AN INVESTIGATION OF DEGENERATE ISOPARAMETRIC FINITE ELEMENTS FOR STRESS INTENSITY COMPUTATIONS

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Abstract

It is now widely accepted that quadratic isoparametric finite elements can be effectively used in fracture mechanics to compute stress intensity factors. To incorporate the theoretical $1/\sqrt{r}$ strain singularity which exists at the crack-tip, the eight-node isoparametric elements at the tip are usually modified by moving the mid-side nodes to the quarter distances from the tip. It has been observed that these elements behave even better when one side of the quadratic quadrilaterals are collapsed to form degenerate triangles.

However, when a quadratic quardilateral degenerates into a triangle, the shape function of the mid-side node at the edge opposite to the collapsed corner creates undesirable effects. Since, the shape function of the mid-side node varies linearly towards the collapsed node while maintaining a quadratic variation along the other natural coordinate, a surface with indeterminate slopes develops at the collapsed node. Although this is not inconsistent with the strain singularity of crack-tip elements, it leads to unstable results if the mid-side nodes across the tip are not located precisely on a straight line, because an incorrect strain singularity is obtained due to perturbations of the mid-side nodes. In addition, the results are sensitive to the particular way a quadrilateral element is degenerated and for correct singularity the collapsed nodes must be at the crack-tip.

For non-singular problems, a modification has been suggested in the literature to remedy this situation related with the troublesome shape function. The modification is realized by multiplying the particular shape function by a dimensionless measure of the distance from the degenerate node. In this way, the indeterminate slope situation is prevented.

In the present investigation, it is shown that when the troublesome shape function is properly modified, the degenerate elements behave considerably better both in singular and non-singular problems.

In order to demonstrate the improvement in results, first a deep cantilever beam is analyzed and it is observed that the modified elements yield far better results in non-singular problems. Second, a single edge crack specimen under uniform tension is considered. It is verified both analytically and numerically that after the modifications the correct $1/\sqrt{r}$ strain singularity is maintained in quarter point collapsed triangles irrespective to the geometry of the side opposite the crack tip, and no instabilities take place as a result of perturbations in the location of the mid-side node. Moreover, the same singularity is achieved in the modified elements independent of the position of the collapsed nodes.

1. Introduction

Degenerate quadratic triangles obtained by collapsing one side of eight-node isoparametric elements and by placing the two mid-side nodes at quarter distances from the collapsed node were shown to contain the correct $1/\sqrt{r}$ strain singularities, and were extensively employed as crack tip elements by many research workers in the past, e.g. {1-3}. Moreover, to compute the stress intensity factors in these elements, an expression consistent with the element interpolation function and the analytical displacement expansion near the crack tip {4,5} has been proposed and favoured by several, e.g. {6-8}. Also, more recently, the collapsed cubic isoparametric elements have been employed as singularity elements in elastic crack problems by changing the positions of side nodes near the tip {9}.

On the other hand, it is known that when a quadratic isoparametric quadrilateral is degenerated into a triangle, the shape function of the mid-side node of the edge opposite to the collapsed corner creates some undesirable effects {10}, since due to improper degeneration of the particular shape function, multivalued displacement gradients are obtained at the collapsed node. Although this is not inconsistent with a crack tip singularity, it has been observed that these elements yield unstable results if mid-side node opposite to the collapsed node is not located precisely on a straight line. Slight perturbations in placing the mid-side node results with considerable errors, since due to perturbations an incorrect strain singularity is obtained in the element {7}. Hence an accurate placement of the mid-side nodes opposite to the crack tip is essential for these elements. The same instability has also been experienced in cubic elements {9}, because a similar difficulty exists in the degeneration of the shape functions.

To maintain the correct degeneration in quadratic elements, a shape function modification has been suggested by Irons {10} for non-singular problems, although no numerical results have been presented. The present paper investigates the effects of this modification on the performance of degenerate triangular elements in singular as well as in non-singular problems, and shows the overall improvement in results.

