FINITE ELEMENT ANALYSIS OF GEOMETRIC AND MATERIAL NONLINEAR DYNAMIC RESPONSE

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SUMMARY

The importance of investigating the nonlinear static and dynamic response of nuclear reactor components for adequate design has been recognized to an increasing extent. In some cases the behavior of the materials is significantly nonlinear and for other components the influence of geometric changes on the response of the structure cannot be neglected.

The finite element analysis of structures has proven to be very effective in linear analysis and with regard to nonlinear analysis practical solutions to many different problems have also been obtained. However, there is still a large number of problems to which either no solution can be calculated or for which it is difficult to predict the response accurately at a reasonable cost. Consequently, a great deal of research is currently undertaken to improve nonlinear finite element analysis procedures.

In essence, there are the following areas which need consideration in the development of nonlinear dynamic analysis methods, namely (1) continuum mechanics formulations and finite element discretizations, (2) development of constitutive relations, (3) effective time integration, and (4) appropriate computer implementation. In addition, extensive experimental results are required. The difficulties encountered in developing more effective techniques lie largely in the interaction that exists between those areas.

The objective in this paper is to present latest developments in the areas (1) to (3) above. A computer program was described in SMiRT-2 Paper M 3/1*. In the first part of the paper recently implemented continuum mechanics and resulting finite element formulations are surveyed which include all nonlinear effects due to large displacements, large strains and material nonlinearities. Recommendations for the use of the formulations are given. Next, specific problems encountered in the implementation of nonlinear hypo-elastic and elastic-plastic material models are surveyed and the important interaction between the implementation of a material model and the continuum mechanics formulation used is described. In the third part of the paper the time integration procedures that have been employed extensively are presented and specific emphasis is given to the use of equilibrium iteration in nonlinear static and dynamic analysis. Finally, to demonstrate the recommendations given a variety of problem solutions are presented. These include the static and dynamic, geometric and material nonlinear analysis of beams, pipes, arches and shells that are subjected to high intensity mechanical and thermal loading conditions.

1. Introduction

Finite element nonlinear analysis requires knowlege in a number of important areas. The objective in this compact is to summarize the continuum mechanics formulations, finite element discretizations, material models and time integration schemes that have been implemented during the development of the static and dynamic linear and nonlinear finite element analysis program ADINA (Automatic Dynamic Incremental Nonlinear Analysis). The computer program ADINA is presently being developed at M.I.T. and is an extension of program NONSAP, which has been documented earlier by Bathe, et. al. [1 - 4]. More detailed discussions and sample solutions will be given at the presentation and in the publication of the complete paper.

2. Continuum Mechanics Formulations

Consider the motion of a general body undergoing small or large displacements and strains with linear or nonlinear material characteristics. Assume that the solution for the static and kinematic variables has been obtained for time points 0, Δt, $2\Delta t$, ..., t. The principle of virtual displacements is used to express the equilibrium of the body in the configuration at time $t+\Delta t$,

$$\int_{^{t+\Delta t}V} {}^{t+\Delta t}\tau_{ij} \; \delta_{t+\Delta t}e_{ij} \; {}^{t+\Delta t}dv = {}^{t+\Delta t}\mathcal{R} \tag{1}$$

where the notation in Fig. 1 is used with

$$^{t+\Delta t}\mathcal{R} = \int_{^0A} {}^{t+\Delta t}_{\;\;0}t_k \; \delta u_k \; {}^0da + \int_{^0V} {}^0\rho \; {}^{t+\Delta t}_{\;\;0}f_k \; \delta u_k \; {}^0dv \tag{2}$$

and ${}^{t+\Delta t}\tau_{ij}$ are Cartesian components of the Cauchy stress tensor at time $t+\Delta t$, ${}^{t+\Delta t}_{\;\;0}t_k$ and ${}^{t+\Delta t}_{\;\;0}f_k$ are surface tractions and body force components at time $t+\Delta t$, but referred to time 0 and ${}^0\rho$ is the specific mass of the material in the original configuration. Also, δ denotes "variation in", δu_k is a variation in the current displacement components ${}^{t+\Delta t}u_k$ and

$$\delta_{t+\Delta t}e_{ij} = \delta \tfrac{1}{2} \left({}_{t+\Delta t}u_{i,j} + {}_{t+\Delta t}u_{j,i} \right) \tag{3}$$

where ${}_{t+\Delta t}u_{l,m} = \partial u_l / \partial\, {}^{t+\Delta t}x_m$. The virtual work principle given above is general and is used in linear and geometric and material nonlinear analysis.

