# Response variability of stochastic viscoelastic finite element systems

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## 1 INTRODUCTION

Structural dynamics problems generally involve two kinds of stochasticity at least to some extent: one in loading conditions and the other in structural system itself. The former gives rise to so-called random vibration problems which have been extensively studied in the last two decades or so. On the other hand, not much attention has been paid to so-called system stochasticity problems which originate from the latter, due primarily to the accompanying mathematical difficulties. The recent development of finite element methods has changed the outlook for alleviating such difficulties. Indeed, a number of papers were written where finite element methods were used to solve static and eigenvalue problems involving stochastic systems.

In this connection, Shinozuka and Dasgupta (1986) also developed the Neumann expansion method within the framework of finite element method. The Neumann expansion method solves system stochasticity problems utilizing a Neumann expansion of the stiffness matrix about its reference value. The reference value may represent the median stiffness, for example. Then, the expected value and variance of the response can be evaluated in approximation, either analytically by truncating the expansion after first few terms, or by means of Monte Carlo simulation. In general, the analytical approach proves to be impractical if we retain more than the first two terms of the Neumann expansion, however. Hence, the Monte Carlo simulation approach was used in conjunction with the algorithm which makes the simulation much more efficient than the direct Monte Carlo simulation where the stiffness matrix is factorized for each sample. The usefulness of this Neumann expansion-based Monte Carlo simulation technique was highlighted by Yamazaki, Shinozuka and Dasgupta (1985) for static problems by comparing its efficiency and accuracy with those of the direct Monte Carlo method and of the perturbation method.

In this paper, the Neumann expansion method is applied for the frequency domain analysis of dynamic problems involving stochastic systems. The response variability of viscoelastic finite element systems resulting from the spatial variation of their material properties is given in terms of the variability of the corresponding frequency response function. By expanding the complex dynamic stiffness matrix, the Neumann expansion method can be easily implemented. Also, a first-order approximation is formulated based on the perturbation method. Finally, numerical examples by those methods are worked out and compared.

## 2 VISCOELASTIC MODEL

Steady-state responses of a linear viscoelastic continuum are considered under harmonic excitations. One of the simplest linear viscoelastic model is known as the Kelvin model and it consists of an elastic solid with linear velocity-dependent damping as shown in Fig.1. Under two-dimensional plane stress condition, the viscoelastic solution for a harmonic motion

satisfies the following relationship:

$$\boldsymbol{\sigma}^0 = \mathbf{D}_{ve}\,\boldsymbol{\varepsilon}^0 \qquad \text{..... (1)}$$

in which $\boldsymbol{\sigma}^0$ and $\boldsymbol{\varepsilon}^0$ are the complex amplitudes of the stress and strain vectors of each element, respectively, and $\mathbf{D}_{ve}$ is the viscoelastic matrix as

$$\mathbf{D}_{ve} = \begin{bmatrix} \dfrac{4G^0(3K^0+G^0)}{3K^0+4G^0} & \dfrac{2G^0(3K^0-2G^0)}{3K^0+4G^0} & 0 \\ & \dfrac{4G^0(3K^0+G^0)}{3K^0+4G^0} & 0 \\ sym. & & G^0 \end{bmatrix} \qquad \text{..... (2)}$$

and

$$G^0 = \bar{G} + i\,w\,G\,\tau_k \qquad ; \qquad K^0 = K \qquad \text{..... (3)}$$

in which $G$ is the shear modulus, $K$ the bulk modulus, $\tau_k$ the retardation time and $\omega$ the circular frequency of the harmonic excitation. This viscoelastic model is implemented in the finite element formulation with the rectangular element. The symbolic manipulation program SMP (1983) is fully utilized in constructing the element stiffness and damping matrices.

