ABSTRACT

LAW, SHIRLEY ELIZABETH. Combinatorial Realization of Certain Hopf Algebras of Pattern-Avoiding Permutations. (Under the direction of Nathan Reading.)

A general lattice theoretic construction of Reading constructs Hopf subalgebras of the Malvenuto-Reutenauer Hopf algebra (MR) of permutations. The products and coproducts of these Hopf subalgebras are defined extrinsically in terms of the embedding in MR. This thesis further develops the understanding of these Hopf subalgebras. The goal is to find an intrinsic combinatorial description of a particular family of these Hopf subalgebras. A simple Hopf algebra in the family has a natural basis given by permutations that we call Pell permutations. The Pell permutations are in bijection with combinatorial objects that we call sashes, that is, tilings of a 1 by n rectangle with three types of tiles: black 1 by 1 squares, white 1 by 1 squares, and white 1 by 2 rectangles. The bijection induces a Hopf algebra structure on sashes. We describe the product and coproduct in terms of sashes, and the natural partial order on sashes. We also describe the dual coproduct and dual product of the dual Hopf algebra of sashes. In general, this family of Hopf subalgebras has a natural basis that is in bijection with combinatorial objects that we call partial evaluations. We give a description of the product and the partial order of partial evaluations, and we give a partial description of the coproduct of partial evaluations.

Combinatorial Realization of Certain Hopf Algebras
of Pattern-Avoiding Permutations

by
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A dissertation submitted to the Graduate Faculty of
North Carolina State University
in partial fulfillment of the
requirements for the Degree of
Doctor of Philosophy

Mathematics

Raleigh, North Carolina

2013

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DEDICATION

A.M.D.G.

(For the greater glory of God)

BIOGRAPHY

Shirley Elizabeth Law was raised in Chapel Hill, North Carolina and graduated from East Chapel Hill High School in 2003. She developed an interest in math from an early age by working logic puzzles with her father. Shirley attended college at Appalachian State University. In 2007 she received a Bachelor of Science in Mathematics, a Bachelor of Arts in Economics, and a minor in Statistics. She continued her studies at North Carolina State University where she received a Master of Science in Mathematics in 2009. From 2009 to 2010, Shirley lived and worked at a university in Henan, China teaching Finance and English. She returned to North Carolina State University to complete a Doctor of Philosophy in Mathematics. Her field of interest is algebraic and enumerative combinatorics.

ACKNOWLEDGEMENTS

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Introduction

1.1 Background

A unital associative *algebra* over a field $\mathbb{K}$ is a vector space V over $\mathbb{K}$ with an associative bilinear product $m : V \otimes V \to V$ and a unit $\epsilon : \mathbb{K} \to V$ such that the diagrams in Figure 1.1 commute. The map I is the identity from V to V. Similarly, a counital coassociative *coalgebra* over $\mathbb{K}$ is a vector space V over $\mathbb{K}$ with a coproduct $\Delta : V \to V \otimes V$ and a counit $\zeta : V \to \mathbb{K}$ such that the diagrams in Figure 1.2 commute.

A *bialgebra* is a vector space that is both a unital associative algebra and a counital coassociative coalgebra, such that the diagram in Figure 1.3 commutes. The map $T : V \otimes V \to V \otimes V$ is the *twist map* that sends $v_2 \otimes v_1 \mapsto v_1 \otimes v_2$, for all $v_1, v_2 \in V$.

A graded vector space over $\mathbb{K}$ is a direct sum $\bigoplus_{n \geq 0} V_n$ where each V_n is a finite dimensional
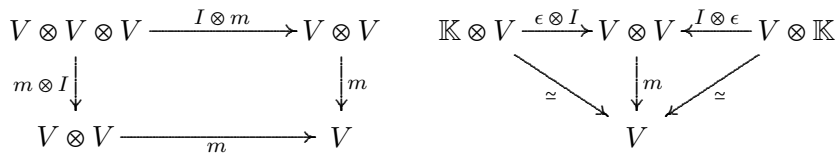
Figure 1.1: Algebra Commutative Diagrams

$$
\begin{array}{ccc}
V & \xrightarrow{\ \Delta\ } & V \otimes V \\
\Delta \downarrow & & \downarrow \Delta \otimes I \\
V \otimes V & \xrightarrow{\ I \otimes \Delta\ } & V \otimes V \otimes V
\end{array}
\qquad\qquad
\mathbb{K} \otimes V \xleftarrow{\zeta \otimes I} V \otimes V \xrightarrow{I \otimes \zeta} V \otimes \mathbb{K}
$$

Figure 1.2: Coalgebra Commutative Diagrams

$$
\begin{array}{ccccc}
V \otimes V & \xrightarrow{\ m\ } & V & \xrightarrow{\ \Delta\ } & V \otimes V \\
\Delta \otimes \Delta \downarrow & & & & \uparrow m \otimes m \\
V \otimes V \otimes V \otimes V & & \xrightarrow{\ I \otimes T \otimes I\ } & & V \otimes V \otimes V \otimes V
\end{array}
$$

Figure 1.3: Bialgebra Commutative Diagram

vector space over $\mathbb{K}$. A bialgebra on a graded vector space $V = \bigoplus_{n \geq 0} V_n$ is a *graded bialgebra* if m maps $V_p \otimes V_q$ to V_{p+q} for all $p \geq 0$ and $q \geq 0$, and if Δ maps V_n to $\bigoplus_{p+q=n} V_p \otimes V_q$.

In general, a Hopf algebra is a bialgebra, with an additional map from V to V called the *antipode*, satisfying certain conditions. However, every graded bialgebra possesses an antipode. Thus a graded Hopf algebra is nothing more than a graded bialgebra. We refer to each graded Hopf algebra by a triple (V, m, Δ), where V is a graded vector space, m is a product map, and Δ is a coproduct map. More information about Hopf algebras (graded or not) can be found in [14].

Example 1.1.1. Let V be a graded vector space of polynomials such that for each grade $n \geq 0$, the basis vector is x^n. We will describe the Hopf algebra of polynomials by defining a product and a coproduct on the basis vectors.

$$
m(x^p \otimes x^q) = x^{p+q}
$$

$$
\Delta(x^n) = \sum_{i=0}^{n} \binom{n}{i} (x^i \otimes x^{n-i})
$$

The focus of this research is on combinatorial Hopf algebras: Hopf algebras such that the basis elements of the underlying vector space are indexed by a family of combinatorial objects. For each $n \geq 0$, let O_n be a finite set of "combinatorial objects". We define a graded vector space over a field $\mathbb{K}$, such that for each grade n the basis vectors of the vector space are indexed by the elements of O_n. That is, the graded vector space is: $\mathbb{K}[O_\infty] = \bigoplus_{n \geq 0} \mathbb{K}[O_n]$. For simplicity, we refer to a basis element of this vector space by the combinatorial object indexing it. There is a more sophisticated approach for defining combinatorial Hopf algebras. For more information see [1].

The Malvenuto-Reutenauer Hopf algebra of permutations (MR) is a graded combinatorial Hopf Algebra $(\mathbb{K}[S_\infty], \bullet, \Delta)$. Given a field $\mathbb{K}$, let $\mathbb{K}[S_n]$ be a vector space whose basis elements are indexed by the elements of S_n, where S_n is the group of permutations of the set $[n] = \{1, 2, \ldots, n\}$. We identify the basis elements with the permutations themselves and thus write elements of $\mathbb{K}[S_n]$ as $\mathbb{K}$-linear combinations of permutations in S_n.

The product in MR is called the shifted shuffle. To multiply two permutations, $x = x_1 x_2 \cdots x_p \in S_p$ and $y = y_1 y_2 \cdots y_q \in S_q$, we begin by shifting y. Define $y' = y'_1 \cdots y'_q$ where $y'_i = y_i + p$. A *shifted shuffle* of x and y is a permutation $z \in S_n$ where $n = p + q$, such that the total order of the entries 1 through p of z, is given by x, and the total order of the entries $p + 1$ through n of z, is given by y'. The product of x and y in MR is the sum of all the shifted shuffles of x and y and is denoted by either $m(x \otimes y)$ or $x \bullet y$.

Example 1.1.2. The product of two basis elements 21 and 312:

$$21 \bullet 312 = \; 21534 + 25134 + 25314 + 25341 + 52134 +$$
$$52314 + 52341 + 53214 + 53241 + 53421$$

The coproduct map is defined by separating a permutation in all possible places and then *standardizing* the result. The notion of standardization is illustrated as part of Example 1.1.3, and a precise definition of standardization is given in Section 1.2. We standardize so that all of the terms of the coproduct are composed of permutations. For example, if we separate the

permutation 32154 between the 2 and the 1 we have 32 and 154. Standardizing these pieces gives the term $21 \otimes 132$.

The coproduct in MR is

$$\Delta(x) = \sum_{i=0}^{n} \operatorname{st}(x_1 \cdots x_i) \otimes \operatorname{st}(x_{i+1} \cdots x_n)$$

where $\operatorname{st}(x_1 \cdots x_0)$ and $\operatorname{st}(x_{n+1} \cdots x_n)$ are both interpreted as the empty permutation $\varnothing$, the unique element of S_0.

Example 1.1.3.

$$\Delta(32154) = \varnothing \otimes 32154 + 1 \otimes 2143 + 21 \otimes 132 + 321 \otimes 21 + 3214 \otimes 1 + 32154 \otimes \varnothing$$

Notice that the product is the sum of the natural ways of combining two basis elements, and the coproduct is the sum of the natural ways of splitting a basis element into two pieces.

Another well-known combinatorial Hopf algebra is NSym [6]. NSym can be represented as the Hopf algebra of noncommutative symmetric functions, the Hopf algebra of compositions, or as the Hopf algebra of subsets, but we will describe it here in terms of another combinatorial object: a tiling of a $1 \times n$ rectangle with black 1×1 squares and/or white 1×1 squares. This representation will be relevant to the description of the Hopf algebra of Sashes in Chapter 2 and to the description of the Hopf algebra of partial evaluations in Chapter 3. More information about this Hopf algebra can be found in [2].

Fix a $1 \times n$ rectangle, and let Υ_n be the set of tilings of that rectangle by black and/or white 1×1 squares, see Figure 1.4. We now describe the Hopf algebra $(\mathbb{K}[\Upsilon_\infty], \bullet_\Upsilon, \Delta_\Upsilon)$, where $\mathbb{K}[\Upsilon_\infty] = \bigoplus_{n \geq 0} \mathbb{K}[\Upsilon_{n-1}]$. Let $A \in \Upsilon_p$ and $B \in \Upsilon_q$. The product map is given by: $A \bullet_\Upsilon B = A \blacksquare B + A \square B$. The coproduct is more complicated, so we will leave the description until Chapter 3.

An important Hopf subalgebra of MR is the Loday-Ronco Hopf algebra, defined on a graded vector space with basis elements indexed by planar binary trees. A *planar binary tree* is a rooted tree (a tree with a distinguished vertex called the *root*) in which every vertex either has no

$$\begin{array}{ll} \Upsilon_{-1} & \varnothing \\ \Upsilon_0 & \| \\ \Upsilon_1 & \blacksquare, \square \\ \Upsilon_2 & \blacksquare\blacksquare, \blacksquare\square, \square\blacksquare, \square\square \\ \Upsilon_3 & \blacksquare\blacksquare\blacksquare, \blacksquare\blacksquare\square, \blacksquare\square\blacksquare, \square\blacksquare\blacksquare, \\ & \blacksquare\square\square, \square\blacksquare\square, \square\square\blacksquare, \square\square\square \end{array}$$
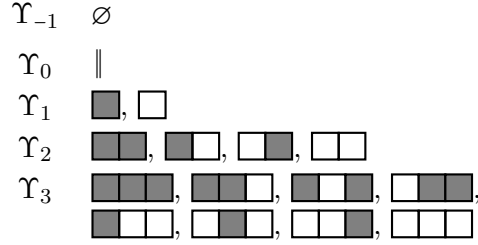
Figure 1.4: Elements of Υ_n for $-1 \le n \le 3$

Figure 1.5: The five planar binary trees in PBT_3.

children or has two children, one designated as a *right* child and one as a *left* child. Vertices with no children are called *leaves*. We draw the trees with the leaves lined up horizontally and the root above them, with edges drawn so that, for each leaf, the path from that leaf to the root is monotone up. Let PBT_n be the set of planar binary trees with $n+1$ leaves. For example, the five trees of PBT_3 are shown in Figure 1.5. A description of the product and coproduct of this combinatorial graded Hopf algebra is found in [9].