In linear elastic or material nonlinear but infinitesimal displacement analysis eq. (1) reduces to

$$\int_{^0V} {}^{t+\Delta t}\sigma_{ij} \; \delta e_{ij} \; {}^0dv = {}^{t+\Delta t}\mathcal{R} \tag{4}$$

where ${}^{t+\Delta t}\sigma_{ij}$ are small displacement stresses at time $t+\Delta t$ and e_{ij} is an infinitesimal strain increment,

$$e_{ij} = \frac{1}{2}({}_0u_{i,j} + {}_0u_{j,i}) \qquad (5)$$

For geometric nonlinear analysis, two formulations which have been termed total Lagrangian (T.L.) and updated Lagrangian (U.L.) formulations have been implemented. In the T.L. formulation, eq. (1) is transformed to refer the stress and strain quantities to the initial configuration to obtain

$$\int_{{}^0V} {}^{t+\Delta t}_{\ 0}S_{ij}\ \delta\, {}^{t+\Delta t}_{\ 0}\varepsilon_{ij}\ {}^0dv = {}^{t+\Delta t}\mathcal{R} \qquad (6)$$

where ${}^{t+\Delta t}_{\ 0}S_{ij}$ are components of the 2nd Piola - Kirchhoff stress tensor and ${}^{t+\Delta t}_{\ 0}\varepsilon_{ij}$ are components of the Green-Lagrange strain tensor using the displacements ${}^{t+\Delta t}u_k$. In the U.L. formulation the stress and strain quantities are referred to the configuration at time t and eq. (1) becomes

$$\int_{{}^tV} {}^{t+\Delta t}_{\ t}S_{ij}\ \delta\, {}^{t+\Delta t}_{\ t}\varepsilon_{ij}\ {}^tdv = {}^{t+\Delta t}\mathcal{R} \qquad (7)$$

In nonlinear analysis it is necessary to linearize the equations of motion. Tables I to III summarize the relations employed to arrive at the linearized equations of motion in the materially nonlinear only, the T.L. and the U.L. formulations. It should be noted that ${}^t\sigma_{ij}$, ${}^t_0S_{ij}$ and ${}^t\tau_{ij}$ are, respectively, known small displacement, 2nd Piola - Kirchhoff and Cauchy stresses acting in the configuration at time t; and ${}_0e_{ij}$, ${}_0\eta_{ij}$ and ${}_te_{ij}$, ${}_t\eta_{ij}$ are the linear and nonlinear incremental strains referred to the configurations at times 0 and t, respectively. If displacements are small, the T.L. and U.L. formulations both reduce to the materially nonlinear only formulation.

The equilibrium relations given in eq. (4) and Tables I to III are used in static or dynamic analysis. In dynamic analysis the applied body forces specified in eq. (2) include inertia forces. Assuming that the mass of the body is preserved, the inertia forces are evaluated using

$$\int_{{}^0V} {}^0\rho\ {}^{t+\Delta t}\ddot{u}_k\ \delta u_k\ {}^0dv\ .$$

3. Finite Element Discretizations

The continuum mechanics relations are very effectively discretized using isoparametric and related finite elements. Considering a single element the following equilibrium matrix equations are obtained:

for linear analysis,

$$M\ {}^{t+\Delta t}\ddot{u} + K\ {}^{t+\Delta t}u = {}^{t+\Delta t}R \qquad (8)$$

for nonlinear analysis, including nonlinear material effects only,

$$M\ {}^{t+\Delta t}\ddot{u} + {}^tK\ u = {}^{t+\Delta t}R - {}^tF \qquad (9)$$

using the T.L. formulation,

$$M\ {}^{t+\Delta t}\ddot{u} + ({}^t_0K_L + {}^t_0K_{NL})u = {}^{t+\Delta t}R - {}^t_0F \qquad (10)$$

and using the U.L. formulation,

$$M\ {}^{t+\Delta t}\ddot{u} + ({}^{t}_{t}K_L + {}^{t}_{t}K_{NL})\,u = {}^{t+\Delta t}R - {}^{t}_{t}F \qquad (11)$$

