Symmetric Primal-Dual Variational Formulations of Coupled Dynamic Systems: Three-Dimensional Elasticity, Shells, Fluid-Structure Problems

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Summary

We present a theory and a methodology for the coupling of domains where primal displacement and dual stress variational formulations are used respectively, in order to obtain a symmetric variational formulation for the linear transient or modal analysis of coupled systems.

This is done by using an auxiliary field variable in the region where the dual stress analysis is carried out, leading, in this region, to a new dynamic mixed variational formulation.

Generalized so-called added mass operators are also introduced in order to eliminate afterwards, in many cases, the auxiliary field variable.

The methodology is described for three-dimensional elasticity and shell problems. Compressible fluid-structure interaction in bounded medium is presented as a particular case of elastic problems.

1. Introduction : The problem of stress computation in mechanical structures is rather badly solved by the application of the displacement finite element method (primal method) in numerous difficult cases (high stress gradients).

In order to obtain more accurate stresses, one should utilize dual or mixed variational principles of mechanics of continuous media [1, 2].

The application of dual or mixed variational formulations to the entire structure may involve too many unknowns. In order to take benefit of those principles without loosing the advantage of simplicity offered by the primal displacement method, the following methodology has been used [2, 3] : the structure is divided into subdomains in which, for example, within a region of high stress gradients, a dual method will be used, the rest of the structure remaining described by the displacement procedure.

A similar procedure has also been used [4] : in a first step, a displacement finite element method is applied to the whole domain ; in a second step, a mixed elastodynamic formulation is applied in a subdomain, using, as entry, data of the first step computations. It is proposed, in the present paper, to couple domains where primal displacement and dual stress variational formulations are used respectively in order to obtain a symmetric variational formulation for elastodynamic analyses of coupled systems.

Let us recall first that, starting from the Hamilton-Toupin principle of complementary variable type, detailed studies on dual elastodynamic principles had been subject to various works [5, 6, 7].

Unfortunately, those dual principles, when coupled with a primal displacement principle, lead to non symmetric, then numerically inconvenient formulations. This is a classical difficulty in fluid-structure linear problems where the primal formulation is reserved to the structure, and the dual one is attached to the fluid.

In the first one, the unknown is the displacement vector field, but in the fluid the unknown is the pressure scalar field, a particular case of the stress tensor [8, 9]. The reference [9] introduces a symmetrization procedure which leads, for instance for modal analysis, to an artificially damped non conservative problem. In the survey paper [10], this question of symmetry is outlined. Using an auxiliary field variable in addition to the primal and dual ones, mixed symmetric formulations have been created for modal analysis of such systems leading to generalized added mass concepts independent of frequency, both for compressibility and gravity - capillarity effects in the fluid [11, 12]. The reference [12] presents in a synthetic way a survey of all the symmetrization procedures developed at ONERA for such problems, with many finite element results and comparison with experiments (aerospace applications namely).

This has been generalized to modal analyses of elastodynamic problems in [13].

The purpose of this paper is to present some developments relative to this kind of symmetric primal-dual formulation for coupled systems in a very general elastodynamic frame.

In a first part, the case of three-dimensional elasticity is considered.

In a second part, we present the generalization of the preceeding results to shell theory (without and with transverse shear strain effects).

In a third part, we present a fluid-structure transient case.

2. <u>Three-dimensional elastodynamic case</u> : The structure is divided into two substructures which occupy respectively the domains Ω_0 and Ω_1. Γ is the interaction boundary. Ω_0(resp. Ω_1) is supposed to be clamped along Σ_{0U} (resp. Σ_{1U}). On Σ_{0F} and Σ_{1F} act given forces.

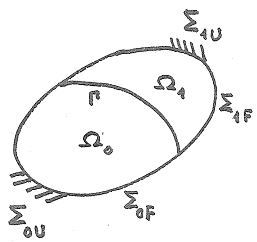

Fig. 1 - Coupled systems scheme.

In Ω_0, the unknown is the stress tensor and in Ω_1 the unknown is the displacement vector. Sympols $\sim$ and $-$ mean transmposition in the 6- dim. stress space and the 3-dim. vector vector space respectively ; ρ is the density, A (resp. B) is the linear operator of the constitutive law in Ω_1 (resp. in Ω_0 ; in fact B will be a flexibility), D is the deformation tensor ; n_0 (resp. n_1) is the unit normal outward to Ω_0(resp. Ω_1).

