VIBRATION OF LIQUID-FILLED THIN SHELLS

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This paper describes the analysis of free and forced vibration of a thin, axisymmetric shell, which contains some liquid. The axis of symmetry is vertical. Only such vibration is considered which can be produced by a horizontal movement of the base of the shell.

The objective of this paper is to examine the response of the coupled shell-liquid system for a frequency range lying between zero and the lowest natural frequency of the empty shell, which is assumed to be above the lowest natural sloshing frequency of the liquid.

The mass of the liquid is modeled by a stationary and one or more sloshing masses. It is shown how the stationary mass can be incorporated in the vibration analysis of the shell and how the natural frequency of the coupled shell-liquid system can be obtained from a simple formula, if the lowest natural frequency of the shell, plus the stationary mass of the liquid, can be determined. A numerical example is given.

1. INTRODUCTION

This paper describes the horizontal vibration of a thin, axisymmetric shell container, holding some liquid, for which the axis of symmetry is vertical (Figure 1). The vibration can be such that either the axis of symmetry moves or that it does not move and remains straight. The first case is called lateral sloshing. When the circumferential variation of the applied loads and the response is described by cos(nθ), then it corresponds to solutions with n=1. (n is called the circumferential wave number and θ is the circumferential coordinate angle.) The second case is called breathing, and it corresponds to all solutions with other circumferential wave numbers but n=1. This paper deals only with lateral sloshing modes with n=1.

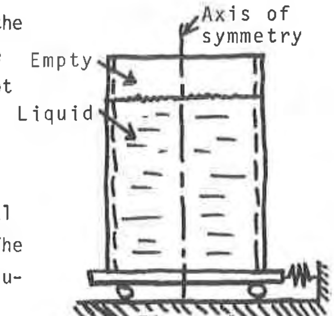


Figure 1

The liquid and the shell structure are two separate systems that are coupled. Each system, acting alone, has an infinite number of modes of free vibration. If the coupled system is excited with some forcing frequency ω, then the response will also have the frequency ω. The magnitude of the response will depend on the location of ω with respect to the natural frequencies of the coupled system.

This paper will deal with low-frequency excitation. The forcing frequency will be assumed not much higher than the lowest natural frequency of the shell when it is vibrating without the liquid. Two types of analyses are involved: (1) the calculation of the natural frequencies of the coupled system; and (2) the determination of the response of the coupled system to given loads that are oscillating with a given forcing frequency. This paper describes the simulation of the coupling of the shell with the liquid and gives specific procedures for the two types of analyses.

2. APPROACH

It is assumed that the liquid can be separated in two distinct layers. The lower layer does not participate in sloshing; it simply moves with the walls of the shell. The mass of liquid of that layer is called the stationary mass. The upper layer participates in sloshing; its mass is called the sloshing mass of the liquid. Then the coupled shell-liquid system is assumed to consist of two separate systems: (1) the shell together with the stationary mass; and (2) the sloshing mass. Each of the separate systems can be represented by a one-degree-of-freedom, spring-mass model.

In most cases, the lowest natural frequency of the liquid in a rigid shell is much lower than that of an elastic shell. This means that in the frequency range considered many natural frequencies of the liquid are encountered. However, it has been observed that the mass of the liquid that participates in the sloshing decreases rapidly with increasing natural frequency. For this reason it will be assumed in this paper that the coupling of the shell with all but the lowest mode of the liquid is negligible.

The assumption that only the lowest mode of the liquid is significant has been made earlier by Housner [1,2]. The best treatment of the complete problem is given by Abramson [3]. Chapter 6 of [3] gives the analysis for all modes of free vibration of the liquid and includes the remark on p. 201 that "... it is generally acceptable to include in the mechanical model only the mass of the fundamental mode...".

With such assumptions, the coupled liquid-shell system can be represented by a two-degree-of-freedom system as shown in Figure 2, where m_1 is the mass of the liquid that participates in sloshing, K_1 is the equivalent spring constant of the sloshing mass, M is the equivalent mass of the shell, including both the mass of the shell material and that of the liquid that does not participate in sloshing, and K is the equivalent stiffness, calculated under the same conditions as M.

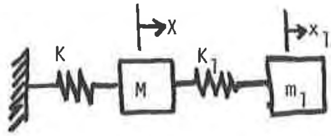


Figure 2

At this stage, the vibration of the coupled system is modeled as shown in Figure 3. The values for m_1, m_0, K_1 and ℓ_1 are given by formulas on p. 204 of [3] for circular cylindrical shells and on pp. 213-214 of [3] for spherical shells. The values of K and M can be obtained from standard computer programs, such as, for example, the KSHEL program.