where M = time independent lumped or consistent mass matrix; K = time independent linear elastic, small displacement stiffness matrix; ${}^{t}_{0}K_L$, ${}^{t}_{t}K_L$ = linear strain incremental stiffness matrices; ${}^{t}_{0}K_{NL}$, ${}^{t}_{t}K_{NL}$ = nonlinear strain (geometric or initial stress) incremental stiffness matrices; ${}^{t}K$ = linear strain incremental stiffness matrix, not including the initial displacement effect; ${}^{t+\Delta t}R$ = vector of externally applied element nodal loads; ${}^{t}_{0}F$, ${}^{t}_{t}F$ = vectors of nodal point forces equivalent to the element stresses at time t; ${}^{t}F$ = vector of nodal point forces equivalent to the element stresses at time t, not including the initial displacement effect; ${}^{t+\Delta t}u$ = vector of nodal point displacements at time t+Δt; ${}^{t+\Delta t}\ddot{u}$ = vector of nodal point accelerations at time t+Δt; and $u = {}^{t+\Delta t}u - {}^{t}u$.

Table IV summarizes the various integrals considered to evaluate the matrices in eqs. (8 - 11), where H_s, H = surface- and volume-displacement transformation matrices; ${}^{t+\Delta t}_{0}t$, ${}^{t+\Delta t}_{0}f$ = vectors of surface and body forces defined per unit area and per unit mass of the body at time 0; B_L , ${}^{t}_{0}B_L$, ${}^{t}_{t}B_L$= linear strain-displacement transformation matrices; ${}^{t}_{0}B_{NL}$, ${}^{t}_{t}B_{NL}$ = nonlinear strain-displacement transformation matrices; C = stress-strain material property matrix (incremental or total); ${}_{0}C$, ${}_{t}C$ = incremental stress-strain material property matrices; ${}^{t}\tau$, ${}^{t}\hat{\tau}$ = matrix and vector of Cauchy stresses; ${}^{t}_{0}S$, ${}^{t}_{0}\hat{S}$ = matrix and vector of 2nd Piola-Kirchhoff stresses; ${}^{t}\hat{\Sigma}$ = vector of stresses in materially nonlinear only analysis. The matrices used in Table IV depend on the specific element considered and have been discussed in references [3 and 4].

Considering the numerical effort involved in the calculation of the element matrices and force vectors in material and geometric nonlinear analysis, it is noted that the linear strain-displacement transformation matrix is a full matrix in the T.L. formulation, and is sparse in the U.L. formulation. On the other hand, an important consideration is that combined geometric and material nonlinear analysis is more effectively implemented using the T.L. formulation provided the constitutive relations can be defined in terms of 2nd Piola-Kirchhoff stresses and Green-Lagrange strains.

4. Constitutive Relations

Program ADINA is presently being developed to perform general one, two and three-dimensional elastic, hypo-elastic, elastic-plastic and creep analysis for mechanical and thermo-mechanical loading conditions. Small or large displacements can be considered.

For elastic analysis the material can be isotropic or orthotropic. A curve description model is available to describe a hypo-elastic material behavior in which cracking of the material can be simulated. The elastic-plastic materials are assumed to obey the Ducker-Prager yield condition with perfect plasticity or the von Mises yield condition with perfectly plastic, isotropic or kinematic hardening conditions. Including temperature effects and creep the incremental stress - total incremental strain law is formulated using, in small displacement analysis [5],

$$\sigma_{ij} = C^E_{ijrs} \left(e_{rs} - e^p_{rs} - e^c_{rs} - e^t_{rs}\right) \qquad (12)$$

where σ_{ij} = stress increment, C^E_{ijrs} = component of the elasticity tensor, e_{rs} = total incremental strain, e^p_{rs} = incremental plastic strain, e^c_{rs} = incremental creep strain and e^t_{rs} = incremental strain due to temperature. Considering large displacement analysis in the T.L. formulation increments in 2nd Piola-Kirchhoff stresses and Green-Lagrange strains are used. If no temperature and creep effects are included, the stress-strain law can be established as used in Tables I to III, whereas, in general, contributions resulting from temperature and creep are added to the current stress components.

5. Time Integration

Once the required element matrices of eqs. (8 to 11) have been calculated, the matrices corresponding to a system of elements are obtained using standard assemblage techniques. Including the effect of velocity dependent damping forces, the equilibrium equations for time t+Δt are

$$M\,{}^{t+\Delta t}\ddot{u} + C\,{}^{t+\Delta t}\dot{u} + {}^{t}K\,u = {}^{t+\Delta t}R - {}^{t}F \qquad (13)$$

where, for the element assemblage, M = mass matrix; C = damping matrix; ${}^tK$ = tangent stiffness matrix at time t which includes the linear and nonlinear strain stiffness matrices; ${}^{t+\Delta t}R$ = vector of externally applied forces at time t+Δt; ${}^tF$ = vector of nodal point forces equivalent to the stresses of the elements at time t, and the displacements and their time derivatives are defined as in eqs. (8 to 11). In static analysis, mass and damping effects are, of course, not included in eq. (13). It should be noted that the total number of elements of an assemblage can be divided into elements which behave linearly (i.e., eq. (8) is the governing equilibrium equation) and elements which are nonlinear. The nonlinear elements can again be subdivided into elements whose response is described including geometric nonlinearities and elements with material nonlinearities only.