2.1. <u>Modal analysis case (free vibrations)</u>

2.1.1. <u>The direct non symmetric weak variational formulation</u> : Let us denote by S (resp. U) the modal amplitude of the stress tensor in Ω_0(resp. displacement in Ω_1).

$\mathcal{U}$ is the space of displacements U which are square-summable and whose first derivative are square-summable in Ω_1, that is $H_1(\Omega_1)$.

$$\mathcal{U} = \{ U \in H_1(\Omega_1) \mid U = 0 \text{ on } \Sigma_{1U} \} \tag{1}$$

$\mathcal{S}$ and $\mathcal{S}'$ are the subspaces of the space of stresses which are square-summable and whose divergence is square-summable in Ω_0, that is H (div, Ω_0).

$$\mathcal{S}' = \{ S' \in H(div., \Omega_0) \mid S' n_0 = 0 \text{ on } \Sigma_{0F} \text{, } div.\, S' = 0 \text{ in } \Omega_0 \} \tag{2}$$

$$\mathcal{S} = \{ S \in \mathcal{S}'^{\perp} \mid S n_0 = 0 \text{ on } \Sigma_{0F} \} \tag{3}$$

$\mathcal{S}'^{\perp}$ is the orthogonal complement to $\mathcal{S}'$ with respect to the scalar product :

$$(S_1, S_2) = \int_{\Omega_0} \tilde{S}_1 B S_2 + \int_{\Omega_0} \frac{1}{2}\, div\, S_1 . \overline{div\, S_2} \tag{4}$$

If ω denotes the circular frequency (ω^2 real), the weak variational formulation for the coupled systems is [17] : Find $\omega^2 \in \mathbb{R}$ and (U, S) $\in \mathcal{U} \times \mathcal{S}$ such that :

$$\left.\begin{array}{l} \int_{\Omega_1} \widetilde{D(\delta U)} A D(U) + \int_{\Gamma} \overline{\delta U} S n_0 - \omega^2 \int_{\Omega_1} \rho \overline{\delta U} U = 0 \\ \int_{\Omega_0} div\, \delta S \frac{\overline{div\, S}}{\rho} + \omega^2 \int_{\Gamma} \widetilde{\delta S n_0} U - \omega^2 \int_{\Omega_0} \widetilde{\delta S} B S = 0 \\ \qquad \forall (\delta U, \delta S) \in \mathcal{U} \times \mathcal{S}. \end{array}\right| \tag{5}$$

leading after finite element discretization to the non symmetric matrix system of the type :

$$\left[\begin{bmatrix} K_1 & C \\ 0 & M_0' \end{bmatrix} - \omega^2 \begin{bmatrix} M_1 & 0 \\ -\overline{C} & K_0 \end{bmatrix} \right] \begin{bmatrix} U \\ S \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6}$$

Physical interpretation : The dual dynamic principle (equation (5_2)) exhibits a potential energy term $\int_{\Omega_0} \widetilde{\delta S}\, B S$ which is multiplied by ω^2. In fact, the dual principle [5, 6, 7] presents a situation which is the opposite of the displacement usual one. Therefore, it is impossible to couple the two systems by the classical mechanical concept of adding potential energies and kinetic energies.

This situation will be physically well re-established by introducing a new field variable, the effect of which will be to put the potential term $\int_{\Omega_0} \widetilde{\delta S}\, B S$ in the proper stiffness place.