2. Eight-Node Isoparametric Element and Collapsed Triangle

2.1 Non-Singular Problems

A typical eight-node isoparametric parent element is shown in Fig. 1a, for which the shape functions in curvilinear coordinates ξ and η are {11} :

a) At mid-side nodes

$$N_i = \frac{1}{2}(1-\xi^2)(1+\eta\eta_i) \quad , \quad i = 5,7 \tag{1.a}$$

$$N_i = \frac{1}{2}(1+\xi\xi_i)(1-\eta^2) \quad , \quad i = 6,8 \tag{1.b}$$

b) At corner nodes

$$N_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i) - \frac{1}{2}(N_{i+3}+N_{i+4}) \quad , \quad i = 2,3,4 \tag{1.c}$$

$$N_1 = \frac{1}{4}(1+\xi)(1+\eta) - \frac{1}{2}(N_5+N_8) \tag{1.d}$$

where ξ_i and η_i are the coordinates of node i.

A collapsed triangular element obtained by superimposing the nodes 2,3 and 6 of the parent square element is shown in Fig. 1b. All shape functions in this case degenerate into those of a regular six-node triangle, except the shape function of the mid-side node across the collapsed corner {10}. Since this shape function varies linearly along radial lines η = constant from the collapsed node and changes quadratically in the other direction η, a nonpolynomial surface with indeterminate slopes is formed at the collapsed node as illustrated in Fig. 1c.

For correct response of the triangle this shape function is to be modified so that it varies quadratically also in the ξ direction. This is accomplished by introducing the following modification on N_8 to become

$$N_8^* = N_8(1 + \xi)/2 \tag{2}$$

for the triangle shown in Fig. 1b. To complete the modifications, N_8 appearing in the shape functions N_1 and N_4 is replaced by N_8^*, while for the remaining nodes we have $N_i^* = N_i$. The new shape functions still satify the constant strain condition $\sum_{i=1}^{8} N_i^* = 1$, and the usual property $N_i^*(\xi_j, \eta_j) = \delta_{ij}$ at the nodal points, and at the same time the indeterminate slope situation at the collapsed corner is prevented.

In order to investigate the effects of this modification first in non-singular problems, a deep cantilever beam shown in Fig.2 is analyzed. For the numerical integration of the stiffness matrix a 3-point Gauss rule per coordinate direction is employed and the stresses are computed at the Gauss integration points. Three cases of degeneracy shown in Fig.2 are considered. The values obtained : a) for vertical deflection at x = 48", y = 6" position, b) for horizontal normal stress at x = 8", y = 9" position, on the beam are given in Table I, where all values are normalized with respect to plane stress elasticity solution {12}. As it is seen, the unmodified triangles behave rather badly, and the results are sensitive to the way the elements are degenerated. The modified triangles, on the other hand, yield results which are invariant for the cases considered, and are also far more accurate.

While the normalized deflection obtained with the modified collapsed triangles is $\bar{v} = 0.989$ at the tip, with the same mesh the regular six-node triangles yield $\bar{v} = 0.955$. Therefore, as far as the accuracy is concerned, the modified collapsed triangles are competitive with the regular six-node triangles.

2.2 Quarter Point Collapsed Triangles for Singular Problems

It can be shown that {1-3}, if mid-side nodes of a collapsed triangle are moved to the quarter positions towards the collapsed node as shown in Fig. 3a, a displacement variation of the form

$$u_i = A_i + B_i \sqrt{r} + C_i r \quad , \quad i = x,y \tag{3}$$

is obtained in the radial r directions emanating from the crack tip, where A_i, B_i and C_i are constants independent of r. Consequently, the strain components vary as

$$\varepsilon_{ij} = a_{ij} + b_{ij}/\sqrt{r} \tag{4}$$

which corresponds to the theoretical crack tip expansion {4,5}. Since it is precisely the $\sqrt{r}$ term in eq. (3) which contributes to $1/\sqrt{r}$ strain singularity in eq. (4), a consistent way of obtaining expressions for stress intensity factors in sufficiently small crack tip elements is to equate the coefficients of the $\sqrt{r}$ term in the analytical and numerical displacement expansions near the tip {6-8}.