Equation (13) represents the incremental equilibrium equations to be solved in each time step using a numerical integration scheme. In program ADINA the Newmark method and Wilson θ-method are presently used.

It is important to realize that in nonlinear analysis eq. (13) is a linear approximation to the actual equation to be solved in each time step, namely eq. (1). Depending on the nonlinearities in the system and the magnitude of the time step Δt, the linearization may introduce serious errors and, indeed, solution instability. A common observation is that the errors introduced as a consequence of the linearization cause the calculated solution to "drift away" from the exact solution. This is much more serious in dynamic analysis than in static analysis, since, in dynamic analysis, the solution for any prescribed load at a specific time is always dependent on the history of solution. In order to avoid large integration errors and instability, we need to iterate until, within the necessary assumptions on the variation of the material constants and the numerical time integration scheme used, eq. (1) is satisfied within a required tolerance [3] [6]. The equations solved in the iteration depend on the nonlinear finite element formulation used, and are extensions of eqs. (9 to 11). The detailed derivations of the equilibrium iterations in the material nonlinear only, T.L. and U.L. formulations have been presented in references [3 and 4]. Table V summarizes the step-by-step algorithm that is employed.

6. Conclusions

Finite element analysis of geometric and material nonlinear dynamic response is a complex combination of a number of disciplines. Some of the work that has been performed during the development of the computer program ADINA has been summarized in this compact. It should be realized that much additional research is required. However, it is the interaction between the areas of numerical analysis, continuum mechanics, experimental investigation and computer implementation that makes the development of general nonlinear analysis procedures an exciting and very rewarding challenge.

References

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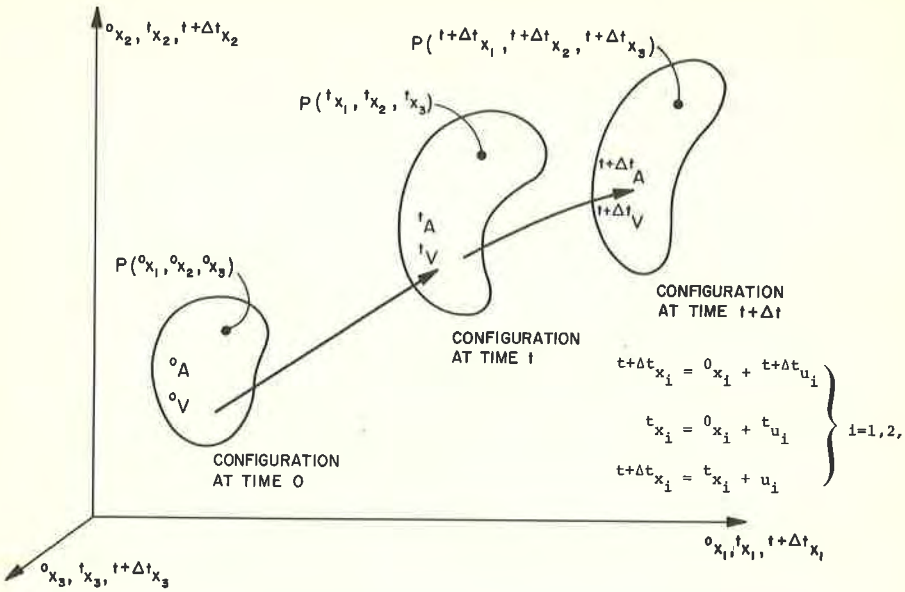

