \Delta^2 E_{\varphi} [\tilde{N}^{(ii)}(\Delta)] \right\} = \Psi(\varphi, \sigma^2), \quad \forall \varphi \left( \neq \frac{1}{2} \right) \in I^*.$$

Now, proceeding as in the proof of Theorem 4.1 it follows that for every  $\eta (> 0)$ , arbitrarily small the OC functions for both  $\tilde{N}^{(ii)}(\Delta)$ ,  $i=1,2$ , can be made close to  $P(\varphi)$ , defined by (4.3), when  $\Delta \rightarrow 0$ . Also, by definition,  $\{\bar{Z}_n\}$  involve an average over independent random variables for which  $E Z_1 = 0$  and  $E Z_1^2 = \sigma^2$ ,  $0 < \sigma < \infty$ . Consequently, (5.15) follows from the classical result of Wald (1947).

The above proof fails when  $\varphi = 1/2$  (as then, in (5.13), both the bounds converge to 0 as  $\Delta \rightarrow 0$ ). However, if we let  $\{\varphi_r = \frac{1}{2} + (-1)^r \epsilon/r, r \geq 1\}$

where  $\epsilon (> 0)$  is arbitrary, then first working with a given  $\phi_r$ , using the above proof, and then applying the L'Hôpital rule, we arrive at our desired result. Q.E.D.

In passing, we may note that if  $\mu$  is some identifiable parameter of the d.f.F (such as the location or scale parameter) and  $\{T_n\}$  correspond to the asymptotically best estimator of  $\mu$ , by the results of Jung and Blom [viz., Chapter 4 of Sarhan and Greenberg (1962)], we note that  $\sigma^2$  equals the Cramér-Rao bound, so that by Theorem 5.1 and the asymptotic equivalence of the SLRT and SPRT, we conclude that, in such a case, the proposed STBLE is asymptotically (as  $\Delta \rightarrow 0$ ) as efficient as the SLRT considered by Bartlett-Cox. In general, for different  $\{J(u), u \in I\}$ , by Theorem 5.1, the ARE (asymptotic relative efficiency) of the different STBLE, as judged by their ASN, will be inversely proportional to the respective  $\sigma^2$ ; this is in agreement with the Pitman-ARE for the non-sequential case.

6. Appendix. First, we consider the following.

Theorem A.1. Under the assumptions of Theorem 4.1, (2.3) holds.

Proof. Defining the  $Z_i, i \geq 1$ , by (5.7) and letting  $\bar{Z}_n = \frac{1}{n} \sum_{i=1}^n Z_i, n \geq 1$ , it follows from the main theorem of Ghosh (1972 a) that

$$(6.1) \quad T_n - \mu = \bar{Z}_n + R_n,$$

where under the assumptions of Theorem 4.1,

$$(6.2) \quad R_n = O(n^{-1}(\log n)^2) \text{ a.s., as } n \rightarrow \infty.$$

Further, the  $Z_i, i \geq 1$ , are i.i.d.r.v. with mean 0, variance  $\sigma^2$  and a finite 4th moment, so that by Theorem 1.5 of Strassen (1967), we claim that as  $n \rightarrow \infty$ .

$$(6.3) \quad n\bar{Z}_n = \sigma W(n) + O((n \log \log n)^{1/4} (\log n)^{1/2}) \text{ a.s.}$$

From (6.1) - (6.3), we obtain that

$$(6.4) \quad \begin{aligned} n(T_n - \mu) &= n\bar{Z}_n + n R_n \\ &= \sigma W(n) + O((n \log \log n)^{1/4} (\log n)^{1/2}) + O((\log n)^2) \text{ a.s.} \\ &= \sigma W(n) + O([n \log \log n]^{1/4} (\log n)^{1/2}) \text{ a.s. Q.E.D.} \end{aligned}$$

Theorem A.2. If (3.1) and (3.3) (with  $r = 2$ ) hold, then (2.6) also holds.