For instance, if we consider a crack tip element shown in Fig. 3a, the displacement variation along the crack surface of length L takes the form

$$u_n = U_n(0) + \{4U_n(L/4) - U_n(L) - 3U_n(0)\} \sqrt{(r/L)} + \{2U_n(L) + 2U_n(0) - 4U_n(L/4)\} r/L \quad (5)$$

where $U_n(0)$, $U_n(L/4)$, $U_n(L)$ are nodal point displacement components normal to the crack surface. On the other hand, for the crack opening mode, the coefficient of the term $\sqrt{r}$ in the analytical displacement expansion {4,5} along the crack surface is given by :

$$B_n = K_I(\kappa + 1)/(G \sqrt{8\pi}) \quad (6)$$

where K_I is mode I stress intensity factor, G is shearing modulus, $\kappa = (3-\nu)/(1+\nu)$ for plane stress, and $\kappa = (3-4\nu)$ for plane strain. Equating the coefficient of the corresponding term in eq. (5) to eq. (6), we obtain

$$K_I = [2G \sqrt{2\pi} / (\kappa+1)] \, [\{4U_n(L/4) - U_n(L) - 3U_n(0)\} / \sqrt{L}\,] \quad (7)$$

as the expression for mode I stress intensity factor.

The above expression proved to yield good results when unmodified collapsed triangles with length approximately equal to one tenth of the crack length were used as crack tip elements {7}. However the results were good only when the mid-side nodes across the crack tip were located on straight lines. For distorted geometries such as in Fig. 3b, an instability was observed and poor results were obtained {7}.

In the following sections we shall show both analytically and numerically that this instability is removed when the shape functions are modified in order to maintain the proper degeneration.

3. Investigation of Singularity : Modified Versus Unmodified Elements

If we consider the distorted collapsed triangle shown in Fig. 3b, the isoparametric mapping $x = N_i x_i$, $y = N_i y_i$ in the unmodified elements yields

$$y/x = y_o/x_o \; \{[\eta(1+\xi)] / [(1+\xi) + 2(\alpha-1)(1-\eta^2)]\} \quad (8)$$

or since $(1+\xi) = 2\sqrt{y/y_o\eta}$, we obtain

$$y/x = y_o/x_o \; \{[\eta\sqrt{y/y_o\eta}\,] / [\,\sqrt{y/y_o\eta} + (\alpha-1)(1-\eta^2)]\} \, . \quad (9)$$

Equation (9) implies that the radial lines $\eta = $ constant in the parent element are mapped into second order curves in the real element, except when $\alpha = 1$. Thus an improper representation of the asymptotic solution is obtained throughout the element unless $\alpha = 1$. This explains why good results can be obtained when the mid-side nodes are located on straight lines only.

For the same element shown in Fig. 3.b, if the modified shape functions N_i^* are used, the mapping becomes

$$y/x = y_o/x_o\{\eta/[1 + (\alpha-1)(1-\eta^2)]\} \, . \quad (10)$$

In this case, for all α values the radial lines η = constant in the parent element are mapped into straight radial lines in the real element. Thus a correct representation of the asymptotic solution is obtained irrespective to the geometry of the side across the collapsed node.

4. Numerical Example - A Single Edge Crack Specimen in Tension

In order to verify the findings of the previous section, we solve a single edge crack specimen under uniform tension. The plate dimensions are W = H = 10", the crack length is a = 4", and as shown in Fig.4 only the upper half of the plate is modeled. Plane stress conditions are assumed and uniform tension is applied at the far end. Also as shown in Fig.4 three cases of local mesh configurations I, II and III are considered near the crack tip, and for each mesh two schemes of node collapsing are employed. In scheme A the collapsed nodes are placed at the crack tip, in scheme B the collapsed nodes are placed away from the tip. In all cases L/a = 0.1 for the crack-tip elements, where L is a measure of element length indicated in Fig.4.

Normalized stress intensity factors $K_I/(\sigma_o \sqrt{\pi a})$ obtained by using eq. (7) are tabulated in Table II. As it is clearly seen, the unmodified elements are extremely sensitive both to the geometry of the side opposite to the tip, and to the location of the collapsed nodes. Whereas, the modified triangles are not affected by these, and even with the rather coarse mesh of 26 elements and 95 nodal points shown in Fig.4, they consistently yield good results. Obtained values are even better in the collapsing scheme B.

5. Conclusions

The shape function modifications incorporated herein to collapsed isoparametric triangles give improved results in non-singular problems. These modifications are essential if the elements are to be used for mesh gradation purposes in non-singular problems as well as in non-singular regions of singular problems {13}.

Unmodified collapsed elements yield good results in singular problems only if the sides across the crack-tip are perfectly straight. Such a restriction does not exist in modified elements, which is an important advantage, because for mesh generation purposes the use of elements with curved sides may be preferred in complicated problems.