FIGURE I MOTION OF BODY IN CARTESIAN COORDINATE SYSTEM

TABLE I. MATERIALLY NONLINEAR ONLY FORMULATION

1. Equations of Motion

$$\int_{^{0}V} {}^{t+\Delta t}\sigma_{ij}\ \delta e_{ij}\ {}^{0}dv = {}^{t+\Delta t}\mathcal{R}$$

where $e_{ij} = \frac{1}{2}({}_{0}u_{i,j} + {}_{0}u_{j,i})$

2. Incremental Decomposition of Stresses

$$^{t+\Delta t}\sigma_{ij} = {}^{t}\sigma_{ij} + \sigma_{ij}$$

3. Equations of Motion with Incremental Stress Decomposition

$$\int_{^{0}V} \sigma_{ij}\ \delta e_{ij}\ {}^{0}dv = {}^{t+\Delta t}\mathcal{R} - \int_{^{0}V} {}^{t}\sigma_{ij}\ \delta e_{ij}\ {}^{0}dv$$

4. Linearization of Equations of Motion

Using the approximation $\sigma_{ij} = C_{ijrs}\ e_{rs}$ we obtain as approximate equations of motion

$$\int_{^{0}V} C_{ijrs}\ e_{rs}\ \delta e_{ij}\ {}^{0}dv = {}^{t+\Delta t}\mathcal{R} - \int_{^{0}V} {}^{t}\sigma_{ij}\ \delta e_{ij}\ {}^{0}dv$$

TABLE II. TOTAL LAGRANGIAN FORMULATION

1. Equations of Motion

$$\int_{{}^0V} {}^{t+\Delta t}_{0}S_{ij}\, \delta\, {}^{t+\Delta t}_{0}\varepsilon_{ij}\, {}^0dv = {}^{t+\Delta t}\mathcal{R} \; ;$$

where $${}^{t+\Delta t}_{0}S_{ij} = \frac{{}^0\rho}{{}^{t+\Delta t}\rho}\; {}^{0}_{t+\Delta t}x_{i,s}\; {}^{t+\Delta t}\tau_{sr}\; {}^{0}_{t+\Delta t}x_{j,r} \; ;$$

$$\delta\, {}^{t+\Delta t}_{0}\varepsilon_{ij} = \delta \frac{1}{2}\left({}^{t+\Delta t}_{0}u_{i,j} + {}^{t+\Delta t}_{0}u_{j,i} + {}^{t+\Delta t}_{0}u_{k,i}\; {}^{t+\Delta t}_{0}u_{k,j} \right)$$

2. Incremental Decompositions

stresses: $${}^{t+\Delta t}_{0}S_{ij} = {}^{t}_{0}S_{ij} + {}_{0}S_{ij}$$

strains: $${}^{t+\Delta t}_{0}\varepsilon_{ij} = {}^{t}_{0}\varepsilon_{ij} + {}_{0}\varepsilon_{ij} \; ; \quad \text{where} \quad {}_{0}\varepsilon_{ij} = {}_{0}e_{ij} + {}_{0}\eta_{ij}$$

$${}_{0}e_{ij} = \frac{1}{2}\left({}_{0}u_{i,j} + {}_{0}u_{j,i} + {}^{t}_{0}u_{k,i}\; {}_{0}u_{k,j} + {}_{0}u_{k,i}\; {}^{t}_{0}u_{k,j} \right) \; ; \quad {}_{0}\eta_{ij} = \frac{1}{2}\, {}_{0}u_{k,i}\; {}_{0}u_{k,j}$$

3. Equations of Motion with Incremental Decompositions

With $\delta\, {}^{t+\Delta t}_{0}\varepsilon_{ij} = \delta\, {}_{0}\varepsilon_{ij}$ and ${}_{0}S_{ij} = {}_{0}C_{ijrs}\; {}_{0}\varepsilon_{rs}$ the equations of motion are

$$\int_{{}^0V} {}_{0}C_{ijrs}\; {}_{0}\varepsilon_{rs}\; \delta\, {}_{0}\varepsilon_{ij}\; {}^0dv + \int_{{}^0V} {}^{t}_{0}S_{ij}\; \delta\, {}_{0}\eta_{ij}\; {}^0dv = {}^{t+\Delta t}\mathcal{R} - \int_{{}^0V} {}^{t}_{0}S_{ij}\; \delta\, {}_{0}e_{ij}\; {}^0dv$$

4. Linearization of Equations of Motion

Using the approximations ${}_{0}S_{ij} = {}_{0}C_{ijrs}\; {}_{0}e_{ij}$, $\delta\, {}_{0}\varepsilon_{ij} = \delta\, {}_{0}e_{ij}$ we obtain as approximate equations of motion

$$\int_{{}^0V} {}_{0}C_{ijrs}\; {}_{0}e_{rs}\; \delta\, {}_{0}e_{ij}\; {}^0dv + \int_{{}^0V} {}^{t}_{0}S_{ij}\; \delta\, {}_{0}\eta_{ij}\; {}^0dv = {}^{t+\Delta t}\mathcal{R} - \int_{{}^0V} {}^{t}_{0}S_{ij}\; \delta\, {}_{0}e_{ij}\; {}^0dv$$