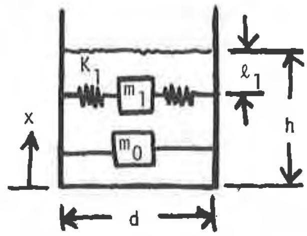


Figure 3

Once the values of m_1, K_1, M, and K are known, the natural frequencies of the coupled system, assuming the coupling as shown in Figure 2, can be computed from well-known formulas. The natural frequencies of a two-mass system are given, for example, in [4] in the form

$$\omega^2 = A \pm \sqrt{A^2 - B} \tag{1}$$

$$A = (K/M + K_1/M + K_1/m_1)/2$$

$$B = (K_1/m_1)(K/M)$$

The most difficult part of the analysis is the calculation of the equivalent mass M and stiffness K for the shell together with the stationary mass, m_0. That will be explained in the following section.

3. EQUIVALENT MASS AND STIFFNESS OF SHELL

The object of this section is the calculation of the equivalent parameters that represent the shell and the stationary mass when they are vibrating in the lowest mode. The equation of motion of mass M (Figure 2) is

$$M\ddot{X} + KX - K_1(x_1 - X) = 0 \tag{2}$$

where X is the displacement of mass M and x_1 is the displacement of mass m_1. The equivalent equation for a shell is obtained from a paper of Kraus and Kalnins [5]. When the shell vibrates in the lowest mode, then its displacement vector can be written as [see Eq.(4) of [5]]

$$\vec{u} = \vec{u}_1 q(t) \tag{3}$$

where $\vec{u}_1$ is the displacement vector of the lowest mode of free vibration and q(t) is determined from [see Eq. (12) of [5]]

$$M_1\ddot{q} + M_1\omega_1^2 q = F_1 \tag{4}$$

where ω_1 is the natural frequency of the lowest mode,

$$M_1 = \int_S \rho h \vec{u}_1 \cdot \vec{u}_1 dS \tag{5}$$

$$F_1 = \int_S \vec{p} \cdot \vec{u}_1 dS \tag{6}$$

and ρ is mass density, h is thickness, S is the reference surface of the shell, and $\vec{p}$ is the surface load vector that is causing the vibration of the shell.

It is assumed that the spring-mass system of the sloshing liquid is attached to the shell at one latitude circle which is located at the distance ℓ_1 as shown in Figure 3. This means that the spring exerts a horizontal force to the shell at that location of the magnitude

$$F = K_1(x_1 - X) \tag{7}$$

The surface load vector, $\vec{p}$, that appears in Eq. (6), must be determined in such a way that it represents the applied force F in Eq. (7). Since F is applied at one value of x, where x is the meridional arclength of the shell, then its distribution with x must be the delta function. It will be assumed also that the force F is distributed uniformly around the circumference of the shell and that F is acting horizontally and parallel to the x_1 axis, as shown in Figure 4. This means that

$$\vec{p} = (F/2\pi R_0)\delta(x-x_0)\vec{i} \tag{8}$$

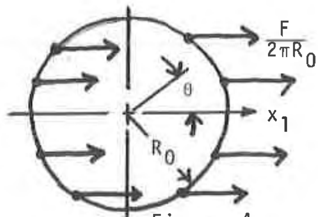


Figure 4

where R_0 is the radius of the latitude circle at the location where the spring is attached, $\delta(x)$ is the delta function, and $\vec{i}$ is a unit vector parallel to the x_1 axis (Figure 4). Insertion of $\vec{p}$ from Eq. (8) into (6) gives

$$F_1 = \int_0^{2\pi} \int_a^b (F/2\pi R_0)\delta(x-x_0)\vec{i}\cdot\vec{u}_1 dx r d\theta \tag{9}$$

where a and b are the endpoints of the meridian of the shell and x_0 is the value of x where the spring is attached. Using the property of the delta function that

$$\int_a^b \delta(x-x_0)f(x)dx = f(x_0) \tag{10}$$

F_1 becomes

$$F_1 = FU_1 \tag{11}$$

where U_1 is the displacement of the shell wall parallel to the x_1-axis (Figure 3) at the location where the spring is attached. U_1 is taken from the free vibration mode shape of the shell and the stationary mass m_0 vibrating in the lowest mode. U_1 is expressed by the usual shell displacements as

$$U_1 = (u_1 - u_\theta)/2 \qquad (12)$$

$$u_1 = w\sin(\phi) + u_\phi\cos(\phi) \qquad (13)$$

w, u_ϕ, u_θ are the normal, meridional, and circumferential displacements of the reference surface of the shell. The direction of u_1 is shown in Figure 5. The circumferential variation of these displacements is $\cos(\theta)$; all other θ-variations vanish when integrated with respect to θ, when going from Eq. (9) to (11). This is a direct consequence of the assumption that F, exerted by the attached spring, is uniformly distributed around the circumference.