2.1.2. Mixed symmetric variational formulation. Introduction of the new field S^* [13, 16, 17] : Let us recall the dynamic equilibrium equation in Ω_0 :

$$\overline{\operatorname{div} S} + \rho\, \omega^2 U = 0 \quad \text{in } \Omega_0 . \tag{7}$$

Applying the deformation operator D to eq. (7) and taking account of the constitutive equation lead to :

$$D\left(\frac{\overline{\operatorname{div} S}}{\rho} \right) + \omega^2 B S = 0 \quad \text{in } \Omega_0 . \tag{8}$$

It is easy to see that a variational formulation of equation (8) exhibits the virtual potential energy "at the wrong place". Let us introduce the new field S^* defined by :

$$S = \omega^2 S^* , \quad \left(S = -\frac{\partial^2 S^*}{\partial t^2} \quad \text{in the transient case} \right). \tag{9}$$

Equation (7) becomes :

$$\frac{\overline{\operatorname{div} S^*}}{\rho} + U = 0 \tag{10}$$

This time, $\int_{\Omega_0} \operatorname{div} \delta S^* . \frac{\overline{\operatorname{div} S^*}}{\rho}$ will represent the virtual kinetic energy $\int_{\Omega_0} \rho\, \overline{\delta U}. U$.

The symmetric formulation is this time :

Find $\omega^2 \in \mathbb{R}$ and (U, S^*, S) such that :

$$\left.\begin{aligned}
&\int_{\Omega_A} \widetilde{D(\delta U)}\, A\, D(U) - \omega^2 \int_{\Omega_A} \rho\, \overline{\delta U}\, U - \omega^2 \int_{\Gamma} \overline{\delta U}\, S^* n_0 = 0, \quad \forall \delta U \\
&-\omega^2 \left[\int_{\Omega_0} \operatorname{div} \delta S^* \frac{\overline{\operatorname{div} S^*}}{\rho} + \int_{\Omega_0} \widetilde{\delta S^*} B S - \int_{\Gamma} \overline{\delta S^* n_0}\, U \right] = 0 , \quad \forall \delta S^* \\
&\int_{\Omega_0} \widetilde{\delta S}\, B S - \omega^2 \int_{\Omega_0} \widetilde{\delta S}\, B S^* = 0 , \quad \forall \delta S
\end{aligned}\right] \tag{11}$$

Here, $U \in \mathcal{U}$, $S^* \in \mathcal{S}^* = \mathcal{S}$, and S belongs to the space of stresses which are now only square-summable and satisfy $S n_0 = 0$ on Σ_{0F}. Equation (11) leads after discretization to a symmetric system.

2.1.3. Generalized added mass concept [13, 16, 17] : The second equation of (11) is independant of ω. This means that we can deduce from eq. (11) a two-field symmetric formulation in U and S, but the connection between U and S is made through a variational extremal property (expressing S^* as a function of U_Γ and S) which is the following :

$$\underset{S^* \in \mathcal{S}^*}{\text{Max.}} \int_{\Omega_0} -\frac{1}{2\rho} \left| \operatorname{div} S^* \right|^2 + \int_{\Omega_0} \widetilde{S^*} B S - \int_{\Gamma} \overline{S^* n_0}\, U \tag{12}$$

U_Γ (U on Γ) and S being (for the moment) supposed to be given and must in consequence satisfy the existence condition :

$$\int_{\Omega_0} \widetilde{\delta S'}\, B\, S - \int_\Gamma \overline{\delta S' n_0}\,.\, U_\Gamma = 0 \;,\; \forall\ \delta S' \in \mathcal{S}' . \qquad (13)$$

The value of the maximum defined by eq. (12) for the solution S^* (U_Γ , S) is precisely the kinetic energy in Ω_0 , and is called the added mass operator of Ω_0(quadratic form) :

$$\mathcal{M}(U_\Gamma, S\,;\, U_\Gamma, S) = \left[\frac{1}{2}\int_{\Omega_0}\frac{|\mathrm{div}\, S^*|^2}{\rho}\right]_{S^*(S,U_\Gamma)} = \frac{1}{2}\int_{\Omega_0}\rho |U|^2 . \qquad (14)$$

The symmetric two-field variational principle is then written : Find $\omega^2 \in \mathbb{R}$ and (U,S) belonging to the appropriate spaces defined above, such that :

$$\int_{\Omega_1} \widetilde{D(\delta U)}\, A\, D(U) + \int_{\Omega_0} \widetilde{\delta S}\, B\, S - \omega^2\left[\int_{\Omega_1}\rho\, \widetilde{\delta U}\, U + \delta\mathcal{M}\right] = 0 \qquad (15)$$

$\forall(\delta U, \delta S) \in$ the appropirate spaces defined above

Remarks :