TABLE III. UPDATED LAGRANGIAN FORMULATION

1. Equations of Motion

$$\int_{{}^tV} {}^{t+\Delta t}_{t}S_{ij}\, \delta\, {}^{t+\Delta t}_{t}\varepsilon_{ij}\, {}^tdv = {}^{t+\Delta t}\mathcal{R} \; ;$$

where $${}^{t+\Delta t}_{t}S_{ij} = \frac{{}^t\rho}{{}^{t+\Delta t}\rho}\; {}^{t}_{t+\Delta t}x_{i,s}\; {}^{t+\Delta t}\tau_{sr}\; {}^{t}_{t+\Delta t}x_{j,r} \; ;$$

$$\delta\, {}^{t+\Delta t}_{t}\varepsilon_{ij} = \delta \frac{1}{2}\left({}_{t}u_{i,j} + {}_{t}u_{j,i} + {}_{t}u_{k,i}\; {}_{t}u_{k,j} \right)$$

2. Incremental Decompositions

stresses: ${}^{t+\Delta t}_{t}S_{ij} = {}^{t}\tau_{ij} + {}_{t}S_{ij}$; strains: ${}^{t+\Delta t}_{t}\varepsilon_{ij} = {}_{t}\varepsilon_{ij}$; where

$${}_{t}\varepsilon_{ij} = {}_{t}e_{ij} + {}_{t}\eta_{ij} \; ; \quad {}_{t}e_{ij} = \frac{1}{2}({}_{t}u_{i,j} + {}_{t}u_{j,i}) \; ; \quad {}_{t}\eta_{ij} = \frac{1}{2}\, {}_{t}u_{k,i}\; {}_{t}u_{k,j}$$

3. Equations of Motion with Incremental Decompositions

Noting that ${}_{t}S_{ij} = {}_{t}C_{ijrs}\; {}_{t}\varepsilon_{rs}$ the equations of motion are

$$\int_{{}^tV} {}_{t}C_{ijrs}\; {}_{t}\varepsilon_{rs}\; \delta\, {}_{t}\varepsilon_{ij}\; {}^tdv + \int_{{}^tV} {}^{t}\tau_{ij}\; \delta\, {}_{t}\eta_{ij}\; {}^tdv = {}^{t+\Delta t}\mathcal{R} - \int_{{}^tV} {}^{t}\tau_{ij}\; \delta\, {}_{t}e_{ij}\; {}^tdv$$

4. Linearization of Equations of Motion

Using the approximations ${}_{t}S_{ij} = {}_{t}C_{ijrs}\; {}_{t}e_{ij}$, $\delta\, {}_{t}\varepsilon_{ij} = \delta\, {}_{t}e_{ij}$ we obtain as approximate equations of motion

$$\int_{{}^tV} {}_{t}C_{ijrs}\; {}_{t}e_{rs}\; \delta\, {}_{t}e_{ij}\; {}^tdv + \int_{{}^tV} {}^{t}\tau_{ij}\; \delta\, {}_{t}\eta_{ij}\; {}^tdv = {}^{t+\Delta t}\mathcal{R} - \int_{{}^tV} {}^{t}\tau_{ij}\; \delta\, {}_{t}e_{ij}\; {}^tdv$$