The final combinatorial Hopf algebra we mention here is the Hopf algebra of diagonal rectangulations $(\mathbb{K}[\mathrm{dRec}_\infty], \bullet_{dR}, \Delta_{dR})$ as described in [8]. A diagonal rectangulation of size n is a $n \times n$ square divided into n rectangles such that the interior of each rectangle intersects the diagonal of the square with negative slope. The bottom left corner of the square is on the origin of the Cartesian plane, and all points of intersection within the rectangulation have integer coordinates. Figure 1.6 shows a diagonal rectangulation of size 20. The diagonal is shown in gray. The number of diagonal rectangulations of size n are counted by the Baxter numbers.

Let dRec_n stand for the set of rectangulations of size n. The set dRec_0 has a single element, the rectangulation of a 0×0 square having no rectangles. This empty rectangulation is repre-

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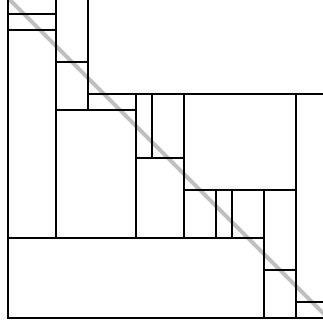
Figure 1.6: A diagonal rectangulation of size 20

Figure 1.7: A product calculation in the Hopf algebra dRec

sented by the symbol $\varnothing$. The set dRec_1 also has a single element, the "division" of a square into a single square. The Hopf algebra dRec is the graded vector space $\mathbb{K}[\mathrm{dRec}_\infty] = \bigoplus_{n \geq 0} \mathbb{K}[\mathrm{dRec}_n]$, with the product and coproduct that we now describe.

Let $R_1 \in \mathrm{dRec}_p$ and $R_2 \in \mathrm{dRec}_q$, where $p + q = n$. The product in dRec is the sum over all *completions* of the following "incomplete rectangulation": A $n \times n$ square with the interior of R_1 in the upper left corner and the interior of R_2 in the lower right corner. That is, we sum over all rectangulations of size n that can be defined by adding additional lines to the incomplete rectangulation. We give two examples in Figures 1.7 and 1.8, marking, for the sake of clarity, the point $(p, n - p)$ in each diagonal rectangulation in the product.

Before we describe the coproduct Δ_{dR}, we first need to define some terminology. Let R be a diagonal rectangulation of size n and consider any path γ from the top-left corner of R to the bottom-right corner of R, traveling only along edges of rectangles, and traveling only downwards and to the right. The path γ divides the rectangles of R into two groups: There are

Figure 1.8: A product calculation in the Hopf algebra dRec

p rectangles below/left of the path and q rectangles above/right of the path, with $p + q = n$.

We associate to the path γ an element $A_\gamma \otimes B_\gamma$ of dRec $\otimes$ dRec. If $p = 0$ then the element A_γ is $\varnothing$. Otherwise, A_γ is obtained as follows: First, delete from R everything that lies on or is above/right of γ, except for the outer edges of the square. Then, scale this incomplete rectangulation down to a $p \times p$ size square. The element A_γ is the sum over all completions of this incomplete diagonal rectangulation. Similarly, $B_\gamma = \varnothing$ if $q = 0$, and otherwise B_γ is obtained by deleting from R everything that lays on or is below/left of γ, except for the outer edges of the square, scaling the result to a $q \times q$ square, and then summing over all completions of the resulting incomplete diagonal rectangulation. The coproduct of R is the sum, over all paths γ, of $A_\gamma \otimes B_\gamma$.

Consider, for example, the coproduct of
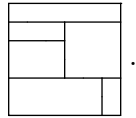
Figure 1.9 shows each path γ and the associated A_γ and B_γ. The figure uses the abbreviation "c." to mean "the sum over all completions of appropriate size."

Figure 1.9: A coproduct calculation in the Hopf algebra dRec

1.2 Hopf Algebra of Pattern-avoiding Permutations

Let S_n be the group of permutations of the set $[n] = \{1, 2, \ldots, n\}$. Also define $[n, n'] = \{n, n+1, \ldots, n'\}$ for $n' \geq n$. For $x = x_1 x_2 \cdots x_n \in S_n$, an *inversion* of x is a pair (x_i, x_j) where $i < j$ and $x_i > x_j$, and the *inversion set of x* is the set of all such inversions. The weak order is the partial order on S_n with $x \leq x'$ if and only if the inversion set of x is contained in the inversion set of x'. The weak order is a lattice. The inverse x^{-1} of a permutation $x \in S_n$ is the permutation $x^{-1} = y = y_1 \cdots y_n \in S_n$ such that $y_i = j$ when $x_j = i$.

Let T be a set consisting of integers $t_1 < t_2 < \cdots < t_n$. Given a permutation $x \in S_n$, the notation $(x)_T$ stands for the permutation of T whose one-line notation has t_j in the ith position when $x_i = j$. On the other hand, given a permutation x of T, the *standardization*, $\mathrm{st}(x)$, is the unique permutation $y \in S_n$ such that $(y)_T = x$.

Now let T be a subset of $[n]$. For $x \in S_n$, the permutation $x|_T$ is the permutation of T obtained by removing from the one-line notation for x all entries that are not elements of T.

Example 1.2.1. Let $x = 31254$, $T_1 = \{2, 3, 6, 8, 9\}$, and $T_2 = \{2, 3, 5\}$. Then, $(x)_{T_1} = 62398$ and thus $\mathrm{st}(62398) = 31254$. Also, $x|_{T_2} = 325$.

The Malvenuto-Reutenauer Hopf algebra MR is a graded Hopf Algebra $(\mathbb{K}[S_\infty], \bullet, \Delta)$, as we explained in Section 1.1. We now reiterate a description of MR with more detail and with some variations to notation. Let $\mathbb{K}[S_\infty] = \bigoplus_{n \geq 0} \mathbb{K}[S_n]$ be a graded vector space. Let $x = x_1 x_2 \cdots x_p \in S_p$ and $y = y_1 y_2 \cdots y_q \in S_q$. Define $y' = y'_1 \cdots y'_q$ to be $(y)_{[p+1, p+q]}$ so that $y'_i = y_i + p$. A *shifted shuffle* of x and y' is a permutation $z \in S_n$ where $n = p + q$, $z|_{[p]} = x$ and $z|_{[p+1,n]} = y'$. The product of x and y in MR is the sum of all the shifted shuffles of x and y. Equivalently,

$$x \bullet y = \sum [x \cdot y', y' \cdot x] \tag{1.1}$$

where $x \cdot y'$ is the concatenation of the permutations x and y', and $\sum [x \cdot y', y' \cdot x]$ denotes the

sum of all the elements in the weak order interval $[x \cdot y', y' \cdot x]$. The coproduct in MR is:

$$\Delta(x) = \sum_{i=0}^{p} \mathrm{st}(x_1 \cdots x_i) \otimes \mathrm{st}(x_{i+1} \cdots x_p) \tag{1.2}$$

where $\mathrm{st}(x_1 \cdots x_0)$ and $\mathrm{st}(x_{p+1} \cdots x_p)$ are both interpreted as the empty permutation $\varnothing$.

Define the map $\mathrm{Inv} : S_n \to S_n$ by $\mathrm{Inv}(x) = x^{-1}$ and extend the map linearly to a map $\mathrm{Inv} : \mathbb{K}S_\infty \to \mathbb{K}S_\infty$. MR is known to be self dual [10] and specifically Inv is an isomorphism from $(\mathbb{K}[S_\infty], \bullet, \Delta)$ to the graded dual Hopf algebra $(\mathbb{K}[S_\infty], \Delta^*, m^*)$. Let $x \in S_p$, $y \in S_q$, and $z \in S_n$, where $p + q = n$. Given a subset T of p elements of $[n]$, T^C denotes the complement of T in $[n]$. The dual product is given by:

$$\Delta^*(x \otimes y) = \mathrm{Inv}(x^{-1} \bullet y^{-1}) = \sum_{\substack{T \subseteq [n], \\ |T|=p}} (x)_T \cdot (y)_{T^C}, \tag{1.3}$$

and the dual coproduct is:

$$m^*(z) = (\mathrm{Inv} \otimes \mathrm{Inv})(\Delta(z^{-1})) = \sum_{i=0}^{n} z|_{[i]} \otimes \mathrm{st}(z|_{[i+1,n]}) \tag{1.4}$$

where $z|_{[0]}$ and $z|_{[n+1,n]}$ are both interpreted as the empty permutation $\varnothing$.

Now that we have explicitly described both the Hopf algebra of permutations and the dual Hopf algebra of permutations, we will present a family of Hopf subalgebras that are defined by a particular pattern-avoidance condition. This family of Hopf algebras is defined by Reading [12].

For some $k \geq 2$, let $V \subseteq [2, k-1]$ such that $|V| = j$ and let V^C be the complement of V in $[2, k-1]$. A permutation $x \in S_n$ *avoids* the pattern $V(k1)V^C$ if for every subsequence $x_{i_1} x_{i_2} \cdots x_{i_k}$ of x with $i_{j+2} = i_{j+1} + 1$, the standardization $\mathrm{st}(x_{i_1} x_{i_2} \cdots x_{i_k})$ is not of the form $v(k1)v'$ for any permutation v of the set V and any permutation v' of V^C. In the notation of Babson and Steingrimsson [4] avoiding $V(k1)V^C$ means avoiding all patterns of the form $v_1 - \cdots - v_j - k1 - v'_1 - \cdots - v'_{k-j-2}$, where $v_1 \cdots v_j$ is a permutation of V and $v'_1 \cdots v'_{k-j-2}$ is a permutation of V^C.

Let U be a set of patterns of the form $V(k1)V^C$, where $|V|$ and k can vary. Define Av_n to be the set of permutations in S_n that avoid all of the patterns in U. We define a graded Hopf algebra $(\mathbb{K}[\mathrm{Av}_\infty], \bullet_{\mathrm{Av}}, \Delta_{\mathrm{Av}})$ as a graded Hopf subalgebra of MR. Let $\mathbb{K}[\mathrm{Av}_n]$ be a vector space, over a field $\mathbb{K}$, with basis vectors indexed by the elements of Av_n, and let $\mathbb{K}[\mathrm{Av}_\infty]$ be the graded vector space $\bigoplus_{n\geq 0} \mathbb{K}[\mathrm{Av}_n]$. The product and coproduct on $\mathbb{K}[\mathrm{Av}_\infty]$ are described below.

We define a map $\pi_\downarrow : S_n \to \mathrm{Av}_n$ recursively. If $x \in \mathrm{Av}_n$ then define $\pi_\downarrow(x) = x$. If $x \in S_n$, but $x \notin \mathrm{Av}_n$, then x contains an instance of a pattern $V(k1)V^C$ in U. That is, there exists some subsequence $x_{i_1} x_{i_2} \cdots x_{i_k}$ of x, where $i_{j+2} = i_{j+1} + 1$ and $j = |V|$, such that $st(x_{i_1} x_{i_2} \cdots x_{i_k}) = vk1v'$ for some permutations v and v' of V and V^C. Exchange $x_{i_{j+1}}$ and $x_{i_{j+2}}$ in x to create a new permutation x', calculate $\pi_\downarrow(x')$ recursively and set $\pi_\downarrow(x) = \pi_\downarrow(x')$. The recursion must terminate because an inversion of x is destroyed at every step, and because the identity permutation is in Av_n. The map $\pi_\downarrow$ is well-defined as explained in [12, Remark 9.5]. We emphasize that the definition of $\pi_\downarrow$ is dependent on U.