For the unmodified elements, the collapsed nodes must be located at the crack tip, otherwise an incorrect singularity is obtained. Although it is verified only numerically here, the modified elements yield the correct singularity independent of the position of collapsed nodes.

Modifications are simple to implement and require no additional computational effort. Similar such modifications may also prevent the instabilities experienced in cubic collapsed triangles.

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Table I : Comparison of normalized deflections and stresses in the cantilever beam - Exact : $\bar{v} = 1$, $\bar{\sigma}_{xx} = 1$

Case	Unmodified		Modified	
	$\bar{v}$	$\bar{\sigma}_{xx}$	$\bar{v}$	$\bar{\sigma}_{xx}$
A	0.983	0.940	0.989	0.943
B	0.919	0.886	0.989	0.943
C	0.756	0.850	0.989	0.943

Table II: Normalized stress intensity factors Exact : $K_I/(\sigma_o \sqrt{\pi a}) = 2.11$ {7}

Mesh	Unmodified		Modified	
	Scheme A	Scheme B	Scheme A	Scheme B
I	2.12	1.55	2.08	2.10
II	2.14	1.68	2.10	2.12
III	1.51	1.81	2.06	2.11

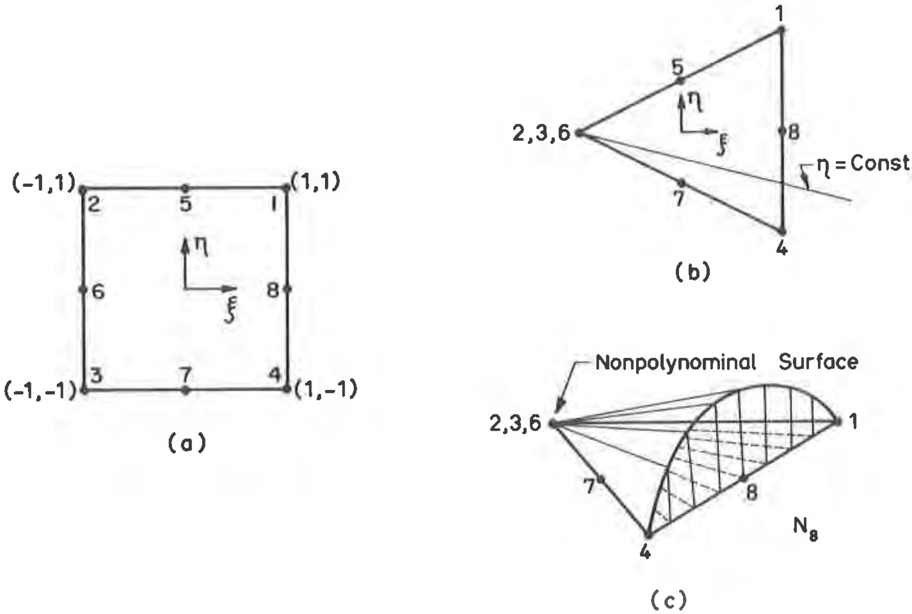

Figure 1. Eight-node isoparametric element : a) Parent square, b) Collapsed triangle (parent), c) Shape function N_8 of collapsed triangle.

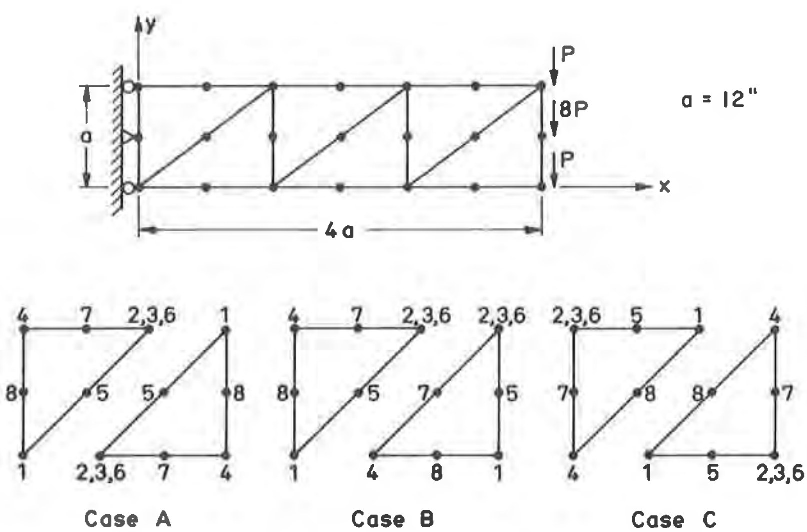

Figure 2. Cantilever plate model and cases of degeneracy.