## 3 FREQUENCY DOMAIN ANALYSIS

The equation of motion under base excitation is represented by

$$\mathbf{M}\ddot{\mathbf{U}} + \mathbf{C}\dot{\mathbf{U}} + \mathbf{K}\mathbf{U} = -\mathbf{M}\mathbf{r}\,\ddot{z}_g \qquad \text{..... (4)}$$

in which **M**, **C** and **K** are the global mass matrix, damping matrix and stiffness matrix, respectively, **r** is the forcing vector, $\ddot{z}_g$ is the base acceleration, **U** is the relative displacement vector and the super dots indicate the time derivatives. The steady state solution is obtained by substituting the following into Eq.4.

$$\ddot{z}_g = e^{iwt} \qquad ; \qquad \mathbf{U} = \mathbf{H}(\omega)\,e^{iwt} \qquad \text{..... (5)}$$

in which $\mathbf{H}(\omega)$ is a complex vector called the frequency response function. Thus, one obtains

$$(\mathbf{K} - \omega^2\mathbf{M} + i\omega\mathbf{C})\,\mathbf{H}(\omega) = -\mathbf{M}\mathbf{r} \qquad \text{..... (6)}$$

It is noted if the above viscoelastic model is adopted, **K** and **C** are real but functions of frequency. In this study, the variability of $\mathbf{H}(\omega)$ is investigated when **K**, **C** and **M** involve randomness due to the spatial variability of material properties.

## 4 SIMULATION OF STOCHASTIC FIELDS BY CHOLESKY DECOMPOSITION

For simulation and discretization of the stochastic field involved, the Cholesky decomposition method is adopted. First, structures are divided into an appropriate number of finite elements. The size of each finite element must be small enough from the spatial variability of the material property as well as strain and stress gradients point of view. From the former point of view, it must be small enough so that the property value can be considered approximately constant within each element. Hence, if there are $n$ finite elements in total, then there are also $n$ material property values associated with these $n$ elements. Considering only the fluctuating component of the variation, the correlational characteristics of these $n$ values, $a$'s, with mean zero can be specified in terms of their covariance matrix, $\mathbf{C}_0$, whose $ij$-component is given by

$$c_{ij} = Cov[\,a_i, a_j\,] = E[\,a_i\,a_j\,] = R_{ff}(\xi_{ij}) \qquad \text{..... (7)}$$

in which $\xi_{ij}$ is a separation between the centroids of elements $i$ and $j$, $R_{ff}(\cdot)$ is an auto-correlation

function of the stochastic field. The following form of isotropic auto-correlation function is considered in the present analysis.

$$R_{ff}(\xi_{ij}) = \sigma_0^2 \exp[-\xi_{ij}^2/d^2] \quad \ldots\ldots\ldots \quad (8)$$

in which $d$ is a real parameter such that the larger it is, the more slowly the correlation disappears, and $\sigma_0$ is the standard deviation of the stochastic field.

A sample vector $\alpha=[\alpha_1\ \alpha_2\ \ldots\ \alpha_n]^T$ can be generated by $\alpha=LZ$ with $Z=[z_1\ z_2\ \ldots\ z_n]^T$ being a vector consisting of independent Gaussian random variables with zero-means and unit standard deviations. L is the lower triangular matrix obtained by the Cholesky decomposition of the covariance matrix : $C_0=LL^T$. Thus, once the Cholesky decomposition is carried out, different sample vectors, $\alpha$'s, are easily obtained by generating different samples of Z.

## 5 NEUMANN EXPANSION METHOD

For each frequency, Eq.6 can be rewritten in the following simpler form by replacing $K-\omega^2 M+i\omega C=D$, $-Mr=F$ and $H(\omega)=X$ as

$$D\ X = F \quad \ldots\ldots\ldots \quad (9)$$

Since D is a function of frequency, Eq.9 must be solved for each frequency. However, for each frequency, Eq.9 is in the same form as that of static problems except for the fact that D, F and X are all complex. Assume that D and F contain spatially variable parameters, and then decompose both D and F into two parts,

$$D = D_0 + \Delta D \quad ; \quad F = F_0 + \Delta F \quad \ldots\ldots\ldots \quad (10)$$

in which $D_0$ and $F_0$ are obtained by substituting the median values of the spatially variable parameters into D and F while $\Delta D$ and $\Delta F$ represent the deviatoric parts of D and F, respectively. The direct Monte Carlo simulation is a method to solve Eq.9 directly for each sample system by substituting the simulated basic random variables into D and F. On the other hand, in the Neumann expansion method, first, $D^{-1}$ is expanded in the following form:

$$D^{-1}=(D_0+\Delta D)^{-1} = (I-P+P^2-P^3+\ldots..)D_0^{-1} \quad \ldots\ldots\ldots \quad (11)$$

where I is an identity matrix of appropriate dimension and $P=D_0^{-1}\Delta D$. Thus, the solution of Eq.9 is obtained as