The map $\pi_\downarrow$ defines an equivalence relation with permutations $x, x' \in S_n$ equivalent if and only if $\pi_\downarrow(x) = \pi_\downarrow(x')$. The set Av_n is a set of representatives of these equivalence classes. This equivalence relation is a lattice congruence on the weak order. Therefore the poset induced on Av_n by the weak order is a lattice (also denoted by Av_n) and the map $\pi_\downarrow$ is a lattice homomorphism from the weak order to Av_n. The congruence classes defined by $\pi_\downarrow$ are intervals, and $\pi_\downarrow$ maps an element to the minimal element of its congruence class. Let $\pi^\uparrow$ be the map that takes an element to the maximal element of its congruence class.

The following proposition is a special case of [12, Proposition 2.2]. The congruence on S_n defined by $\pi_\downarrow$ is denoted by Θ. For $x \in S_n$, the congruence class of $x \bmod \Theta$ is denoted by $[x]_\Theta$.

Proposition 1.2.2. *Given S_n a finite lattice, Θ a congruence on S_n, and $x \in S_n$, the map $y \to [y]_\Theta$ restricts to a one-to-one correspondence between elements of S_n covered by $\pi_\downarrow(x)$ and elements of Av_n covered by $[x]_\Theta$.*

Example 1.2.3. A well known lattice is the Tamari lattice which is isomorphic to Av_n for $U = \{2(31)\}$. Let G_n be the set of objects of this lattice. The elements of G_n are strings of n factors $\square$ grouped into pairs by $n-1$ sets of parentheses. The partial order relation for the Tamari lattice is $(A(BC)) \lessdot ((AB)C)$, where A, B, and C are elements of G_h, G_i, and G_j respectively, for some $1 \le h, i, j \le n-2$ such that $h + i + j = n$.

Both $\pi_\downarrow$ and $\pi^\uparrow$ are order preserving and $\pi^\uparrow \circ \pi_\downarrow = \pi^\uparrow$ and $\pi_\downarrow \circ \pi^\uparrow = \pi_\downarrow$. A $\pi_\downarrow$-*move* is the result of switching two adjacent entries of a permutation in the manner described above. That is, it changes $\cdots k1 \cdots$ to $\cdots 1k \cdots$ for some pattern in U. A $\pi^\uparrow$-*move* is the result of switching two adjacent entries of a permutation in a way such that a $\pi^\uparrow$-move undoes a $\pi_\downarrow$-move. That is, it changes $\cdots 1k \cdots$ to $\cdots k1 \cdots$.

We define a map $r : \mathbb{K}[S_\infty] \to \mathbb{K}[\text{Av}_\infty]$ that identifies the representative of a congruence class. Given $x \in S_n$,

$$r(x) = \begin{cases} x & \text{if } x \in \text{Av}_n \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we define a map $c : \mathbb{K}[\text{Av}_\infty] \to \mathbb{K}[S_\infty]$ that takes an avoider to the sum of its congruence class:

$$c(x) = \sum_{\substack{y \text{ such that} \\ \pi_\downarrow(y)=x}} y.$$

We now describe the product and coproduct in $(\mathbb{K}[\text{Av}_\infty], \bullet_{\text{Av}}, \Delta_{\text{Av}})$. Let $x \in \text{Av}_p$, and let $y \in \text{Av}_q$. Then:

$$m_{\text{Av}}(x \otimes y) = x \bullet_{\text{Av}} y = r(x \bullet y). \tag{1.5}$$

Just as the product in MR is $\sum [x \cdot y', y' \cdot x]$, we can view this product as:

$$x \bullet_{\text{Av}} y = \sum [x \cdot y', \pi_\downarrow(y' \cdot x)], \tag{1.6}$$

where $[x \cdot y', \pi_\downarrow(y' \cdot x)]$ is an interval on the lattice Av_n.

The coproduct is:

$$\Delta_{\mathrm{Av}}(z) = (r \otimes r)(\Delta(c(z))). \tag{1.7}$$

We now describe the Hopf algebra $(\mathbb{K}[\mathrm{Av}_\infty], \Delta^*_{\mathrm{Av}}, \bullet^*_{\mathrm{Av}})$ that is dual to $(\mathbb{K}[\mathrm{Av}_\infty], \bullet_{\mathrm{Av}}, \Delta_{\mathrm{Av}})$. We extend the map $\pi_\downarrow$ linearly, so $\pi_\downarrow$ is a map from $\mathbb{K}[S_\infty]$ to $\mathbb{K}[\mathrm{Av}_\infty]$. The map that is dual to the map c is $c^* : \mathbb{K}[S_\infty] \to \mathbb{K}[\mathrm{Av}_\infty]$, where $c^*(x) = \pi_\downarrow(x)$ for $x \in \mathbb{K}[S_\infty]$. The map that is dual to the map r is $r^* : \mathbb{K}[\mathrm{Av}_\infty] \to \mathbb{K}[S_\infty]$, where $r^*(x) = x$ for $x \in \mathbb{K}[\mathrm{Av}_\infty]$.

Let $z \in \mathrm{Av}_n$, where $n = p + q$. The dual coproduct is given by dualizing Equation (1.5), so that:

$$m^*_{\mathrm{Av}}(z) = m^*(z). \tag{1.8}$$

The dual product Δ^*_{Av} is given by dualizing Equation (1.7):

$$\Delta^*_{\mathrm{Av}}(x \otimes y) = \pi_\downarrow \Delta^*(x \otimes y). \tag{1.9}$$

Combining Equation (1.9) with Equation (1.3), we have:

$$\Delta^*_{\mathrm{Av}}(x \otimes y) = \sum_{\substack{T \subseteq [n] \\ |T| = p}} \pi_\downarrow((x)_T \cdot (y)_{T^C}) \tag{1.10}$$

Equation (1.10) leads to the following order theoretic description of the coproduct Δ_{Av}, which was worked out jointly with Nathan Reading.

Given $z \in \mathrm{Av}_n$, a subset $T \subseteq [n]$ is *good with respect to* z if there exists a permutation $z' = z'_1 \cdots z'_n$ with $\pi_\downarrow(z') = z$ such that $T = \{z'_1, \ldots, z'_{|T|}\}$. Suppose T is good with respect to z, let $p = |T|$ and let $q = n - p$. Let $z_{\min}$ be minimal, in the weak order on S_n, among permutations equivalent to z and whose first p entries are the elements of T. Let $z_{\max}$ be maximal, in the weak order, among such permutations. Define I_T to be the sum over the elements in the interval $[\mathrm{st}(z_{\min}|_T), \pi_\downarrow \mathrm{st}(z_{\max}|_T)]$ in Av_p and define J_T to be the sum over the elements in the interval $[\mathrm{st}(z_{\min}|_{T^C}), \pi_\downarrow \mathrm{st}(z_{\max}|_{T^C})]$ in Av_q.

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Theorem 1.2.4. *Let $z \in \mathrm{Av}_n$. Then*

$$\Delta_{\mathrm{Av}}(z) = \sum_{T \text{ is good}} I_T \otimes J_T$$

where $I_T = \sum [\mathrm{st}(z_{\min}|_T), \pi_\downarrow \mathrm{st}(z_{\max}|_T)]$, $J_T = \sum [\mathrm{st}(z_{\min}|_{T^C}), \pi_\downarrow \mathrm{st}(z_{\max}|_{T^C})]$.

To prove Theorem 1.2.4, we first need several lemmas.

Lemma 1.2.5. *The elements in the interval $[z_{\min}, z_{\max}]$ are equivalent to z and their first p entries are the elements of T.*

Proof. All of the elements in the interval $[z_{\min}, z_{\max}]$ are equivalent to z because equivalence classes are intervals in the weak order. To prove the rest of the lemma, suppose for the sake of contradiction that there is an element $z' \in [z_{\min}, z_{\max}]$ whose first p entries are not the elements of T. That is, z' has some $y \in T^C$ before some $x \in T$. If $x < y$, then $(y, x) \in \mathrm{Inv}(z')$, but $(y, x) \notin \mathrm{Inv}(z_{\max})$, so $z' \notin [z_{\min}, z_{\max}]$. If $y < x$, then $(x, y) \in \mathrm{Inv}(z_{\min})$, but $(x, y) \notin \mathrm{Inv}(z')$, so $z' \notin [z_{\min}, z_{\max}]$. Therefore the first p entries of elements in the interval $[z_{\min}, z_{\max}]$ are the elements of T. $\square$

Lemma 1.2.6. *Suppose $T \subseteq [n]$ with $|T| = p$. Let $q = n - p$. Suppose also that $x_1 \leq x_2 \leq x_3$ in Av_p, and that $y_1 \leq y_2 \leq y_3$ in Av_q. If $\pi_\downarrow((x_1)_T \cdot (y_1)_{T^C}) = \pi_\downarrow((x_3)_T \cdot (y_3)_{T^C}) = z$, then $\pi_\downarrow((x_2)_T \cdot (y_2)_{T^C}) = z$.*

Proof. If $x_1 \leq x_2 \leq x_3$, and $y_1 \leq y_2 \leq y_3$, then $(x_1)_T \cdot (y_1)_{T^C} \leq (x_2)_T \cdot (y_2)_{T^C} \leq (x_3)_T \cdot (y_3)_{T^C}$. Since $\pi_\downarrow$ is an order preserving map, $\pi_\downarrow((x_1)_T \cdot (y_1)_{T^C}) \leq \pi_\downarrow((x_2)_T \cdot (y_2)_{T^C}) \leq \pi_\downarrow((x_3)_T \cdot (y_3)_{T^C})$. The assertion of the lemma follows. $\square$

Lemma 1.2.7. *Suppose $x_1, x_2 \in S_p$ and $y_1, y_2 \in S_q$. Suppose $T \subseteq [n]$, where $n = p + q$, and with $|T| = p$. The following identities hold:*

$$(x_1)_T \cdot (y_1)_{T^C} \vee (x_2)_T \cdot (y_2)_{T^C} = (x_1 \vee x_2)_T \cdot (y_1 \vee y_2)_{T^C}$$

$$(x_1)_T \cdot (y_1)_{T^C} \wedge (x_2)_T \cdot (y_2)_{T^C} = (x_1 \wedge x_2)_T \cdot (y_1 \wedge y_2)_{T^C}$$

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Proof. First we consider the identity with joins. There are three different kinds of inversions in $(x_1)_T \cdot (y_1)_{T^C}$: inversions within x_1, inversions within y_1, and inversions between T and T^C. The inversion set of the permutation on the left hand side of the equation is the union of: inversions within x_1 in terms of T, inversions within y_1 in terms of T^C, inversions within x_2 in terms of T, inversions within y_2 in terms of T^C, and inversions between T and T^C. Similarly, the inversion set of the permutation on the right hand side of the equation is the union of: inversions within x_1 or x_2 in terms of T, inversions within y_1 or y_2 in terms of T^C, and inversions between T and T^C. Therefore the permutation on the left hand of the equation and the permutation on the right hand of the equation have identical inversion sets and are thus the same.