TABLE IV. FINITE ELEMENT MATRICES

ANALYSIS TYPE	INTEGRAL	MATRIX EVALUATION
IN ALL ANALYSES	$\int_{^0V} {}^0\rho\; {}^{t+\Delta t}\ddot{u}_k\; \delta u_k\; {}^0dv$	$M\; {}^{t+\Delta t}\ddot{u} = {}^0\rho\left(\int_{^0V} H^T\, H\; {}^0dv\right)\; {}^{t+\Delta t}\ddot{u}$
	${}^{t+\Delta t}\mathcal{R} = \int_{^0A} {}^{t+\Delta t}_{0}t_k\; \delta u_k\; {}^0da + \int_{^0V} {}^0\rho\; {}^{t+\Delta t}_{0}f_k\; \delta u_k\; {}^0dv$	${}^{t+\Delta t}R = \int_{^0A} H_S^T\; {}^{t+\Delta t}_{0}t\; {}^0da + {}^0\rho \int_{^0V} H^T\; {}^{t+\Delta t}_{0}f\; {}^0dv$
LINEAR ANALYSIS	$\int_{^0V} C_{ijrs}\; {}^{t+\Delta t}e_{rs}\; \delta e_{ij}\; {}^0dv$	$K\; {}^{t+\Delta t}u = \left(\int_{^0V} B_L^T\, C\, B_L\; {}^0dv\right)\; {}^{t+\Delta t}u$
MATERIAL NONLINEARITY ONLY	$\int_{^0V} C_{ijrs}\; e_{rs}\; \delta e_{ij}\; {}^0dv$	${}^tK\; u = \left(\int_{^0V} B_L^T\, C\, B_L\; {}^0dv\right)\; u$
	$\int_{^0V} {}^t\sigma_{ij}\; \delta e_{ij}\; {}^0dv$	${}^tF = \int_{^0V} B_L^T\; {}^t\hat{\Sigma}\; {}^0dv$
TOTAL LAGRANGIAN FORMULATION	$\int_{^0V} {}_0C_{ijrs}\; {}_0e_{rs}\; \delta\, {}_0e_{ij}\; {}^0dv$	${}^t_0K_L\; u = \left(\int_{^0V} {}^t_0B_L^T\; {}_0C\; {}^t_0B_L\; {}^0dv\right)\; u$
	$\int_{^0V} {}^t_0S_{ij}\; \delta\, {}_0\eta_{ij}\; {}^0dv$	${}^t_0K_{NL}\; u = \left(\int_{^0V} {}^t_0B_{NL}^T\; {}^t_0S\; {}^t_0B_{NL}\; {}^0dv\right)\; u$
	$\int_{^0V} {}^t_0S_{ij}\; \delta\, {}_0e_{ij}\; {}^0dv$	${}^t_0F = \int_{^0V} {}^t_0B_L^T\; {}^t_0\hat{S}\; {}^0dv$
UPDATED LAGRANGIAN FORMULATION	$\int_{^tV} {}_tC_{ijrs}\; {}_te_{rs}\; \delta\, {}_te_{ij}\; {}^tdv$	${}^t_tK_L\; u = \left(\int_{^tV} {}^t_tB_L^T\; {}_tC\; {}^t_tB_L\; {}^tdv\right)\; u$
	$\int_{^tV} {}^t\tau_{ij}\; \delta\, {}_t\eta_{ij}\; {}^tdv$	${}^t_tK_{NL}\; u = \left(\int_{^tV} {}^t_tB_{NL}^T\; {}^t\tau\; {}^t_tB_{NL}\; {}^tdv\right)\; u$
	$\int_{^tV} {}^t\tau_{ij}\; \delta\, {}_te_{ij}\; {}^tdv$	${}^t_tF = \int_{^tV} {}^t_tB_L^T\; {}^t\hat{\tau}\; {}^tdv$

TABLE V. STEP-BY-STEP INTEGRATION

Initial Calculations

1. Form linear stiffness matrix K, mass matrix M and damping matrix C; initialize ${}^{0}u$, ${}^{0}\dot{u}$, ${}^{0}\ddot{u}$
2. Calculate the following constants:
   tol $\leq$ 0.01 ; nitem $\geq$ 3 ; in static analysis $\theta = 1$ and go to 3.

   Wilson θ-method : $\theta \geq 1.37$, usually $\theta = 1.4$, $\tau = \theta\Delta t$

$a_0 = 6/\tau^2$	$a_1 = 3/\tau$	$a_2 = 2a_1$	$a_3 = 2$
$a_4 = 2$	$a_5 = \tau/2$	$a_6 = a_0/\theta$	$a_7 = -a_2/\theta$
$a_8 = 1 - 3/\theta$	$a_9 = \Delta t/2$	$a_{10} = \Delta t^2/6$	

   Newmark method: $\theta = 1.0$, $\delta \geq 0.50$, $\alpha \geq 0.25\,(0.5 + \delta)^2$, $\tau = \Delta t$

$a_0 = 1/(\alpha\Delta t^2)$	$a_1 = \delta/(\alpha\Delta t)$	$a_2 = 1/(\alpha\Delta t)$	$a_3 = 1/(2\alpha) - 1$
$a_4 = \delta/\alpha - 1$	$a_5 = \Delta t(\delta/\alpha - 2)/2$	$a_6 = a_0$	$a_7 = -a_2$
$a_8 = -a_3$	$a_9 = \Delta t(1 - \delta)$	$a_{10} = \delta\Delta t$	