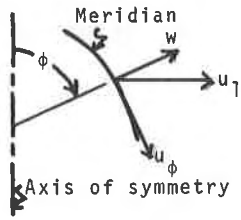


Figure 5

Putting F from Eq. (7) into (11), yields

$$F_1 = K_1(x_1 - X)U_1 \qquad (14)$$

and then into (4) gives

$$M_1\ddot{q} + M_1\omega_1^2 q = K_1(x_1 - X)U_1 \qquad (15)$$

The relationship between X and U_1 is given from Eq. (3) as

$$X = U_1 q(t) \qquad (16)$$

which gives Eq. (15) the form

$$(M_1/U_1^2)\ddot{X} + (M_1/U_1^2)\omega_1^2 X = K_1(x_1 - X) \qquad (17)$$

This equation reveals the equivalence of the vibration of the shell with that of the two-degree-of-freedom system shown in Figure 2. It becomes identical to Eq. (2) if

$$M = M_1/U_1^2 \quad , \quad K = \omega_1^2(M_1/U_1^2) \qquad (18)$$

These relationships give the equivalent parameters for the representation of the shell and the mass m_0 by a spring-mass system. The calculation of the lowest mode of free vibration is required. The KSHEL computer program gives ω_1, M_1, and U_1 directly on the output. When M and K are known, then the natural frequencies of the coupled system (shell plus m_0 with sloshing mass m_1) can be calculated from Eq. (1).

4. JOINING OF SHELL WITH STATIONARY AND SLOSHING MASSES

If the shell is assumed rigid, then it makes no difference whether the stationary mass m_0 and the sloshing mass m_1 are attached to the shell at single points or distributed over portions of the shell wall, as they actually are. For an elastic shell, it does make a difference.

Attaching the stationary mass at one point is neither reasonable nor convenient. The precise distribution of the stationary mass along the shell wall can be determined by considering the effect of the motion of the liquid from the point of view of fluid mechanics.

Such a determination can be carried out*, but is regarded as lying outside the scope of this paper. A simple procedure is to distribute the stationary mass m_0, as calculated from the formulas of [3], uniformly along the shell wall, from the bottom to the point where the spring of the sloshing mass is attached. Admittedly, this is a very rough approximation, but the addition of m_0 can then be effected very simply by increasing the mass density of the shell material by the amount of m_0/V, where V is the volume of the shell wall from the bottom to the point where the sloshing spring is attached. Such a procedure will give much better results for M_1, ω_1, and U_1 of the lowest mode of free vibration because it will not distort the mode shape, as a mass will, when attached at one point.

The attachment of the sloshing mass at one point as located by the distance ℓ_1 (Figure 3) seems reasonable, because that location is needed only for the determination of U_1 in the shell-m_0 free vibration problem. Even though the spring of the sloshing mass should be distributed over a portion of the shell wall, it seems reasonable to assume that its effect on the shell will not change appreciably when that portion is narrowed. Natural frequencies are overall characteristics of systems, and usually do not change much by local redistributions of properties.

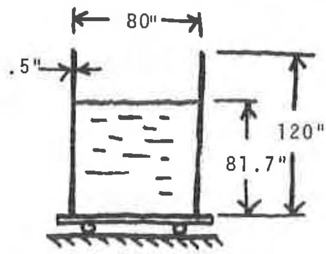
5. FORCED RESPONSE OF COUPLED SYSTEM

Once the location of the sloshing mass is determined, and the manner of the distribution of the stationary mass is decided upon, the forced response of the coupled system can be obtained by using standard computer programs, such as the KSHEL program, which does admit the attachment of a uniformly distributed spring-mass system at any latitude circle and an arbitrary mass distribution.

In most cases, the sloshing frequency is well below that of the shell. When plotting some maximum response of the shell vs. the forcing frequency, then a sharp peak will occur in the response at the sloshing frequency, indicating a resonating sloshing layer of the liquid beating against the wall of the shell. For higher frequencies, the effect of the sloshing on the response of the shell becomes negligible and the resonances of the shell with the stationary mass play a dominant role.

6. EXAMPLE

A cylindrical shell, shown in Figure 6, is filled with LNG, with mass density of $\rho=0.467\times10^{-4}$ lb-sec^2/in^4, up to the height of 81.7 inches. With the values of d=80", h=81.7", acceleration due to gravity g=385.9 in/sec^2, $\rho=0.467\times10^{-4}$ lb-sec^2/in^4, the following is calculated from Table 6.1 on p. 204 of [3]:



* In private communications, Dr. Y. N. Chen of the American Bureau of Shipping in New York, and Dr. Dieter Fischer of VÖEST in Linz, Austria, have indicated to the author that they have worked out more accurate distributions for the stationary mass.