The proof for the identity with meets is identical except for examining intersections of the inversion sets instead of unions. $\square$

Proof of Theorem 1.2.4. In light of Equation (1.10), $\Delta_{\mathrm{Av}}(z)$ is the sum, over $T \subseteq [n]$, of terms $x \otimes y \in \mathrm{Av}_p \otimes \mathrm{Av}_q$ such that $\pi_{\downarrow}((x)_T \cdot (y)_{T^C}) = z$. Some terms $x \otimes y$ may appear in $\Delta_{\mathrm{Av}}(z)$ with coefficient greater than 1, but for each T, a term $x \otimes y$ occurs at most once. Let $\mathrm{terms}(z, T)$ be the set $\{x \otimes y : \pi_{\downarrow}((x)_T \cdot (y)_{T^C}) = z\}$. It is immediate that when $\mathrm{terms}(z, T)$ is nonempty, T is good with respect to z. On the other hand, if T is good with respect to z, then let z' have $\pi_{\downarrow}(z') = z$ and $\{z'_1, z'_2, \ldots, z'_{|T|}\} = T$. Let $x \in S_p$ and $y \in S_q$ be such that $z' = (x)_T \cdot (y)_{T^C}$. Then $\pi_{\downarrow}(x) \in \mathrm{Av}_p$ and $\pi_{\downarrow}(y) \in \mathrm{Av}_q$. Since $\pi_{\downarrow}(x)$ is obtained from x by a sequence of $\pi_{\downarrow}$-moves, and $\pi_{\downarrow}(y)$ is obtained similarly from y, we see that $(\pi_{\downarrow}(x))_T \cdot (\pi_{\downarrow}(y))_{T^C}$ is obtained from $z' = (x)_T \cdot (y)_{T^C}$ by a sequence of $\pi_{\downarrow}$-moves. Thus, $\pi_{\downarrow}((\pi_{\downarrow}(x))_T \cdot (\pi_{\downarrow}(y))_{T^C}) = \pi_{\downarrow}(z') = z$, so $\pi_{\downarrow}(x) \otimes \pi_{\downarrow}(y) \in \mathrm{terms}(z, t)$ and in particular $\mathrm{terms}(z, T)$ is nonempty.

Next, we need to show that, for each good subset T, the set $\mathrm{terms}(z, T)$ is of the form $I_T \otimes J_T$. For convenience, we consider each $x \otimes y$ as an element of $\mathrm{Av}_p \times \mathrm{Av}_q$ without rewriting $x \otimes y$ as (x, y).

Suppose $x_1 \otimes y_1$ and $x_2 \otimes y_2$ are in $\mathrm{terms}(z, T)$. Then by Lemma 1.2.7,

$$\pi_{\downarrow}\big((x_1 \vee x_2)_T \cdot (y_1 \vee y_2)_{T^C}\big) = \pi_{\downarrow}\big((x_1)_T \cdot (y_1)_{T^C} \vee (x_2)_T \cdot (y_2)_{T^C}\big).$$

Since $\pi_\downarrow$ is a lattice homomorphisim, the latter is

$$\pi_\downarrow\big((x_1)_T \cdot (y_1)_{T^C}\big) \vee \pi_\downarrow\big((x_2)_T \cdot (y_2)_{T^C}\big) = z \vee z = z.$$

Thus $(x_1 \vee x_2) \otimes (y_1 \vee y_2)$ is in terms(z, T). The same argument holds for meets, so terms(z, T) is closed under meets and joins in the product order $\mathrm{Av}_p \times \mathrm{Av}_q$. Lemma 1.2.6 implies that terms(z, T) is order-convex in $\mathrm{Av}_p \times \mathrm{Av}_q$. An order-convex subset that is closed under meets and joins is necessarily an interval.

Suppose $x \otimes y < \mathrm{st}(z_{\min}|_T) \otimes \mathrm{st}(z_{\min}|_{T^C})$ in $\mathrm{Av}_p \times \mathrm{Av}_q$. Then $(x)_T \cdot (y)_{T^C} < z_{\min}$ in S_n. Thus $\pi_\downarrow\big((x)_T \cdot (y)_{T^C}\big) \neq z$, by the definition of $z_{\min}$, and therefore $x \otimes y \notin$ terms(z, T). Thus $\mathrm{st}(z_{\min}|_T) \otimes \mathrm{st}(z_{\min}|_{T^C})$ is the minimal element of terms(z, T).

Now suppose $x \otimes y > \pi_\downarrow \mathrm{st}(z_{\max}|_T) \otimes \pi_\downarrow \mathrm{st}(z_{\max}|_{T^C})$ in $\mathrm{Av}_p \times \mathrm{Av}_q$. Then since $\pi^\uparrow$ is order-preserving and $\pi^\uparrow \circ \pi_\downarrow = \pi^\uparrow$, we see that $\pi^\uparrow(x) \otimes \pi^\uparrow(y) > \pi^\uparrow \mathrm{st}(z_{\max}|_T) \otimes \pi^\uparrow \mathrm{st}(z_{\max}|_{T^C})$. Thus on the lattice S_n,

$$\big(\pi^\uparrow(x)\big)_T \cdot \big(\pi^\uparrow(y)\big)_{T^C} > \big(\pi^\uparrow \mathrm{st}(z_{\max}|_T)\big)_T \cdot \big(\pi^\uparrow \mathrm{st}(z_{\max}|_{T^C})\big)_{T^C}. \tag{1.11}$$

The right side of Equation (1.11) is obtained from $z_{\max}$ by standardizing the first part of the permutation, doing some $\pi^\uparrow$-moves, unstandardizing, and then repeating for the last part of the permutation. The same result can be obtained by simply applying the corresponding $\pi^\uparrow$-moves to $z_{\max}$, without standardizing and unstandardizing. In particular, the right side of Equation (1.11) is greater than or equal to $z_{\max}$. Now Equation (1.11) implies that $\big(\pi^\uparrow(x)\big)_T \cdot \big(\pi^\uparrow(y)\big)_{T^C}$ is strictly greater than $z_{\max}$. The definition of $z_{\max}$ says that $\pi^\uparrow(x) \otimes \pi^\uparrow(y) \notin$ terms(z, T). Thus $\big(\pi^\uparrow(x)\big)_T \cdot \big(\pi^\uparrow(y)\big)_{T^C}$ is not equivalent to z.

But $\big(\pi^\uparrow(x)\big)_T \cdot \big(\pi^\uparrow(y)\big)_{T^C}$ is obtained from $(x)_T \cdot (y)_{T^C}$ by standardizing the first part, doing some $\pi^\uparrow$-moves, unstandardizing, and then repeating for the last part. The same result is again obtained by simply applying the the corresponding $\pi^\uparrow$-moves to $(x)_T \cdot (y)_{T^C}$, without standardizing and unstandardizing. Thus $\big(\pi^\uparrow(x)\big)_T \cdot \big(\pi^\uparrow(y)\big)_{T^C}$ is equivalent to $(x)_T \cdot (y)_{T^C}$,

which is therefore not equivalent to z. We have shown that $x \otimes y \notin \text{terms}(z, T)$.

Thus, we have shown that $\text{terms}(z, T)$ equals

$$[\text{st}(z_{\min}|_T) \otimes \text{st}(z_{\min}|_{T^C}), \pi_\downarrow \text{st}(z_{\max}|_T) \otimes \pi_\downarrow \text{st}(z_{\max}|_{T^C})].$$

Any interval in $\text{Av}_p \times \text{Av}_q$ is the product of an interval in Av_p with an interval in Av_q. Thus $\text{terms}(z, T)$ is $I_T \otimes J_T$. $\square$

The proof of Theorem 1.2.4 also establishes the following more detailed statement.

Proposition 1.2.8. *For some $T \subseteq [n]$, $x \otimes y \in \text{terms}(z, T)$ if and only if $x \otimes y$ is a term of $I_T \otimes J_T$ in $\Delta_{\text{Av}}(z)$.*

Proof. Since $x \otimes y \in \text{terms}(z, T)$ means that $\pi_\downarrow\big((x)_T \cdot (y)_{T^C}\big) = z$, we see from Equation(1.10) and Theorem 1.2.4 that for a fixed set T, $x \otimes y$ is a term of the summand indexed by T in $\Delta_{\text{Av}}(z)$ if and only if z is the summand indexed by T in $\Delta^*_{\text{Av}}(x \otimes y)$. $\square$

1.3 Combinatorial Realizations

All of the Hopf algebras described in Section 1.1 are Hopf subalgebras of MR. Also, each is isomorphic to $(\mathbb{K}[\text{Av}_\infty], \bullet_{\text{Av}}, \Delta_{\text{Av}})$ for choices of U shown here.

Object	U
polynomial	$\{(21)\}$
black/white square tilings	$\{2(31), (31)2\}$
planar binary trees	$\{2(31)\}$
diagonal rectangulations	$\{2(41)3, 3(41)2\}$

In Chapter 2, we consider a particular Hopf algebra given by $U = \{2(31), (41)23\}$ and describe it in terms of a natural combinatorial object. To begin, we compute how many basis vectors the Hopf subalgebra has for each grade. Next we find a combinatorial object that is in bijection with those basis vectors. Then we compute a product and coproduct on the

combinatorial objects by using the bijection to map the combinatorial objects to permutations, computing the operation on the permutations, and then mapping the permutations back to the combinatorial object. However, the goal of this research is to find an intrinsic representation of these operations; an operation directly on the objects that will produce the same output as the method described.

For $U = \{2(31), (41)23\}$, the set Av_n is counted by the Pell numbers, so $(\mathbb{K}[\mathrm{Av}_\infty], \bullet_{\mathrm{Av}}, \Delta_{\mathrm{Av}})$ might be called the Hopf algebra of Pell permutations. In Section 2.1 we define Pell permutations, and we also introduce a combinatorial object called sashes. There is a bijection from Pell permutations to sashes which is used to determine the operations on the Hopf algebra of sashes.

In Section 2.2 we describe the Hopf algebra of sashes and the dual Hopf algebra of sashes by defining the product, dual coproduct, dual product, and coproduct intrinsically. These four operations are all defined directly in terms of sashes.

In Chapter 3, we consider a family of Hopf subalgebras of which the Hopf Algebra of sashes is a member. This family is indexed by a parameter k, and each Hopf Algebra is $(\mathbb{K}[\mathrm{Av}_\infty], \bullet_{\mathrm{Av}}, \Delta_{\mathrm{Av}})$, for $U = \{2(31), (k1)23\cdots k-1\}$. We realize these Hopf algebras via combinatorial objects we call partial evaluations. In Section 3.1, we describe the avoiders Av_n. Then we define a bijection from avoiders to partial evaluations. In Section 3.2 we describe the product in the Hopf algebra of partial evaluations and give partial results on the coproduct.

Chapter 2

The Hopf Algebra of Sashes

2.1 Pell Permutations and Sashes

In this section, we describe a particular combinatorial Hopf algebra that is isomorphic to a Hopf subalgebra of pattern-avoiding permutations. To begin, we define a set of permutations called Pell permutations.

Given a permutation $x = x_1 x_2 \cdots x_n \in S_n$, for each $i \in [n-1]$, there is a nonzero integer j such that $x_i = x_{i+1} + j$. If $j > 0$, then there is an *descent* of size j in the ith position of x. A Pell permutation is a permutation of $[n]$ with no descents of size larger than 2, and such that for each descent $x_i = x_{i+1} + 2$, the element $x_{i+1} + 1$ is to the right of x_{i+1}. We write P_n for the set of Pell permutations in S_n.

Let us consider how many Pell permutations of length n there are. Given $x \in P_{n-1}$, we can place n at the end of x or before $n-1$. We can also place n before $n-2$, but only if $n-1$ is the last entry of x. Therefore $|P_n| = 2|P_{n-1}| + |P_{n-2}|$. This recursion, with the initial conditions $|P_0| = 0$ and $|P_1| = 1$, defines the Pell numbers as defined by [13, Sequence A000129].

Lemma 2.1.1. $P_n = \mathrm{Av}_n$ *for* $U = \{2(31), (41)23\}$.