Proof. Let  $E^* = \{(x, y) : -\infty < x < y < \infty\}$ . Then, we observe that

$$V(X_1) = 2 \iint_{E^*} F(x)[1 - F(y)] dx dy \text{ and } s_n^2 = n^{-1} \sum_{i=1}^n (X_i - \bar{X}_n)^2 =$$

$2 \iint_{E^*} F_n(x)[1 - F_n(y)] dx dy$ , where  $F_n$  is defined by (2.4). Since,  $s_n^2 \rightarrow \sigma^2$  a.s., as  $n \rightarrow \infty$ , from the above, we conclude that

$$(6.5) \quad \iint_{E^*} F_n(x)[1 - F_n(y)] dx dy \rightarrow \iint_{E^*} F(x)[1 - F(y)] dx dy \text{ a.s., as } n \rightarrow \infty.$$

Now, by (2.2) and (2.5), we have

$$(6.6) \quad \frac{1}{2} (\sigma_n^2 - \sigma^2) = I_{n1} + I_{n2} + I_{n3},$$

where

$$(6.7) \quad I_{n1} = \iint_{E^*} [J(F_n(x)) - J(F(x))] J(F_n(y)) F_n(x)[1 - F_n(y)] dx dy,$$

$$(6.8) \quad I_{n2} = \iint_{E^*} J(F(x)) [J(F_n(y)) - J(F(y))] F_n(x)[1 - F_n(y)] dx dy,$$

$$(6.9) \quad I_{n3} = \iint_{E^*} J(F(x)) J(F(y)) \{ F_n(x)[1 - F_n(y)] - F(x)[1 - F(y)] \} dx dy.$$

Now, by the Glivenko-Cantelli Theorem,  $\sup_x |F_n(x) - F(x)| \rightarrow 0$  a.s.,

as  $n \rightarrow \infty$ , while by (3.1),  $J(u)$  is uniformly continuous (and bounded) for

$u \in I$ , so that  $\sup_x |J(F_n(x)) - J(F(x))| \rightarrow 0$  a.s., as  $n \rightarrow \infty$ , and

$\sup_x |J(F_n(x))| \leq \sup_x |J(F(x))| < K_0 < \infty, \forall n \geq 1$ . Consequently, by (6.5),

both  $I_{n1}$  and  $I_{n2}$  converge to 0 a.s., as  $n \rightarrow \infty$ .

To tackle  $I_{n3}$ , we employ the truncation technique. For every  $\epsilon > 0$ , there exists a positive  $c (= c_\epsilon < \infty)$ , such that on defining

$$E_\epsilon^* = \{(x, y) : -c \leq x \leq y \leq c\} \text{ and } \varphi = \iint_{E_\epsilon^*} F(x)[1-F(y)] dx dy, \quad |\varphi - V(X_1)| < \epsilon.$$

Define, then

$$(6.9) \quad X_i = \begin{cases} X_i, & \text{if } |X_i| \leq c, \\ 0, & \text{if } |X_i| > c, \end{cases} \quad i \geq 1,$$

and let  $\bar{X}_n^* = n^{-1} \sum_{i=1}^n X_i^*$ ,  $s_n^{*2} = n^{-1} \sum_{i=1}^n (X_i^* - \bar{X}_n^*)^2$ . Now,  $s_n^{*2} \rightarrow \varphi$  a.s., as  $n \rightarrow \infty$ ,  $s_n^2 \rightarrow V(X_1)$  a.s., as  $n \rightarrow \infty$ , and  $|\varphi - V(X_1)| < \epsilon$ , so that we have

$$(6.10) \quad |s_n^{*2} - s_n^2| < \epsilon \text{ a.s., as } n \rightarrow \infty.$$