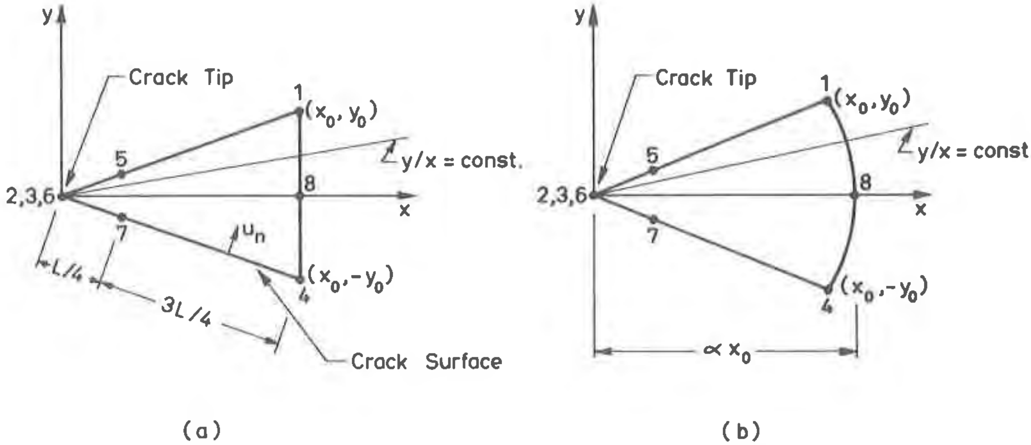

Figure 3. Quarter point collapsed triangles : a) Straight, b) Curved.

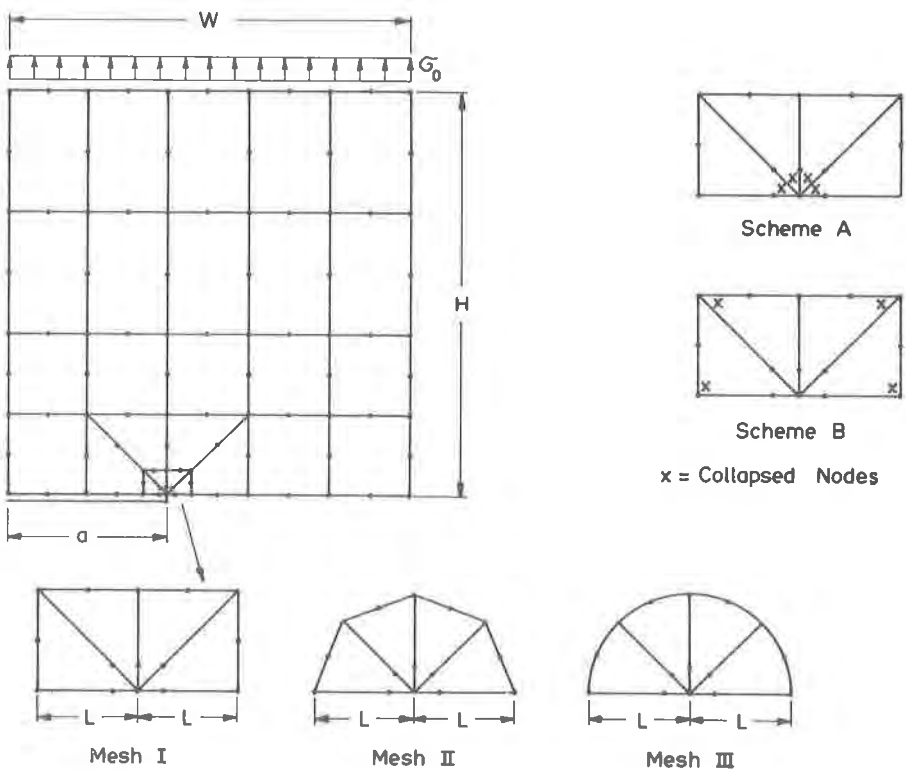

Figure 4. Single edge crack specimen in tension (H = W, a/W = 0.4, L/a = 0.1).