Proof. Suppose $x \in P_n$. Since x does not have any descents larger than 2, it avoids $(41)23$. For each descent $x_i = x_{i+1} + 2$ in x, the element $x_{i+1} + 1$ is to the right of x_{i+1}. Thus x also avoids

(a) Σ_3

(b) Σ_4
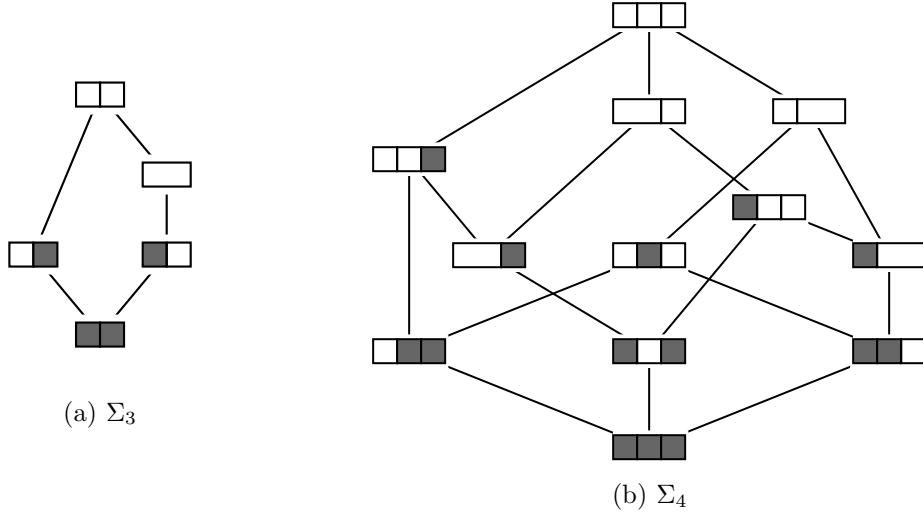
Figure 2.1: The elements of Σ_3 and Σ_4.

$2(31)$. Now suppose $x \in \mathrm{Av}_n$. Suppose x has a descent $x_i = x_{i+1} + j$. Because x avoids $2(31)$, the entries $x_{i+1} + 1, \ldots, x_{i+1} + j - 1$ are to the right of the x_{i+1}. Thus, since x avoids $(41)23$ we see that $j \leq 2$ and conclude that $x \in P_n$. $\qquad\square$

The poset induced on P_n by the weak order is a lattice (also denoted by P_n). As a consequence of Lemma 2.1.1, there is a Hopf algebra $(\mathbb{K}[\mathrm{Av}_\infty], \bullet_{\mathrm{Av}}, \Delta_{\mathrm{Av}})$ of Pell permutations. For the rest of this chapter we fix $U = \{2(31), (41)23\}$.

There is a combinatorial object in bijection with Pell permutations that will allow us to have a more natural understanding of the Hopf algebra of Pell permutations.

A *sash* of length n is a tiling of a $1 \times n$ rectangle by black 1×1 squares, white 1×1 squares, and/or white 1×2 rectangles. The set of sashes of length n is called Σ_n. There are no sashes of length -1 so $\Sigma_{-1} = \varnothing$, and there is one sash of length 0, a 1 by 0 rectangle denoted $\|$, so $|\Sigma_0| = 1$. There are two sashes of length 1: ■ and □. The five sashes of length 2 and the twelve sashes of length 3 are shown in Figure 2.1. The poset structure of these sashes will be explained later in this section.

A sash of length n starts with either a black square, a white square, or a rectangle. Thus

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$|\Sigma_n| = 2|\Sigma_{n-1}| + |\Sigma_{n-2}|$. Since $|\Sigma_{-1}| = 0$ and $|\Sigma_0| = 1$, there is a bijection between Pell permutations of length n and sashes of length $n - 1$. We now describe a bijection that we use to induce a Hopf Algebra structure on sashes.

Definition 2.1.2. We define a map σ from S_n to Σ_{n-1}. Let $x \in S_n$. We build a sash $\sigma(x)$ from left to right as we consider the entries in x from 1 to $n - 1$. For each value $i \in [n - 1]$, if $i + 1$ is to the right of i, place a black square on the sash, and if $i + 1$ is to the left of i, place a white square on the sash. There is one exception: If $i + 1$ is to the right of i, and $i + 2$ is to the left of i (and of $i + 1$), then place a rectangle in the ith and $(i + 1)^{st}$ positions of the sash. We also define $\sigma(1) = \|$ and $\sigma(\varnothing) = \varnothing$.

From the definition of the map σ we see that σ sometimes involves replacing an adjacent black square and white square by a rectangle. Later, we will sometimes break a rectangle into a black square and a white square.

Example 2.1.3. Here is the procedure for computing $\sigma(421365)$.

2 is to the left of 1	$\rightarrow$	☐
3 is to the right of 2	$\rightarrow$	☐▪
4 is to the left of 3 and also to the left of 2	$\rightarrow$	☐▭
5 is to the right of 4	$\rightarrow$	☐▭▪
6 is to the left of 5 but to the right of 4	$\rightarrow$	☐▭▪☐
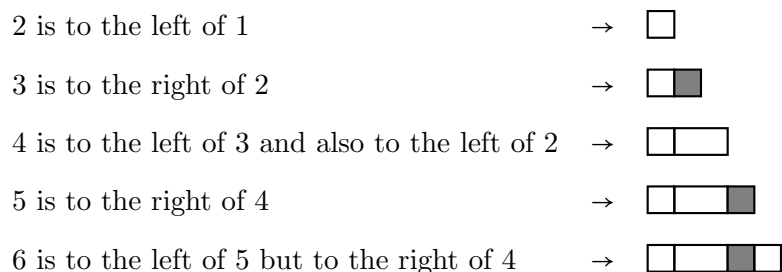
Let T be a set of n integers and let x be a permutation of T. We define $\sigma(x) = \sigma(\mathrm{st}(x))$.

Example 2.1.4. $\sigma(742598) = \sigma(\mathrm{st}(742598)) = \sigma(421365) = $ ☐▭▪☐

Definition 2.1.5. We define a map $\eta : \Sigma_{n-1} \to P_n$. To calculate $\eta(A)$ for a sash $A \in \Sigma_{n-1}$, we place the numbers 1 through n one at a time. Place the number 1 to begin and let i run from 1 to $n - 1$. If A has either a black square or the left half of a rectangle in the ith position, place $i + 1$ at the right end of the permutation. If A has either a white square or the right half of a rectangle in the ith position, place $i + 1$ immediately to the left of i or $i - 1$ respectively. We also define $\eta(\|) = 1$ and $\eta(\varnothing) = \varnothing$.

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It is immediate that this construction yields a Pell permutation because the output has no descents of size larger than 2, and for each descent of size 2, the value in between the values of the descent is to the right of the descent.

Example 2.1.6. Here are the steps to calculate $\eta(A)$ for $A =$ ⬛.

$$
\begin{array}{rcl}
& \to & 1 \\
\square & \to & 21 \\
\square & \to & 213 \\
\square & \to & 4213 \\
\blacksquare & \to & 42135 \\
\square & \to & 421365
\end{array}
$$

Theorem 2.1.7. *The restriction of σ to the Pell permutations is a bijection $\sigma : P_n \to \Sigma_{n-1}$ whose inverse is given by $\eta : \Sigma_{n-1} \to P_n$.*

Proof. Let $A \in \Sigma_{n-1}$. We first show that $\sigma(\eta(A)) = A$. If A has a black square in position i, then $\eta(A)$ has $i+1$ to the right of i and $i+2$ not to the left of i. So $\sigma(\eta(A))$ also has a black square in the ith position. If A has a white square in position i, then $\eta(A)$ has $i+1$ immediately to the left of i. So $\sigma(\eta(A))$ also has a white square in the ith position. If A has a rectangle in positions i and $i+1$, then $\eta(A)$ has $i+1$ to the right of i, and $i+2$ immediately to the left of i. So $\sigma(\eta(A))$ also has a rectangle in the ith and $(i+1)^{st}$ positions. We conclude that $\sigma(\eta(A)) = A$.

We have constructed a Pell permutation $\eta(A)$ that maps to A under σ, therefore σ is surjective. Since we know $|P_n| = |\Sigma_{n-1}|$, the map σ restricted to Pell permutations is a bijection. The inverse map of σ is η. □

Proposition 2.1.8. $x, y \in S_n$ *are equivalent if and only if $\sigma(x) = \sigma(y)$.*

Proof. The permutations $x = x_1 \cdots x_n$ and $y = y_1 \cdots y_n$ are equivalent if and only if $\pi_\downarrow(x) = \pi_\downarrow(y)$. Thus to prove the forward direction of the proposition, it is enough to consider the case where y is obtained from x by a single $\pi_\downarrow$-move. Consider a $\pi_\downarrow$-move switching x_i and x_{i+1} of x. First we suppose that $x_i \geq x_{i+1} + 3$. The relative position of x_i with regard to x_{i+1} is irrelevant to the map

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σ, thus $\sigma(x) = \sigma(y)$. Now we suppose that $x_i = x_{i+1} + 2$. There can only be a $\pi_\downarrow$-move switching x_i and x_{i+1} of x if $x_{i+1} + 1$ is to the left of x_i. In this case, both $\sigma(x)$ and $\sigma(y)$ have a white square in the ith position and a black square in the $(i+1)^{st}$ position. Therefore $\sigma(x) = \sigma(y)$.

To prove the reverse implication suppose that x and y are not equivalent, that is $\pi_\downarrow(x) \neq \pi_\downarrow(y)$. Since $\pi_\downarrow(x)$ and $\pi_\downarrow(y)$ are Pell permutations, and σ is a bijection from Pell permutations to sashes, $\sigma(\pi_\downarrow(x)) \neq \sigma(\pi_\downarrow(y))$. But by the previous paragraph, $\sigma(\pi_\downarrow(x)) = \sigma(x)$ and $\sigma(\pi_\downarrow(y)) = \sigma(y)$. □

The partial order on Σ_{n-1} is such that the map $\sigma : P_n \to \Sigma_{n-1}$ is an order isomorphism from the lattice of Pell permutations to Σ_{n-1}. We refer to this lattice as Σ_{n-1}.

From Proposition 1.2.2, the cover relations in Σ_{n-1} are exactly the relations $\sigma(y) \lessdot \sigma(x)$ where $x \in P_n$ and y is covered by x in S_n.

Proposition 2.1.9. *The cover relations on sashes are*

1. *$A\blacksquare B \lessdot A\square B$ for any sash A and for a sash B whose leftmost tile is not a white square*

2. *$A\blacksquare\square B \lessdot A\square\square B$ for any sash A and any sash B*

3. *$A\square\square B \lessdot A\square\square\square B$ for any sash A and any sash B*

Proof. Let $x \in P_n$ and let $y \in S_n$ such that y is covered by x in the weak order. That is, $x = x_1 \cdots x_i x_{i+1} \cdots x_n$ and $y = x_1 \cdots x_{i+1} x_i \cdots x_n \in S_n$ for some $x_i > x_{i+1}$.

Suppose $x_i = x_{i+1} + 1$ and $x_i + 1$ is not to the left of x_i. Let $A = \sigma(x|_{[1,x_{i+1}]}) = \sigma(y|_{[1,x_{i+1}]})$ and let $B = \sigma(x|_{[x_i,n]}) = \sigma(y|_{[x_i,n]})$. Thus, $A\blacksquare B = \sigma(y) \lessdot \sigma(x) = A\square B$, where the leftmost tile of B is not a white square.

Suppose $x_i = x_{i+1} + 1$ and $x_i + 1$ is to the left of x_i. Let $A = \sigma(x|_{[1,x_{i+1}]}) = \sigma(y|_{[1,x_{i+1}]})$ and let $B = \sigma(x|_{[x_i+1,n]}) = \sigma(y|_{[x_i+1,n]})$. Thus, $A\square B = \sigma(y) \lessdot \sigma(x) = A\square\square B$.

Suppose $x_i = x_{i+1} + 2$. Let $A = \sigma(x|_{[1,x_{i+1}]}) = \sigma(y|_{[1,x_{i+1}]})$ and let $B = \sigma(x|_{[x_i,n]}) = \sigma(y|_{[x_i,n]})$. Thus, $A\blacksquare\square B = \sigma(y) \lessdot \sigma(x) = A\square B$. □

Example 2.1.10. See Figure 2.1 for the poset on Σ_3 and Σ_4.