1) Mechanical interpretation of the space $\mathcal{S}^*$and of the existence condition (eq. (13)) :

A way to interprete the space $\mathcal{S}^*$is given through the Poincaré's theorem [1, 2] :

$$[S^* \in \mathcal{S}^*] \Leftrightarrow [\exists\ U^*,\ U^* = 0 \text{ on } \Sigma_{0U} \text{ and } \Gamma,\ \text{such that } S^* = B^{-1} D(U^*)] \qquad (16)$$

The mechanical interpretation of the existence condition (13) is the following :
It exists a field U in Ω_0 , rebuilt from U_Γ , such that

$$S = B^{-1}\, D(U) \;,\; U = 0 \text{ on } \Sigma_{0U} . \qquad (17)$$

U is precisely the displacement field in Ω_0,and this is a way to find the primal principle in Ω_0 from those concepts. De Rham's theorem allows to write global closure conditions in case of non simply-connected domain [1, 14, 15].

2) Discretization, by the finite element method, of the extremal property (eq. 12) .

The search of appropriate interpolations in order to discretize $\mathcal{S}^*$are under development (taking perhaps account of remark 1).

2.2. Transient analysis case

2.2.1. External forces acting on Ω_1 : All the preceding results are valid. It is only necessary, for example for the formulation (11), to add a second member in equation (11) , namely $\int_{\Omega_1} f\, \delta U + \int_{\Sigma_{1F}} F\, \delta U$, where f and F are given volume and surface forces.

Every expression like $-\omega^2 U$, $-\omega^2 S$, $-\omega^2 S^*$ are replaced by $\frac{\partial^2 U}{\partial t^2}$, $\frac{\partial^2 S}{\partial t^2}$, $\frac{\partial^2 S^*}{\partial t^2}$; the variations $\delta U, \delta S, \delta S^*$ remaining, as usual, independent of the time t. We must of course add the initial conditions which are deduced, very easily from the displacement one through the constitutive linear law and from eq. (9). The added mass concept is the same, the time t being only a parameter (see [17] for details).

2.2.2. External forces acting on Ω_0: This case is much more complicated, because the mass operator will depend upon those external forces. But the mixed three-field formulation (11) will be changed as follows : $-\omega^2$ is replaced by $\frac{\partial^2}{\partial t^2}$ as in paragraph 2.2.1., but we must add two contributions in eq. (11) , namely :

$$\int_{\Omega_0} \frac{\partial}{\partial M}\frac{f}{\rho} \cdot \delta S^* n_0 \quad \text{and} \quad \int_{\Gamma} \frac{f}{\rho}\, \delta S^* n_0 .$$

The admissible class which S^* belongs to must also contain the force equilibrium relation :

$S n_0 = -\frac{\partial^2 S^*}{\partial t^2} n_0 = F$ on Σ_{0F} , (see [17] for details).

3. Case of shells : generalized added mass concept (eq. (12)).

In the following, we shall mainly transpose the 3-dim. 2.1.3. concept.

3.1. The case of Kirchoff-Love shells : We neglect therefore the shear strain and the normal strain energies, that is to say that all (kinematic) Kirchhoff-Love assumptions apply [1].

We shall call Σ_{m_0} the middle surface (corresponding to Ω_0 in paragraph 2), C_{m_0} its external boundary and Γ the interaction boundary ; s is the line abscissa, τ the unit tangent on C_{m_0} and Γ, and ν_0 the unit normal to C_{m_0} in the tangent plane ; N_0 the unit normal at a point $m_0 \in \Sigma_{m_0}$ to the tangent plane at m_0.

Here, instead of U_Γ on Γ , we have the kinematic variables : the displacement V of m_0 and the normal derivative $\frac{\partial \overline{N_0 V}}{\partial \nu_0}$. Using the theory developed in ref. [1], the 3-dim. stress tensor S is replaced by the surface stresses $\mathbb{n}$ and $\mathbb{m}$ (dual quantities of the extension γ and the curvature variation K of the middle surface of the shell). The 3-dim. div S, expression which appears in the classical equilibrium eq. (7), becomes

$$\widehat{\mathrm{div}}\ t = 0 \tag{18}$$

with

$$t = \mathbb{n} + \mathbb{m}\frac{\partial N_0}{\partial m_0} + N_0\, \widehat{\mathrm{div}}\ \mathbb{m} . \tag{19}$$

$\widehat{\mathrm{div}}$ is the surface divergence operator which applies to a 3-dim. vector, whose range belongs to the tangent plane at a point $m_0 \in \Sigma_{m_0}$.