3. Form effective linear stiffness matrix: $\hat{K} = K + a_0 M + a_1 C$
4. In linear analysis triangularize $\hat{K}$

For Each Timestep

A. IN LINEAR ANALYSIS

(i) Form effective load vector:

$${}^{t+\tau}\hat{R} = {}^{t}R + \theta\,({}^{t+\Delta t}R - {}^{t}R) + M(a_0\,{}^{t}u + a_2\,{}^{t}\dot{u} + a_3\,{}^{t}\ddot{u})$$

$$+ C(a_1\,{}^{t}u + a_4\,{}^{t}\dot{u} + a_5\,{}^{t}\ddot{u})$$

(ii) Solve for displacement increments:

$$\hat{K}\ {}^{t+\tau}u = {}^{t+\tau}\hat{R}\ ;$$

$$u = {}^{t+\tau}u - {}^{t}u$$

(iii) Go to C.

TABLE V. (Continued)

B. IN NONLINEAR ANALYSIS

(i) If a new stiffness matrix is to be formed, update $\hat{K}$ for nonlinear stiffness effects to obtain ${}^t\hat{K}$; triangularize ${}^t\hat{K}$: ${}^t\hat{K} = LDL^T$

(ii) Form effective load vector

$$ {}^{t+\tau}\hat{R} = {}^tR + \theta({}^{t+\Delta t}R - {}^tR) + M\,(a_2{}^t\dot{u} + a_3{}^t\ddot{u}) + C\,(a_4{}^t\dot{u} + a_5{}^t\ddot{u}) - {}^tF $$

(iii) Solve for displacement increments using latest D,L factors:

$$ LDL^T\,u = {}^{t+\tau}\hat{R} $$

(iv) If required, iterate for dynamic equilibrium; then initialize

$$ u^{(0)} = u\ ,\ i = 0 $$

(a) $i = i + 1$

(b) Calculate (i-1)st approximation to accelerations, velocities, and displacements: ${}^{t+\tau}\ddot{u}^{(i-1)} = a_0 u^{(i-1)} - a_2{}^t\dot{u} - a_3{}^t\ddot{u}$;

${}^{t+\tau}\dot{u}^{(i-1)} = a_1 u^{(i-1)} - a_4{}^t\dot{u} - a_5{}^t\ddot{u}$; ${}^{t+\tau}u^{(i-1)} = {}^tu + u^{(i-1)}$

(c) Calculate (i-1)st effective out-of-balance loads:

$$ {}^{t+\tau}\hat{R}^{(i-1)} = {}^tR + \theta({}^{t+\Delta t}R - {}^tR) - M\,{}^{t+\tau}\ddot{u}^{(i-1)} - C\,{}^{t+\tau}\dot{u}^{(i-1)} - {}^{t+\tau}F^{(i-1)} $$

(d) Solve for i'th correction to displacement increments:

$$ LDL^T\,\Delta u^{(i)} = {}^{t+\tau}\hat{R}^{(i-1)} $$

(e) Calculate new displacement increments: $u^{(i)} = u^{(i-1)} + \Delta u^{(i)}$

(f) Iteration convergence if $\| \Delta u^{(i)} \|_2 / \| u^{(i)} + {}^tu \|_2 < tol$

If convergence : $u = u^{(i)}$ and go to C;

If no convergence and $i <$ nitem: go to (a); otherwise restart using new stiffness matrix and/or a smaller time step size.

C. CALCULATE NEW ACCELERATIONS, VELOCITIES, AND DISPLACEMENTS

Wilson θ-method:

$$ {}^{t+\Delta t}\ddot{u} = a_6 u + a_7{}^t\dot{u} + a_8{}^t\ddot{u} $$

$$ {}^{t+\Delta t}\dot{u} = {}^t\dot{u} + a_9\,({}^{t+\Delta t}\ddot{u} + {}^t\ddot{u}) $$

$$ {}^{t+\Delta t}u = {}^tu + \Delta t\,{}^t\dot{u} + a_{10}\,({}^{t+\Delta t}\ddot{u} + 2\,{}^t\ddot{u}) $$

Newmark method:

$$ {}^{t+\Delta t}\ddot{u} = a_6 u + a_7{}^t\dot{u} + a_8{}^t\ddot{u} $$

$$ {}^{t+\Delta t}\dot{u} = {}^t\dot{u} + a_9{}^t\ddot{u} + a_{10}{}^{t+\Delta t}\ddot{u} $$

$$ {}^{t+\Delta t}u = {}^tu + u $$