2.2 The Hopf Algebra (and Dual Hopf Algebra) of Sashes

The bijection σ allows us to carry the Hopf algebra structure on Pell permutations to a Hopf algebra structure $(\mathbb{K}[\Sigma_\infty], \bullet_S, \Delta_S)$ on sashes and a dual Hopf algebra $(\mathbb{K}[\Sigma_\infty], \Delta_S^*, m_S^*)$ on sashes, where $\mathbb{K}[\Sigma_\infty]$ is a vector space, over a field $\mathbb{K}$, whose basis elements are indexed by sashes. In order to do this, we extend σ and η to linear maps. For each grade n of the vector space, the basis elements are represented by the sashes of length $n - 1$. Recall that the sash of length -1 is represented by $\varnothing$, and the sash of length 0 is represented by $\|$. Let A, B, and C be sashes. Using σ, we define a product, coproduct, dual product, and dual coproduct of sashes:

$$m_S(A, B) = A \bullet_S B = \sigma\big(\eta(A) \bullet_{\mathrm{Av}} \eta(B)\big) \tag{2.1}$$

$$\Delta_S(C) = (\sigma \otimes \sigma)\big(\Delta_{\mathrm{Av}}(\eta(C))\big) \tag{2.2}$$

$$\Delta_S^*(A \otimes B) = \sigma\Big(\Delta_{\mathrm{Av}}^*\big(\eta(A) \otimes \eta(B)\big)\Big) \tag{2.3}$$

$$m_S^*(C) = (\sigma \otimes \sigma)\big(m_{\mathrm{Av}}^*(\eta(C))\big) \tag{2.4}$$

These operation definitions are somewhat unsatisfying because they require computing the operation in MR. That is, calculating a product or coproduct in this way requires mapping sashes to permutations, performing the operations in MR, throwing out the non-avoiders in the result, and then mapping the remaining permutations back to sashes. In the rest of this chapter we show how to compute these operations directly in terms of sashes.

2.2.1 Product

Proposition 2.2.1. *The empty sash $\varnothing$ is the identity for the product $\bullet_S$. For sashes $A \neq \varnothing$ and $B \neq \varnothing$, the product $A \bullet_S B$ equals:*

$$\begin{cases} \sum \left[A\blacksquare B, A'\boxed{}B \right] & \text{if } A = A'\blacksquare \\[2ex] \sum \left[A\blacksquare B, A\square B \right] & \text{if } A \neq A'\blacksquare \end{cases}$$

where $\sum [D, E]$ is the sum of all the sashes in the interval $[D, E]$ on the lattice of sashes.

To clarify, here are some more specific cases of the product on sashes. If $A = A'\blacksquare$, then $\sum \left[A\blacksquare B, A'\boxed{}B \right]$ equals:

$$\begin{cases} A\blacksquare B + A\boxed{}B' + A\square B + A'\boxed{}B & \text{if } B = \square B' \\[2ex] A\blacksquare B + A\square B + A'\boxed{}B & \text{if } B \neq \square B' \end{cases}$$

and if $A \neq A'\blacksquare$, then $\sum \left[A\blacksquare B, A\square B \right]$ equals:

$$\begin{cases} A\blacksquare B + A\boxed{}B' + A\square B & \text{if } B = \square B' \\[2ex] A\blacksquare B + A\square B & \text{if } B \neq \square B' \end{cases}$$

The case where $A = \|$ is an instance of $A \neq A'\blacksquare$, and similarly for $B = \|$.

Proof. We begin by computing the product of Pell permutations. We showed in Section 1.2 that the product of Pell permutations is the sum over the interval $[x \cdot y', \pi_\downarrow(y' \cdot x)]$ in the lattice of Pell permutations, where $x \in P_p$, $y \in P_q$, and $y' = (y)_{[p+1,n]}$.

To compute the product of sashes we can apply the map σ to the product of Pell permu-tations. Let $\sigma(x) = A$ and $\sigma(y) = B$, thus $A \bullet_S B = \sigma(x \bullet_P y) = \sum[\sigma(x \cdot y'), \sigma(\pi_\downarrow(y' \cdot x))] = \sum[\sigma(x \cdot y'), \sigma(y' \cdot x)]$. The map σ takes the first p values of $x \cdot y'$ to A and the last q values to B. Because $p + 1$ is to the right of p and since $p + 2$ is not to the left of p, $\sigma(x \cdot y') = A\blacksquare B$.

Similarly, σ takes the first p values of $y' \cdot x$ to A and the last q values to B. Since $p+1$ is to the left of p, to compute $\sigma(y' \cdot x)$ we need to consider whether or not $p-1$ is before p in x.

Suppose $p-1$ is before p in x. Thus, A ends with a black square so $\sigma(y' \cdot x)$ replaces the last black square of A with a rectangle in positions $p-1$ and p. That is $\sigma(y' \cdot x) = A'\,\rule[1pt]{18pt}{6pt}\,B$, where $A = A'\,\blacksquare$.

Suppose $p-1$ is not before p in x. Thus, A either ends with a white square, the right half of a rectangle, or $p-1$ does not exist. Thus, $\sigma(y' \cdot x)$ places a white square after A and before B, so $\sigma(y' \cdot x) = A\,\square\,B$. $\qquad\qquad\square$

In informal terms, the product of two sashes is the sum of the sashes created by joining the two sashes with a black square and a white square, and if by so doing an adjacent black square to the left of a white square is created, then the product has additional terms with rectangles in the places of the adjacent black square and white square.

Example 2.2.2. Let $A = \boxed{\,\square\,\blacksquare\,}$ and let $B = \square$. Notice that for $A' = \square$ and $B' = \|$, both $A = A'\,\blacksquare$ and $B = \square\,B'$.

$$A \bullet_S B = \quad A\,\blacksquare\,B + \quad A\,\rule[1pt]{18pt}{6pt}\,B' + \quad A\,\square\,B + \quad A'\,\rule[1pt]{18pt}{6pt}\,B$$

$$\square\,\blacksquare \bullet_S \square = \square\,\blacksquare\,\blacksquare + \square\,\square\,\blacksquare\,\square \quad + \square\,\blacksquare\,\square\,\square + \square\,\square\,\square\,\square$$

2.2.2 Dual Coproduct

From Equation (1.8) and Equation (1.4), it follows that:

$$m_S^*(C) = \sum_{i=0}^{n} \sigma\big(\eta(C)|_{[i]}\big) \otimes \sigma\big(\eta(C)|_{[i+1,n]}\big) \qquad (2.5)$$

Proposition 2.2.3. *The dual coproduct on a sash $C \in \Sigma_n$ is given by:*

$$m_S^*(C) = \sum_{i=-1}^{n} C_i \otimes C^{n-i-1}$$

Where $C_i \in \Sigma_i$ is a sash identical to the first i positions of C (unless C has $\rule[1pt]{14pt}{0.5pt}$ in position i,

in which case C_i ends with ▪), and $C^{n-i-1} \in \Sigma_{n-i-1}$ is a sash identical to the last $n - i - 1$ positions of C (unless C has ⬚ in position $i + 2$, in which case C^{n-i-1} begins with ☐), and we define $C_0 = C^0 = \| $ and $C_{-1} = C^{-1} = \varnothing$.

Proof. We need to show that C is a term of $A \bullet_S B$ if and only if $A \otimes B$ is a term of $m_S^*(C)$.

Suppose that $C \in \Sigma_n$ is a term of $A \bullet_S B$, with $A \in \Sigma_p$, $B \in \Sigma_q$, and $p + q = n - 1$. Thus C is one of the following: $A▪B$, $A☐B$, $A'\boxed{}B$, or $A\boxed{}B'$, for A' and B' as in Proposition 2.2.1. In any case, $m_S^*(C)$ has a term $A \otimes B$ because $C_p = A$ and $C^{n-p-1} = B$.

Now suppose that for $A \in \Sigma_p$ and $B \in \Sigma_q$, $A \otimes B$ is a term of $m_S^*(C)$, where $C \in \Sigma_n$ and $p + q = n - 1$. Thus $C_p = A$ and $C^{n-p-1} = C^q = B$. If C is $A▪B$ or $A☐B$, then C is a term of $A \bullet_S B$. If C is $A'\boxed{}B$, then $A = A'▪$, and C is a term of $A \bullet_S B$. If C is $A\boxed{}B'$, then $B = ☐B'$, and C is a term of $A \bullet_S B$. Therefore in all cases C is a term of $A \bullet_S B$ and we have shown that the map m_S^* is the dual coproduct on sashes. □

2.2.3 Dual Product

From Equation (1.9), it follows that:

$$\Delta_S^*(A \otimes B) = \sum_{\substack{T \subseteq [n] \\ |T| = p}} \sigma\Big(\big(\eta(A)\big)_T \cdot \big(\eta(B)\big)_{T^C}\Big) \tag{2.6}$$

We now prepare to describe the dual product Δ_S^* directly on sashes.

Definition 2.2.4. Given a set $T \subseteq [n]$ such that $|T| = p$ and $n = p + q$, and given sashes $D \in \Sigma_{p-1}$ and $E \in \Sigma_{q-1}$, define a sash $\gamma_T(D \otimes E)$ by the following steps. First, write D above E. Then, label D with T, by placing the elements of T in increasing order between each position of D, including the beginning and end. Label E similarly using the elements of T^C.

Example 2.2.5. Let $T = \{1, 2, 4, 7, 8, 9, 12, 13\}$, $D =$ ▭▯■▭▭■, and $E =$ ▭▯▭■▭
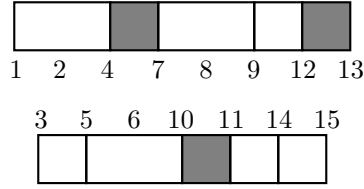
Next, draw arrows from i to $i + 1$ for all $i \in [n - 1]$. Lastly, follow the path of the arrows placing elements in a new sash based on the following criteria:

Place a rectangle in the ith and $(i + 1)^{st}$ positions of the new sash if either of the following conditions are met:

1. if the ith arrow is from D to E, the $(i + 1)^{st}$ arrow is from E to D, and there is a ▯ or ▯ in D in between i and $i + 2$

2. if the ith arrow is from E to E, the $(i + 1)^{st}$ arrow is from E to D, and there is a ■ or ▯ in E in between i and $i + 1$

If the above criteria are not met, then the following rules apply:

1. if the ith arrow is from D to D (or from E to E), place whatever is in between i and $i + 1$ in D (or in E) in the ith position.

2. if the ith arrow is from D to E, place a black square in the ith position.

3. if the ith arrow is from E to D, place a white square in the ith position.

Note that it may be necessary to replace the left half of a rectangle by a black square or to replace the right half of a rectangle by a white square (as in the first step of the example below).

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Example 2.2.6. Let T, D, and E be as in Example 2.2.5. Then we compute $\gamma_T(D \otimes E)$ to obtain: $\gamma_{\{1,2,4,7,8,9,12,13\}}(\,\rule{2.5cm}{0pt}\, \otimes \,\rule{1.8cm}{0pt}\,) = \,\rule{3cm}{0pt}\,.$
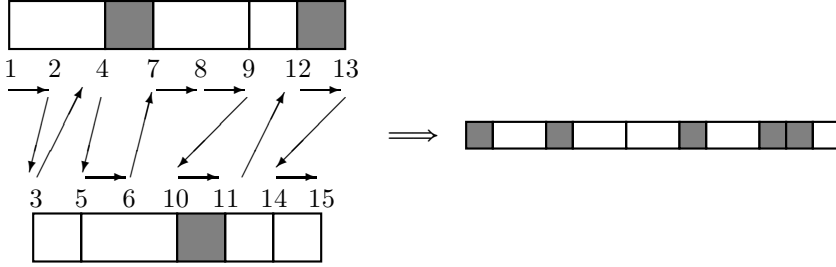
Theorem 2.2.7. *The dual product of sashes $D \in \Sigma_{p-1}$ and $E \in \Sigma_{q-1}$, for $p + q = n$, is given by:*

$$\Delta_S^*(D \otimes E) = \sum_{\substack{T \subseteq [n], \\ |T| = p}} \gamma_T(D \otimes E)$$

Proof. For $D \in \Sigma_{p-1}$ and $E \in \Sigma_{q-1}$ such that $\eta(D) = x \in P_p$ and $\eta(E) = y \in P_q$, where $p + q = n$, we consider Equation (2.6) to define the dual product of sashes.