As previously, we introduce an auxiliary field variable (like S^*) :

$$\begin{aligned} t &= \omega^2 t^* \quad \left(\text{or } t = -\frac{\partial^2 t^*}{\partial t^2} \text{ in the transient case}\right). \\ t^* &= \mathbb{n}^* + \mathbb{m}^*\frac{\partial N_0}{\partial m_0} + N_0\, \widehat{\mathrm{div}}\ \mathbb{m}^* . \end{aligned} \tag{21}$$

$\mathbb{n}^*, \mathbb{m}^*$ and t^* satisfy equations corresponding to some manipulations, similar to the one described in paragraph 2.1.3.

The extremal variational property of the added mass operator is then defined as follows (like eq. (12)) :

$$\underset{S^*\in\mathcal{S}^*}{\mathrm{Max.}} \int_{\Sigma_{m_0}} \left[-\frac{1}{2\rho}\left|\widehat{\mathrm{div}}\ t^*\right|^2 + \tilde{S}^* B S\right] - \int_\Gamma \Big[\overline{\nu_0}\left[\mathbb{n}^* + 2\mathbb{m}^*\frac{\partial N_0}{\partial m_0} + \overline{N_0\, \widehat{\mathrm{div}}\ \mathbb{m}^*}\right] V_\Gamma + \frac{d}{ds}\left[\overline{\tau_0}\ \mathbb{m}^* \nu_0\right]\overline{N_0} V_\Gamma + \overline{\nu_0}\ \mathbb{m}^* \nu_0 \frac{\partial \overline{N_0 V}}{\partial \nu_0}\Big|_\Gamma \Big] ds , \tag{22}$$

with $t^*(S^*)$, (eq. (21)) and $S^* = \begin{bmatrix}\mathbb{n}^*\\ \mathbb{m}^*\end{bmatrix}$.

The corresponding existence condition, (like eq. (13)), concerning S, V, $\frac{\partial \overline{N_0 V_\Gamma}}{\partial \nu_0}$ is obtained by setting div $t^* = 0$ in eq. (22) and equating, after derivation, the remaining terms to zero. The remarks of paragraph 2.1.3. apply to this case [17].

3.2. Case of shell with transverse shear strain and stresses : Let us recall that, in this case, the assumption according to which material normals are identified with geometric normals is left aside. The surface stresses are no more symmetric as it is well known [1].

If E_3 is the 3-dim. Euclidian space and $\overrightarrow{E_{20}}$ the 2-dim. tangent space, we must consider the stresses : $\mathbb{N} \in \mathcal{L}(\overrightarrow{E_3}, \overrightarrow{E_{20}})$ and $\mathbb{M} \in \mathcal{L}(\overrightarrow{E_{20}}, \overrightarrow{E_{20}})$. The stress $\mathbb{N}$ has a normal component $\mathbb{N} N_0$ which is the transverse shear stress. Now $S = \begin{bmatrix}\mathbb{N}\\ \mathbb{M}\\ \mathbb{N}N_0\end{bmatrix}$ As before, it is necessary to define the auxiliary variable $S^* = \begin{bmatrix} \mathbb{N}^* \\ \mathbb{M}^* \\ \mathbb{N}^* N_0 \end{bmatrix}$ satisfying equations derived from the dynamic equilibrium equations of the shell [17]. If we define the vector U^* and the matrix ξ by :

$$U^* = -\begin{bmatrix} \widehat{\mathrm{div}}\, \mathbb{N}^* \\ \widehat{\mathrm{div}}\, \mathbb{M}^* - \mathbb{N}^* N_0 \end{bmatrix} \quad , \qquad \xi = \begin{bmatrix} \frac{1}{\rho_m} & 0 \\ 0 & \frac{1}{\rho_N} \end{bmatrix} ,$$