$$X=D^{-1}F = (I-P+P^2-P^3+\ldots..)D_0^{-1}(F_0+\Delta F) \quad \ldots\ldots\ldots \quad (12)$$

$$= X_0 - X_1 + X_2 - X_3 + \ldots..$$

If the L-U decomposition of $D_0$ is carried out once as $D_0=L_0U_0$, $X_i$'s ($i=1, 2, \ldots$) can be obtained very efficiently without requiring further matrix factorization. The series in Eq.12 can be terminated after a few terms if the convergence of the series is confirmed. The expected value and variance of X are estimated by taking the statistics of resulting sample solutions.

## 6 PERTURBATION METHOD

The response variability can also be evaluated by the perturbation method, which has been intensively studied by several researches (e.g., Hisada and Nakagiri 1985). Recent work by Yamazaki, Shinozuka and Dasgupta (1985) has demonstrated that the second-order perturbation approximation is obtained only after inordinate amount of computational efforts arising from the four-fold summations involved. Thus, first-order approximation is used in this study. First, D and F in Eq.9 involving random variables, $a_i$'s ($i=1,2, \ldots, n$), are expanded about their expectations as

$$D = D^0 + \sum_{i=1}^{n} D_i^I a_i + \cdots \quad ; \quad F = F^0 + \sum_{i=1}^{n} F_i^I a_i + \cdots \quad \ldots\ldots\ldots \quad (13)$$

in which the partial derivatives of D and F are calculated as

$$\mathbf{D}_i^{\mathrm{I}} = \left.\frac{\partial \mathbf{D}}{\partial a_i}\right|_{a=0} \quad ; \qquad \mathbf{F}_i^{\mathrm{I}} = \left.\frac{\partial \mathbf{F}}{\partial a_i}\right|_{a=0} \qquad \cdots\cdots\cdots (14)$$

Then, the solution **X** is assumed to be given in a similar series form :

$$\mathbf{X} = \mathbf{X}^0 + \sum_{i=1}^{n} \mathbf{X}_i^{\mathrm{I}} a_i + \cdots\cdots \qquad \cdots\cdots\cdots (15)$$

Substituting Eqs.13 and 15 into Eq.9 and comparing the zero- and first-order terms of the both sides of the equation, the coefficients vectors in Eq.15 are obtained by

$$\mathbf{X}^0 = (\mathbf{D}^0)^{-1}\mathbf{F}^0 \quad ; \quad \mathbf{X}_i^{\mathrm{I}} = (\mathbf{D}^0)^{-1}(\mathbf{F}_i^{\mathrm{I}} - \mathbf{D}_i^{\mathrm{I}}\mathbf{X}^0) \qquad \cdots\cdots\cdots (16)$$

Truncating the right-side term of Eq.15 after the second term, the expected value of **X** is evaluated by $E[\mathbf{X}] = \mathbf{X}^0$. Also, the covariance matrix of **X** is obtained by the analytical form :

$$Cov[\mathbf{X},\mathbf{X}^*] = E[(\mathbf{X}-E[\mathbf{X}])(\mathbf{X}^*-E[\mathbf{X}^*])^T] = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{X}_i^{\mathrm{I}}(\mathbf{X}^{*\mathrm{I}}_j)^T E[a_i\, a_j] \quad \cdots. (17)$$

in which asterisks indicate complex conjugates and $E[a_i\, a_j]$ is determined from Eq.7. In this study, only the diagonal components of the covariance matrix are calculated.

## 7 NUMERICAL EXAMPLE

The finite element model shown in Fig.2 is adopted as an example. The model consists of 36 plane stress square elements with 49 nodes. The nodes along the lower edge of the model are fixed and the fixed base is excited by harmonic motions in the horizontal direction. The dimensions of the model are chosen to represent a realistic shear wall-type building which is often seen in nuclear facilities. The medians of the material properties (Young's modulus, Poisson's ratio, the mass density and retardation time) are also chosen to represent ordinary reinforced concrete structures as: $E_0 = 5.18\times10^5$ kips/ft$^2$, $\nu_0 = 0.2$, $\rho_0 = 0.00467$ kips·sec$^2$/ft$^4$ and $\tau_{k0} = 8.4\times10^{-4}$sec. The frequency response function using these median values is shown in Fig.3. Although the model has 84 degrees-of-freedom, the model is close to a single-degree-of-freedom system because all the other natural frequencies are out of the practical range for the earthquake response analysis.