Let $T \subseteq [n]$ such that $|T| = p$. It is just left to show that $\gamma_T(D \otimes E) = \sigma((x)_T \cdot (y)_{T^C})$.

Case 1: $i, i + 2 \in T$ and $i + 1 \in T^C$ such that $i + 2$ is to the left of i in $(x)_T$.
$\gamma_T(D \otimes E)$ outputs a rectangle in the ith and $(i+1)^{st}$ positions because the ith arrow is from D to E, the $(i+1)^{st}$ arrow is from E to D, and there is a $\square$ or $\sqsupset$ in D in between the labels i and $i + 2$. The sash $\sigma((x)_T \cdot (y)_{T^C})$ has a rectangle in the ith and $(i+1)^{st}$ positions because $i + 1$ is to the right of i, and because $i + 2$ is to the left of i.

Case 2: $i, i + 1 \in T^C$ and $i + 2 \in T$ such that $i + 1$ is to the right of i in $(y)_{T^C}$.
$\gamma_T(D \otimes E)$ outputs a rectangle in the ith and $(i+1)^{st}$ positions because the ith arrow is from E to E, the $(i+1)^{st}$ arrow is from E to D, and there is a $\blacksquare$ or $\sqsubset$ in E in between the labels i and $i + 1$. The sash $\sigma((x)_T \cdot (y)_{T^C})$ has a rectangle in the ith and $(i+1)^{st}$ positions because $i + 1$ is to the right of i, and because $i + 2$ is to the left of i.

Case 3: $i \in T$, $i + 1 \in T^C$, and if $i + 2 \in T$, then $i + 2$ is to the right of i in $(x)_T$.
$\gamma_T(D \otimes E)$ outputs a black square in the ith position because the ith arrow is from D to E, and the criteria to place a rectangle is not met. The sash $\sigma((x)_T \cdot (y)_{T^C})$ has a black square in the

i^{th} position because $i + 1$ is to the right of i, and because $i + 2$ is not to the left of i.

Case 4: $i \in T^C$, $i+1 \in T$, if $i-1 \in T$ then $i+1$ is to the right of $i-1$ in $(x)_T$, and if $i-1 \in T^C$ then i is to the left of $i-1$ in $(y)_{T^C}$.

$\gamma_T(D \otimes E)$ outputs a white square in the ith position because the ith arrow is from E to D, and the criteria to place a rectangle is not met. The sash $\sigma((x)_T \cdot (y)_{T^C})$ has a white square in the ith position because $i+1$ is to the left of i, and because $i-1$ is not in between $i+1$ and i.

Case 5: $i, i+1 \in T$.

$\gamma_T(D \otimes E)$ outputs whatever is in between the labels i and $i+1$ in D in the ith position. The sash $\sigma((x)_T \cdot (y)_{T^C})$ has the same.

Case 6: $i, i+1 \in T^C$, and if $i+2 \in T$ then $i+1$ is to the left of i in $(y)_{T^C}$.

$\gamma_T(D \otimes E)$ outputs whatever is in between the labels i and $i+1$ in E in the ith position. The sash $\sigma((x)_T \cdot (y)_{T^C})$ has the same.

Therefore we have shown that $\sigma((x)_T \cdot (y)_{T^C})$ and $\gamma_T(D \otimes E)$ have the same object in every position. $\qquad\square$

2.2.4 Coproduct

We now describe the coproduct in the Hopf algebra of sashes and we begin with some definitions.

Definition 2.2.8. For $C \in \Sigma_{n-1}$, a *dotting of C* is C with a dot in any subset of the $n - 1$ positions of C. An *allowable dotting of C* is a dotting of C that meets all of the following conditions

1. has at least one dot

2. the first dot can be in any position, and dotted positions alternate between a black square (or the left half of a rectangle) and a white square (or the right half of a rectangle)

3. has no instances of $\boxed{\bullet\ \ }$ or $\blacksquare\boxed{\bullet}$
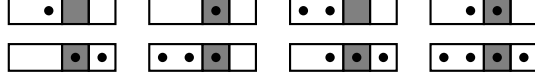
Figure 2.2: The allowable dottings of a sash

Figure 2.2 shows the allowable dottings of the sash ▭.

Consider an allowable dotting $d = c_1 \bullet_1 c_2 \bullet_2 \cdots c_j \bullet_j c_{j+1}$ of a sash C, where each c_i is a sub sash of C without any dots, and $\bullet_i$ is a single dotted position. If any $\bullet_i$ is on the right half of a rectangle, then the the left half of the rectangle in the last position of c_i is replaced by a black square. If $\bullet_i$ and $\bullet_{i+1}$ are in adjacent positions, then $c_{i+1} = \|$. (If any $\bullet_i$ is on the left half of a rectangle, then $\bullet_{i+1}$ is on the right half of the same rectangle, so $c_{i+1} = \|$.)

We use C and d to define two objects A and B that are similar to sashes, but have an additional type of square $\boxed{?}$, which we call a mystery square. If $\bullet_1$ is on a black square or the left half of a rectangle, then let A be the concatenation of the *odd* c_i with a mystery square in between each c_i (where i is odd), and let B be the concatenation of the *even* c_i with a mystery square in between each c_i (where i is even). For example, if $\bullet_1$ is on a black square and j is even, then $A = c_1 \boxed{?} c_3 \boxed{?} \cdots \boxed{?} c_{j+1}$ and $B = c_2 \boxed{?} c_4 \boxed{?} \cdots \boxed{?} c_j$. If $\bullet_1$ is on a white square or the right half of a rectangle, then let A be the concatenation of the *even* c_i with a mystery square in between each c_i, and let B be the concatenation of the *odd* c_i with a mystery square in between each c_i.

We use the objects A and B to define four sashes $\underline{A}$, $\overline{A}$, $\underline{B}$, and $\overline{B}$.

To compute $\underline{A}$, consider each mystery square in A. If the mystery square follows c_i and the ith and $(i+1)^{st}$ dots of d are on the same rectangle, $\boxed{\bullet \ \bullet}$, then replace the mystery square after c_i with a white square. Otherwise replace the mystery square with a black square.

To compute $\overline{A}$, consider each mystery square in A from left to right. If the mystery square follows c_i and the ith and $(i+1)^{st}$ dots of d are on an adjacent black square and white square, $\boxed{\bullet}\boxed{\bullet}$, then we check to see whether or not the mystery square is followed by a white square.

If the mystery square is followed by a white square (*i.e.* if c_{i+2} starts with a white square), then replace the mystery square and the white square with a rectangle. Otherwise replace the mystery square with a black square. If the mystery square follows c_i and the ith and $(i+1)^{st}$ dots of d are *not* on an adjacent black square and white square, then we check to see whether or not the mystery square is preceded by a black square. If either c_i ends in a black square or $c_i = \|$ and the previous mystery square has been changed to a black square, then replace the mystery square after c_i and the black square before it with a rectangle. Otherwise replace the mystery square after c_i with a white square.

To compute $\underline{B}$, replace all mystery squares of B with black squares.

To compute $\overline{B}$, replace all mystery squares of B with white squares, unless the mystery square is preceded by a black square, in which case replace both the black square and the mystery square with a rectangle.

Example 2.2.9. If $d = $, then $c_1 = $, $c_2 = c_3 = $, $c_4 = c_5 = c_6 = \|$, $c_7 = $, $c_8 = $, $c_9 = c_{10} = c_{11} = \|$, and $\bullet_1$ is on a black square. Thus, $A = c_1 \boxed{?} c_3 \boxed{?} c_5 \boxed{?} c_7 \boxed{?} c_9 \boxed{?} c_{11}$ and $B = c_2 \boxed{?} c_4 \boxed{?} c_6 \boxed{?} c_8 \boxed{?} c_{10}$. Using the rules above to compute A, B, and the four sashes $\underline{A}$, $\overline{A}$, $\underline{B}$, and $\overline{B}$, we have:

Given an allowable dotting d of a sash C we define $I_d = \sum [\underline{A}, \overline{A}]$ and $J_d = \sum [\underline{B}, \overline{B}]$ for $\underline{A}$, $\overline{A}$, $\underline{B}$, and $\overline{B}$ computed as above. Thus the notation $I_d \otimes J_d$ denotes $\sum_{\substack{D \in [\underline{A}, \overline{A}] \\ E \in [\underline{B}, \overline{B}]}} D \otimes E$.

Theorem 2.2.10. *Given $C \in \Sigma_{n-1}$:*

$$\Delta_S(C) = \varnothing \otimes C + C \otimes \varnothing + \sum_{\substack{allowable \\ dottings \\ d \ of \ C}} I_d \otimes J_d$$

To prove Theorem 2.2.10, we need to introduce some more terminology. Given an allowable dotting d of a sash C and the objects A and B defined above, we define two more objects $\hat{A}$ and $\hat{B}$. These objects are similar to sashes, but have three additional types of squares: ⊞, ⊞, and ？. We call these squares: black-plus square, white-plus square, and mystery square respectively. If $\bullet_i$ and $\bullet_{i+1}$ are on an adjacent black square and white square, *i.e.* ●●, then replace the ？ after c_i on A with a ⊞ on $\hat{A}$. If $\bullet_i$ and $\bullet_{i+1}$ are on a rectangle, *i.e.* ●●, then replace the ？ after c_i on A with a ⊞ on $\hat{A}$. The resulting objects are $\hat{A}$ and $\hat{B}$. Note $B = \hat{B}$.

We say that a sash D is of the form $\hat{A}$ if D is identical to $\hat{A}$ except for the following allowable substitutions:

- A black-plus square on $\hat{A}$ is replaced by a black square on D.

- A white-plus square on $\hat{A}$ is replaced by a white square on D.

- A mystery square on $\hat{A}$ is replaced by a either a black square or a white square on D.

- A black-plus square or a mystery square on $\hat{A}$, and a white square, a white-plus square, or a mystery square following it, are replaced by a rectangle on D.

- A white-plus square or a mystery square on $\hat{A}$, and a black square, a black-plus square, or a mystery square preceding it, are replaced by a rectangle on D.

Similarly, a sash E is of the form $\hat{B}$ if it follows the same rules as above. Since $\hat{B}$ does not have any black-plus squares or white-plus squares, E is of the form $\hat{B}$ if E is identical to $\hat{B}$ except for the following allowable substitutions:

- A mystery square on $\hat{B}$ is replaced by a either a black square or a white square on E.

- A mystery square on $\hat{B}$, and a white square or a mystery square following it, are replaced by a rectangle on E.

- A mystery square on $\hat{B}$, and a black square or a mystery square preceding it, are replaced by a rectangle on E.

Lemma 2.2.11. *The sash $\underline{A}$ is minimal with respect to sashes of the form $\hat{A}$.*

Proof. The sash $\underline{A}$ is of the form $\hat{A}$, because every white-plus square on $\hat{A}$ is replaced by a white square on $\underline{A}$ and every black-plus square and mystery square on $\hat{A}$ is replaced by a black square on $\underline{A}$.

Suppose the sash A' is obtained from $\underline{A}$ by going down by a cover relation. We want to show that A' is not of the form $\hat{A}$.

Case 1: $A' = A_1\,\blacksquare\,A_2$ and $\underline{A} = A_1\,\square\,A_2$, where the leftmost tile of A_2 is not a white square. Let $|A_1| = i - 1$, so A' has a black square in the ith position and $\underline{A}$ has a white square in the ith position. Thus, $\hat{A}$ either has a white square or a white-plus square in the ith position. Either way, A' is not of the form $\hat{A}$.