where ρ_m, ρ_N are the surface densities relative to the displacement and to the normal rotation, the generalized added mass operator is defined then like eq. (22) as :

$$\underset{S^* \in \mathcal{S}^*}{\mathrm{Max.}} \int_{\Sigma_{m_0}} \left[-\frac{1}{2} \overline{U^*} \xi U^* + \overline{S^*} B S \right] - \int_\Gamma \overline{\vec{v}_0} \left[\mathbb{N}^* V_\Gamma + \mathbb{M}^* \Delta N_\Gamma \right] ds , \tag{23}$$

where $\Delta N \in \vec{E}_{20}$ defines the vector difference between the material and geometric normal vectors [1].

4. Transient compressible fluid-structure interaction : Let us consider the transient elasto-acoustic interaction : Ω_1 is the elastic structure and Ω_0 (of boundary Γ) contains an inviscid compressible gas. The stress S becomes the scalar pressure field tensor - p. The auxiliary variable S^* becomes the displacement potential of the gas, and eq. (9) becomes :

$$p = -\rho_f \frac{\partial^2 \varphi}{\partial t^2} ; \tag{24}$$

(in order to retrieve the physical meaning of S^*, we have introduced the density ρ_f). The flexibility B becomes here a scalar operator with the coefficient $\rho_f c^2$, where c is the second velocity in the gas :

$$p = -\rho_f c^2 \mathrm{div}\, U_f \quad ,$$ where U_f is the displacement of the fluid.

The variational formulation (11) together with the transient situation described in paragraph 2.2.1. becomes :

Find $(U, p, \varphi) \in$ appropriate mathematical espaces, such that :

$$\left.\begin{aligned} &\int_{\Omega_1} \overline{D(\delta U)} A D(U) + \int_{\Omega_1} \rho_s \overline{\delta U} \frac{\partial^2 U}{\partial t^2} + \int_\Gamma \rho_f \overline{\delta U}\, n \frac{\partial^2 \varphi}{\partial t^2} = \int_{\Sigma_{1F}} F \delta U , \forall \delta U \\ &\int_{\Omega_0} \rho_f \overline{\mathrm{grad}\, \delta\varphi}\, \mathrm{grad} \frac{\partial^2 \varphi}{\partial t^2} + \frac{1}{c^2} \int_{\Omega_0} \frac{\partial^2 p}{\partial t^2} \delta\varphi + \int_\Gamma \rho_f \delta\varphi \frac{\partial^2}{\partial t^2} [\overline{U} n] = 0 , \forall \delta\varphi \\ &\int_{\Omega_0} \frac{1}{\rho_f c^2} p \delta p + \frac{1}{c^2} \int_{\Omega_0} \frac{\partial^2 \varphi}{\partial t^2} \delta p = 0 , \qquad \forall \delta p \end{aligned}\right| \tag{25}$$

plus initial conditions.

ρ_s is the density of the structure ; $\delta U \in H^1(\Omega_1)$, $\delta\varphi \in H^1(\Omega_0)$, $\delta p \in L^2(\Omega_0)$.

The finite element discretization of (25) leads to the symmetric matrix system (26) :

$$\begin{bmatrix} K_1 & 0 & 0 \\ 0 & K_0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} U \\ p \\ \varphi \end{bmatrix} + \begin{bmatrix} M_1 & 0 & G \\ 0 & 0 & E \\ \overline{G} & \overline{E} & -F \end{bmatrix} \begin{bmatrix} \ddot{U} \\ \ddot{p} \\ \ddot{\varphi} \end{bmatrix} = \begin{bmatrix} \\ 0 \\ 0 \end{bmatrix} , \tag{26}$$

plus initial conditions.

Under some conditions, φ can be eliminated and we retrieve the results of paragraph 2.1.3. (see [17] for details). The added mass computation is the same as in the modal analysis [12].

5. Conclusion : In elastodynamics, the computation of refined stresses in elastic bodies, or pressures in fluids, can be carried out through a direct stress analysis in the concerned domains of a complex system. The introduction of auxiliary variables leads to symmetric formulations in the transient case as well as in the modal case. Details will be found in an extended paper [17].