As the first example of the probabilistic analysis, Young's modulus is assumed to be log-normally distributed and all the other parameters are assumed to be deterministic. The logarithmic standard deviation $\zeta = 0.1$ and the correlation parameter $d = 30$ ft are used throughout the analysis. The sample fields simulated by the Cholesky decomposition algorithm are depicted in Fig.4. The sample size of Monte Carlo simulation is selected as 100, and it was proved to be large enough for the present case. The expected value and standard deviation of the frequency response function are shown in Figs.5 and 6. The results by the Neumann expansion method are almost identical to that of the direct Monte Carlo method, because the convergence is confirmed for each sample. The results by the first-order perturbation method show a large difference from those of the direct Monte Carlo method around the natural frequency of the median system ($\omega = 119$ rad/sec). Note that the results by the direct Monte Carlo simulation are considered to be exact. The cumulative probability of $H(\omega)$ by the direct Monte Carlo method at $\omega = 119$ rad/sec is plotted on a log-normal probability paper in Fig.7. Although the input basic random variables are log-normal, the response is found to be highly skewed.

As the second example of the spatial variability, the damping parameter (the retardation time) is assumed to follow a log-normal distribution. The results by the Neumann expansion method, the direct Monte Carlo method and the first-order perturbation method are shown in Figs.8 and 9, and all of them coincide excellently in this case. The CPU time to obtain those results are listed in Table 1. It is noted that the first-order perturbation method is the fastest. Also, the Neumann expansion method is very efficient for reducing the large CPU time which the direct Monte Carlo method requires. The third example involves the mass density

variation. The results in this case exhibit the same characteristics as those derived from Young's modulus variation.

## 8 CONCLUSION AND ACKNOWLEDGMENT

The Neumann expansion method is presented as a new tool to solve system stochasticity problems. By expanding the dynamic stiffness matrix about its reference values, the Neumann expansion method can be introduced in the dynamic finite element analysis. Since the matrix factorization is required only once for all the samples in the Neumann expansion method, Monte Carlo simulation becomes very efficient. The usefulness of this Neumann expansion-based Monte Carlo approach is highlighted by comparing its efficiency and accuracy with those of the direct Monte Carlo method and of the first-order perturbation method.

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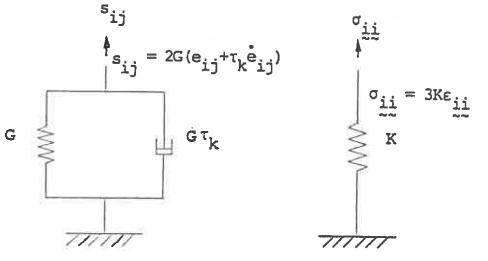