Table 6.1. - Model Parameters for Cylindrical Tank

Spring-mass analogy

$m_T = 19.18$

$K_1 = 75.96$

$m_1 = 4.26$

$m_0 = 14.92$

$\ell_1 = 21.7$

$$K_1 = m_T\left(\frac{g}{1.19h}\right)\left[\tanh\left(3.68\frac{h}{d}\right)\right]^2$$

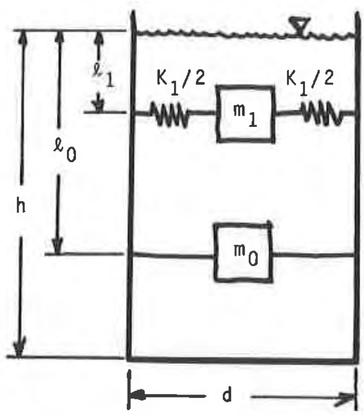
$$m_1 = m_T\left(\frac{d}{4.4h}\right)\tanh\left(3.68\frac{h}{d}\right)$$

$$m_0 = m_T - m_1 \qquad m_T = \frac{1}{4}\pi\rho d^2 h$$

$$\ell_1 = \frac{d}{3.68}\tanh\left(3.68\frac{h}{d}\right)$$

$$\ell_0 = \frac{m_T}{m_0}\left[\frac{h}{2} - \frac{d^2}{8h}\right] - \ell_1\frac{m_1}{m_0}$$

m_T = total mass of liquid



The total volume of the shell wall, from bottom to the point where m_1 is attached, is

$$V = \pi 80 \times 60 \times 0.5 = 7540 \text{ in}^3$$

The added density of the stationary mass m_0 is then

$$\rho_{m_0} = m_0/V = 19.8\times10^{-4}\text{lb-sec}^2/\text{in}^4$$

The mass density of the shell material is $\rho_{shell} = 7.3\times10^{-4}$ lb-sec^2/in^4, Young's Modulus $E = 30\times10^6$ psi, and Poisson's Ratio $\nu = 0.3$.

These mass densities are used for the calculation of the lowest mode of free vibration and for the response to some applied loads of the shell-m_0 system. The density for the lower 60" of the shell is

$$\rho = 19.8\times10^{-4} + 7.3\times10^{-4} = 27.1\times10^{-4} \text{ lb-sec}^2/\text{in}^4$$

and for the upper 60" of the shell it is

$$\rho = 7.3\times10^{-4} \text{ lb-sec}^2/\text{in}^4$$

The lowest mode of free vibration of the shell-m_0 system was calculated with the KSHEL computer program with the following results:

$$\omega_1 = 117.7 \text{ cps} = 739.5 \text{ radians/sec}$$

$$M_1 = \pi \times 2.062 = 6.478 \text{ (lb-sec}^2\text{/in)in}^2$$

The displacements at the point where the sloshing mass m_1 is attached (at the point 60" from the bottom of the shell) are given as

$w = u_1 = 0.6012$ (Note that ϕ=90° for a cylindrical shell; ϕ is needed in Eq. (13))

$u_\theta = -0.5558$

which gives the horizontal displacement from Eq. (12) as

$$U_1 = (0.6012 + 0.5558)/2 = 0.5785$$

Then the equivalent mass M and the stiffness K of the shell-m_0 system, that are needed to represent it as a spring-mass system, are given by Eq. (18) as

$$M = 6.478/0.5785^2 = 19.36 \text{ lb-sec}^2\text{/in}$$

$$K = 19.36\text{x}739.5^2 = 10.59\text{x}10^6 \text{ lb/in}$$

Finally, the two natural frequencies of the coupled shell-m_0 system and the sloshing mass m_1, when regarded as a two-degree-of-freedom system, are obtained from Eq. (1):

$$A = (10.59\text{x}10^6/19.36 + 75.96/19.36 + 75.96/4.26)/2 = 273.5\text{x}10^3$$

$$B = (75.96/4.26)(10.59\text{x}10^6/19.36) = 9754\text{x}10^3$$

Then

$$\omega_1 = 4.22 \text{ rad/sec}$$

$$\omega_2 = 739.6 \text{ rad/sec}$$

These results show that the sloshing of the fluid was much too weak to affect the free vibration of the shell-m_0 system. This is immediately evident by comparing the natural frequencies of the sloshing mass ($\omega=\sqrt{K_1/m_1}$=4.22 rad/sec) with that of the shell-m_0 system (ω=739.5 rad/sec). The stationary mass, m_0, however, affected the natural frequency considerably because the natural frequency of the shell alone, without the added mass m_0, was calculated at 869.8 rad/sec or 138.4 cps.

No example is displayed for the forced response problem because its results can be anticipated. At forcing frequencies near the 4.22 rad/sec value, there is some effect of the sloshing mass, but its effect becomes negligible at higher frequencies. Of course, the added stationary mass, m_0, has a considerable effect on the response.

7. REFERENCES

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