6. References :

/ 1 / VALID, R., "Mechanics of Continuous Media and Analysis of Structures", North-Holland Ed., (1981).

/ 2 / VALID, R., "Le Principe des Travaux Virtuels et les Principes Variationnels Associés", Publication ONERA 1982-1, (1982).

/ 3 / VALID, R., "Tendances Actuelles en Calcul des Structures à l'ONERA", 2ièmes Journées Tendances Actuelles en Calcul des Structures, 1-3 Février 1982, Sophia Antipolis, TP-ONERA N° 1982-2, La Rech. Aérosp. 1983-1. French and English eds.

/ 4 / GIBERT, P., "The mixed Variational Principles in Vibrational Mechanics", Thesis Dissertation, N.T. ONERA N° 1981-4, (1981).

/ 5 / FRAIJS de VEUBEKE, B.M., "The dual Principles of Elastodynamics, Finite Element Applications", NATO Advanced Study Institute on Finite Element Method in Continuum Mechanics, Lisbon, Portugal (Sept. 1971).

/ 6 / IDELSOHN, S., GERARDIN, M., "On the Self-Stressing Modes in Free Vibration Analysis", Journal of Sound and Vibration, Vol. 83, N° 2 (July 1982).

/ 7 / TABARROK, B., SAKAGUCHI, R.L., "Calculations of Plate Frequencies from Complementary Energy Formulations", Int. J. Num. Meth. in Engng., Vol. 2, (1970).

/ 8 / CRAGGS, A., "Some Practical Applications of Elasto-Acoustic Finite Elements", Journal of Sound and Vibration, Vol. 11, N° 2, (1976).

/ 9 / EVERSTINE, G.C., "A Symmetric Potential Formulation for Fluid-Structure Interaction", Journal of Sound and Vibration, 79 (1), p. 157-160, (1981).

/ 10 / ZIENKIEWICZ, O.C., BETTESS, P., "Fluid-Structure Interaction", Ocean Structural Dynamics Symposium 82, Sept. 8-10, Oregon State University, Corvalis, Oregon (1982).

/ 11 / VALID, R., OHAYON, R., BERGER, H., "Le Calcul des Réservoirs Elastiques Partiellement Remplis de Liquide pour la Prévision de l'Effet Pogo", 23rd Int. Astr. Fed. Congress, Vienne, Autriche, 5-8 Oct. 1972, TP-ONERA 1154 (1974). La Recherche Aérospatiale 1974-6, p. 367-379,

/ 12 / OHAYON, R., VALID, R., "True Symmetric Formulations of Free Vibrations of Fluid-Structure Interaction. Applications and Extensions", Int. Conf. on Num. Methods for Coupled Problems, University of Swansea, U.K., Sept. 7-11, 1981, Proceedings edited by Pineridge Press, Swansea (U.K.). To appear as a chapter of a book, "Numerical Methods for Coupled Problems", R.W. Lewis, E. Hinton, P. Bettess Ed., (John Wiley and Sons, 1983).

/ 13 / OHAYON, R., "Symmetric Variational Formulation of Harmonic Vibration Problems by Coupling Primal and Dual Principles. Application to Fluid-Structure Coupled Systems", La Rech. Aérosp., N° 1979-3, p. 207-211, (1979). English translation, ESA-TT 609.

/ 14 / GURTIN, M.E., "A Generalization of the Beltrami Stress Functions for General Continua", Arch. Rat. Mech. Analysis, Vol. 4, N° 1, p. 1-29 (1959).

/ 15 / VALID, R., "Closure Conditions in the Case of Multiply Connected Shells", 14th. IUTAM Congress, Delft (The Netherlands), August 30 - Sept. 4, 1976. La Rech. Aérosp. N° 1976-5, p. 289-299, TP-ONERA 1976-82 E (1976). English translation ESA-TT. 298.

/ 16 / OHAYON, R., VALID, R., "Principes Variationnels Symétriques Couplés de Type Primal-Dual en Elastodynamique", (Proposition de Comptes Rendus à l'Académie des Sciences, Paris).

/ 17 / OHAYON, R., VALID, R., Extended version of the present paper to be submitted to Int. J. of Nuclear Engng.