Figure 1. Viscoelastic model

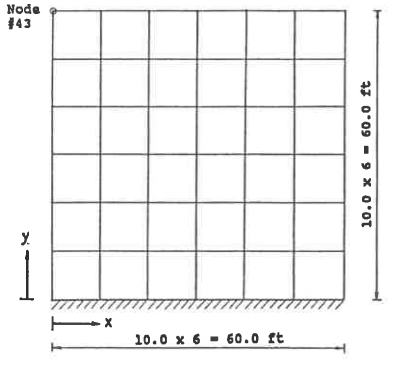


Figure 2. Finite element model

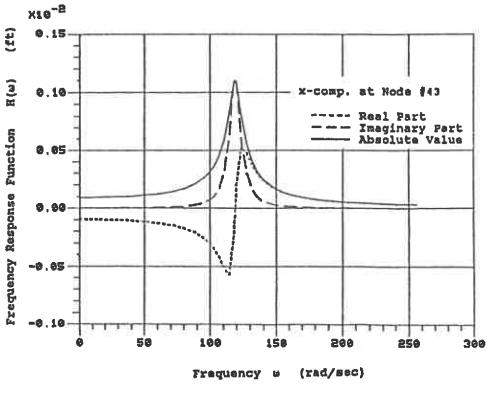


Figure 3. Frequency response function for model with median material properties

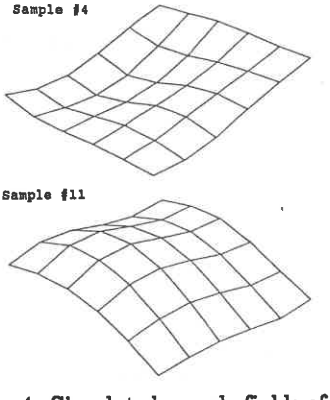


Figure 4. Simulated sample fields of Young's modulus

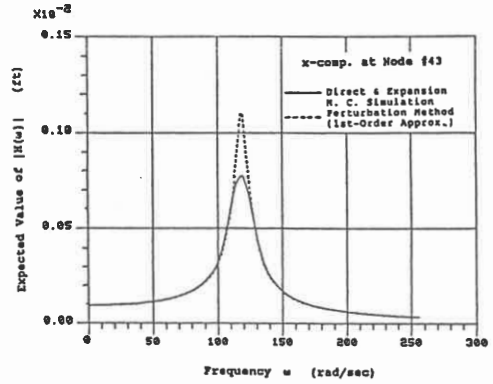


Figure 5. Expected value of frequency response function ( variation of $E$ )

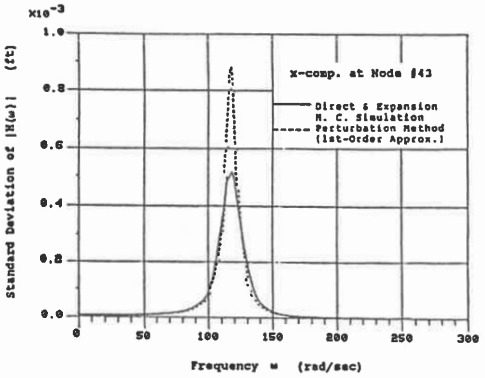


Figure 6. Standard deviation of frequency response function ( variation of $E$ )

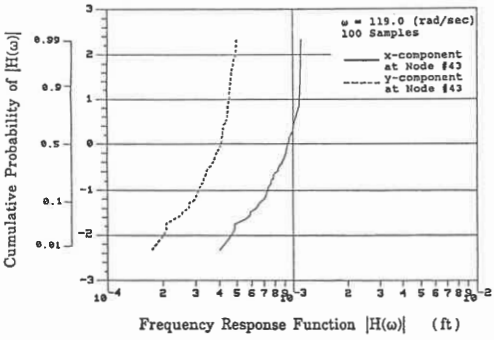


Figure 7. Cumulative probability of frequency response function ( variation of $E$ )

Table 1. Comparison of CPU time

| Monte Carlo simulation (sample size = 100) | | Perturbation method (first-order approx.) |
|---|---|---|
| Direct method | Neumann expansion | |
| 28:38:55 | 04:48:10 | 00:35:27 |

VAX-11/750 computer is used. ( hr : min : sec )

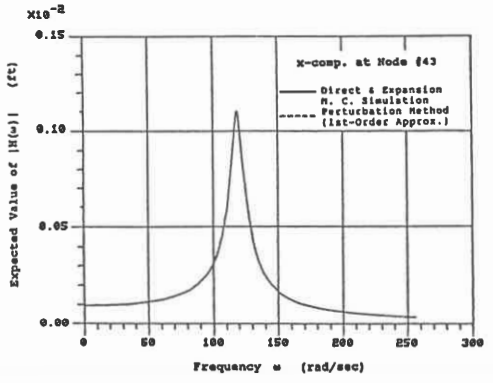


Figure 8. Expected value of frequency response function ( variation of $\tau_k$ )

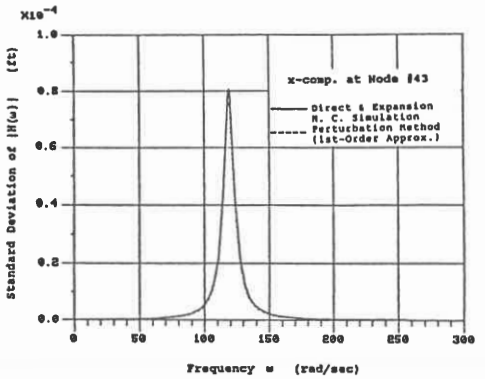


Figure 9. Standard deviation of frequency response function ( variation of $\tau_k$ )