As a result, writing  $E^* = E_\epsilon^* + (E^* - E_\epsilon^*)$ , we have for  $I_{n3}$ ,

$$(6.11) \quad \left| \iint_{E_\epsilon^*} J(F(x)) J(F(y)) [F_n(x)\{1-F_n(y)\} - F(x)\{1-F(y)\}] dx dy \right| \\ \leq \left\{ \sup_x |J(F(x))| \right\}^2 \cdot (2c)^2 \cdot \sup_{(x,y) \in E_\epsilon^*} |F_n(x)\{1-F_n(y)\} - F(x)\{1-F(y)\}| \\ \rightarrow 0. \text{ a.s., as } n \rightarrow \infty;$$

$$(6.12) \quad \left| \iint_{E^* - E_\epsilon^*} J(F(x)) J(F(y)) [F_n(x)\{1-F_n(y)\} - F(x)\{1-F(y)\}] dx dy \right| \\ \leq \left\{ \sup_x |J(F(x))| \right\}^2 \left[ \iint_{E^* - E_\epsilon^*} F_n(x)[1-F_n(y)] dx dy + \iint_{E^* - E_\epsilon^*} F(x)[1-F(y)] dx dy \right] \\ \leq \left\{ \sup_x |J(F(x))| \right\}^2 \left[ |s_n^2 - s_n^{*2}| + |V(X) - \varphi| \right] \\ \leq \left\{ \sup_x |J(F(x))| \right\}^2 \cdot 2\epsilon, \text{ a.s., as } n \rightarrow \infty.$$

Since,  $J$  is bounded in  $I$  and  $\epsilon$  is arbitrary, the rhs of (6.12) can be made arbitrarily small, as  $n \rightarrow \infty$ . Hence  $I_{n3} \rightarrow 0$  a.s., as  $n \rightarrow \infty$ . Q.E.D.

Theorem A.3. Under the hypotheses of Theorem 5.1, (5.5) and (5.6)  
hold; for (5.6), one may even take  $r > 2$ , in (3.3).

Prof. We prove only (5.5); the treatment of (5.6) follows similarly  
 by noting that by (1.1) and (1.2)

$$(6.13) \quad T_n - \mu = \int_{-\infty}^{\infty} x [J(F_n(x)) - J(F(x))] dF_n(x) \\
 + \int_{-\infty}^{\infty} x J(F(x)) d[F_n(x) - F(x)].$$

We need to point out only the modifications in the proof of  
 Theorem A.2. It follows from Kiefer (1961) that for every  $\lambda > 0$ ,

$$(6.14) \quad P\left\{ \sup_x |F_n(x) - F(x)| > \lambda(n^{-1} \log n)^{1/2} \right\} \leq 2 \exp\{-2\lambda^2 \log n\}.$$

Also, by (3.1) and (3.2),  $J'(u)$  is uniformly continuous and bounded for  
 $u \in I$ , so that by (6.14), we have for some  $K^* (< \infty)$ ,

$$(6.15) \quad \sup_x |J(F_n(x)) - J(F(x))| \leq \left\{ \sup_{0 \leq n \leq 1} |J'(u)| \right\} \left\{ \sup_x |F_n(x) - F(x)| \right\} \\
 \leq K^* n^{-1/2} (\log n)^{1/2}, \text{ with probability } \geq 1 - 2n^{-2\lambda^2}.$$

Further,  $E|X_1|^r < \infty$  for some  $r > 4$ , so that for some  $\delta > 0$ , for every  
 $\epsilon > 0$ , there exists a  $K_\epsilon (< \infty)$ , such that

$$(6.16) \quad P\left\{ |s_n^2 - \sigma^2| > \epsilon \right\} \leq K_\epsilon n^{-1-\delta}.$$

Thus, if in (6.15), we let  $2\lambda^2 = 1 + \delta$ , we obtain that both  $I_{n1}$  and  
 $I_{n2}$  can be made smaller than  $n/3$ ,  $n > 0$ , with a probability  $\geq 1 - O(n^{-1-\delta})$ .

For  $I_{n3}$ , we note that  $E|X_1|^r < \infty \Rightarrow E|X_1^*|^r < \infty$ ,  $r > 4$ , so that (6.16)  
 holds for  $s_n^2$  and  $\sigma^2$  being replaced by  $s_n^{*2}$  and  $\sigma$ , respectively. Thus, in  
 (6.12), we attach a probability  $\geq 1 - O(n^{-1-\delta})$ , while in (6.11), (6.15)  
 applies. Hence, the result follows

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