Case 2: $A' = A_1\,\blacksquare\,\square\,A_2$ and $\underline{A} = A_1\,\rule[0.1em]{1.6em}{0.05em}\,A_2$.
Let $|A_1| = i - 1$, so A' has a black square and white square in the ith and $(i+1)^{st}$ positions and $\underline{A}$ has a rectangle in the ith and $(i+1)^{st}$ positions. Thus, $\hat{A}$ has a rectangle in the ith and $(i+1)^{st}$ positions, and A' is not of the form $\hat{A}$.

Case 3: $A' = A_1\,\rule[0.1em]{1.6em}{0.05em}\,A_2$ and $\underline{A} = A_1\,\square\,\square\,A_2$.
Let $|A_1| = i - 1$, so A' has a rectangle in the ith and $(i+1)^{st}$ positions and $\underline{A}$ has a white square in both the ith and $(i+1)^{st}$ positions. There are four possibilities of what occupies the ith and $(i+1)^{st}$ positions of $\hat{A}$: $\square\,\square$, $\boxplus\,\square$, $\square\,\boxplus$, or $\boxplus\,\boxplus$. In any case, A' is not of the form $\hat{A}$. $\square$

Lemma 2.2.12. *The sash $\overline{A}$ is maximal with respect to sashes of the form $\hat{A}$.*

Proof. The sash $\overline{A}$ is of the form $\hat{A}$, because black-plus squares followed by white-plus squares, mystery squares, or white squares on $\hat{A}$ are replaced by a rectangle on $\overline{A}$, white-plus squares and mystery squares preceded by black-plus squares or black squares on $\hat{A}$ are replaced by a rectangle on $\overline{A}$, all other black-plus squares on $\hat{A}$ are replaced by black squares on $\overline{A}$, and all other white-plus squares and mystery squares on $\hat{A}$ are replaced by white squares on $\overline{A}$.

Suppose the sash A' is obtained from $\overline{A}$ by going up by a cover relation. We want to show that A' is not of the form $\hat{A}$.

Case 1: $\overline{A} = A_1 \blacksquare A_2$ and $A' = A_1 \square A_2$, where the leftmost tile of A_2 is not a white square.

Let $|A_1| = i - 1$, so $\overline{A}$ has a black square in the ith position and A' has a white square in the ith position. Thus, $\hat{A}$ either has a black square or a black-plus square in the ith position. Either way, A' is not of the form $\hat{A}$.

Case 2: $\overline{A} = A_1 \blacksquare\square A_2$ and $A' = A_1 \boxed{} A_2$.

Let $|A_1| = i - 1$, so $\overline{A}$ has a black square and white square in the ith and $(i+1)^{st}$ positions and A' has a rectangle in the ith and $(i + 1)^{st}$ positions. Thus, $\hat{A}$ has a black square and white square in the ith and $(i + 1)^{st}$ positions, and A' is not of the form $\hat{A}$.

Case 3: $\overline{A} = A_1 \boxed{} A_2$ and $A' = A_1 \square\square A_2$.

Let $|A_1| = i - 1$, so $\overline{A}$ has a rectangle in the ith and $(i+1)^{st}$ positions and A' has a white square in both the ith and $(i + 1)^{st}$ positions. There are five possibilities of what occupies the ith and $(i+1)^{st}$ positions of $\hat{A}$: $\boxed{+\ }$, $\boxed{+\,+}$, $\boxed{+\,?}$, $\boxed{\ +}$, or $\boxed{\ ?}$. In any case, A' is not of the form $\hat{A}$. $\square$

Lemma 2.2.13. *The sash $\underline{B}$ is minimal with respect to sashes of the form $\hat{B}$.*

Proof. The sash $\underline{B}$ is of the form $\hat{B}$, because every mystery square on $\hat{B}$ is replaced by a black square on $\underline{B}$.

Suppose the sash B' is obtained from $\underline{B}$ by going down by a cover relation. We want to show that B' is not of the form $\hat{B}$.

Case 1: $B' = B_1 \blacksquare B_2$ and $\underline{B} = B_1 \square B_2$, where the leftmost tile of B_2 is not a white square.

Let $|B_1| = i - 1$, so B' has a black square in the ith position and $\underline{B}$ has a white square in the ith position. Thus, $\hat{B}$ has a white square in the ith position, and B' is not of the form $\hat{B}$.

Case 2: $B' = B_1 \blacksquare\square B_2$ and $\underline{B} = B_1 \boxed{} B_2$.

Let $|B_1| = i - 1$, so B' has a black square and white square in the ith and $(i + 1)^{st}$ positions and $\underline{B}$ has a rectangle in the ith and $(i + 1)^{st}$ positions. Thus, $\hat{B}$ has a rectangle in the ith and $(i + 1)^{st}$ positions, and B' is not of the form $\hat{B}$.

Case 3: $B' = B_1 \boxed{} B_2$ and $\underline{B} = B_1 \square\square B_2$.

Let $|B_1| = i - 1$, so B' has a rectangle in the ith and $(i+1)^{st}$ positions and $\underline{B}$ has a white square

in both the ith and $(i+1)^{st}$ positions. Thus, $\hat{B}$ has a white square in both the ith and $(i+1)^{st}$ positions, and B' is not of the form $\hat{B}$. $\qquad\square$

Lemma 2.2.14. *The sash $\overline{B}$ is maximal with respect to sashes of the form $\hat{B}$.*

Proof. The sash $\overline{B}$ is of the form $\hat{B}$, because every mystery square on $\hat{B}$ is replaced by a black square on $\overline{B}$, unless it is preceded by a black square, in which case the black square and the mystery square are replaced by a rectangle on $\overline{B}$.

Suppose the sash B' is obtained from $\overline{B}$ by going up by a cover relation. We want to show that B' is not of the form $\hat{B}$.

Case 1: $\overline{B} = B_1 \blacksquare B_2$ and $B' = B_1 \square B_2$, where the leftmost tile of B_2 is not a white square. Let $|B_1| = i - 1$, so $\overline{B}$ has a black square in the ith position and B' has a white square in the ith position. Thus, $\hat{B}$ has a black square in the ith position, and B' is not of the form $\hat{B}$.

Case 2: $\overline{B} = B_1 \blacksquare\square B_2$ and $B' = B_1 \boxed{\ \ } B_2$.
Let $|B_1| = i - 1$, so $\overline{B}$ has a black square and white square in the ith and $(i+1)^{st}$ positions and B' has a rectangle in the ith and $(i+1)^{st}$ positions. Thus, $\hat{B}$ has a black square and white square in the ith and $(i+1)^{st}$ positions, and B' is not of the form $\hat{B}$.

Case 3: $\overline{B} = B_1 \boxed{\ \ } B_2$ and $B' = B_1 \square\square B_2$.
Let $|B_1| = i - 1$, so $\overline{B}$ has a rectangle in the ith and $(i+1)^{st}$ positions and B' has a white square in both the ith and $(i+1)^{st}$ positions. Thus, $\hat{B}$ either has a rectangle or a black square and mystery square in the ith and $(i+1)^{st}$ positions. Either way, B' is not of the form $\hat{B}$. $\qquad\square$

Proposition 2.2.15. *If a sash D is of the form $\hat{A}$, then $D \in [\underline{A}, \overline{A}]$.*

Proof. Suppose that D is of the form $\hat{A}$ and that $D \neq \underline{A}$. We want to show that there exists a sash D' such that $D' \lessdot D$ and D' is of the form $\hat{A}$.

Case 1: For some white-plus square of $\hat{A}$, it is not replaced by a white square on D.
Since D is of the form $\hat{A}$, the white-plus square of $\hat{A}$ is preceded by either a black square, a black-plus square, or a mystery square and is replaced by the right half of a rectangle on D. Thus $D = D_1 \boxed{\ \ } D_2$. Let $D' = D_1 \blacksquare\square D_2$, so that $D' \lessdot D$. In any case, D' is of the form $\hat{A}$.

Case 2: For some black-plus square of $\hat{A}$, it is not replaced by a black square on D.

Since D is of the form $\hat{A}$, the black-plus square of $\hat{A}$ is followed by either a white square, a white-plus square, or a mystery square and is replaced by the left half of a rectangle on D. Thus $D = D_1\boxed{}D_2$. Let $D' = D_1\blacksquare\boxed{}D_2$, so that $D' \lessdot D$. In any case, D' is of the form $\hat{A}$.

Case 3: For some mystery square of $\hat{A}$, it is not replaced by a black square on D.

Subcase 3a: The mystery square of $\hat{A}$ is replaced by a white square on D. If $D = D_1\square D_2$, where the first tile of D_2 is not a white square, then let $D' = D_1\blacksquare D_2$, so that $D' \lessdot D$. If $D = D_1\boxed{}D_3$, then let $D' = D_1\boxed{}D_3$, so that $D' \lessdot D$. Either way, the sash D' is of the form $\hat{A}$.

Subcase 3b: The mystery square of $\hat{A}$ is replaced by the left half of a rectangle on D.

Since D is of the form $\hat{A}$, the mystery square of $\hat{A}$ is followed by either a white square, a white-plus square, or another mystery square. Thus $D = D_1\boxed{}D_2$. Let $D' = D_1\blacksquare\boxed{}D_2$, so that $D' \lessdot D$. In any case, D' is of the form $\hat{A}$.

Subcase 3c: The mystery square of $\hat{A}$ is replaced by the right half of a rectangle on D.

Since D is of the form $\hat{A}$, the mystery square of $\hat{A}$ is preceded by either a black square, a black-plus square, or another mystery square. Thus $D = D_1\boxed{}D_2$. Let $D' = D_1\blacksquare\boxed{}D_2$, so that $D' \lessdot D$. In any case, D' is of the form $\hat{A}$.

Now, suppose that D is of the form $\hat{A}$ and that $D \neq \overline{A}$. We want to show that there exists a sash D' such that $D \lessdot D'$ and D' is of the form $\hat{A}$. Consider a black-plus square, white-plus square, or mystery square in the ith position of $\hat{A}$.

Case 1: $D = D_1\blacksquare D_2$ where $|D_1| = i - 1$, and the $(i+1)^{st}$ position of D is not a white square. Let $D' = D_1\square D_2$, so that $D \lessdot D'$. If the ith position of $\hat{A}$ is a black-plus square, then $D = \overline{A}$. So, the ith position of $\hat{A}$ is a mystery square. The sash D' is of the form $\hat{A}$.

Case 2: $D = D_1\blacksquare\boxed{}D_2$ where $|D_1| = i - 1$.

Let $D' = D_1\boxed{}D_2$, so that $D \lessdot D'$. The ith position of $\hat{A}$ is either a black-plus square or a mystery square. The $(i+1)^{st}$ position of $\hat{A}$ is either a white square, a white-plus square, or a mystery square. Thus, the sash D' is of the form $\hat{A}$.

Case 3: $D = D_1 \square\!\square D_2$ where $|D_1| = i - 1$.

Let $D' = D_1 \square\square D_2$, so that $D \lessdot D'$. The ith position of $\hat{A}$ is either a black-plus square or a mystery square. Suppose that the ith position of $\hat{A}$ is a black-plus square. If the $(i + 1)^{st}$ position of $\hat{A}$ is either a white square, a white-plus square, or a mystery square, then $D = \overline{A}$. If the $(i + 1)^{st}$ position of $\hat{A}$ is any other object, then D is not of the form $\hat{A}$. Thus, the ith position of $\hat{A}$ is a mystery square, and the $(i + 1)^{st}$ position of $\hat{A}$ is either a white square, a white-plus square, or a mystery square. The sash D' is of the form $\hat{A}$.

Case 4: $D = D_1 \square\!\square D_2$ where $|D_1| = i - 2$.

Let $D' = D_1 \square\square D_2$, so that $D \lessdot D'$. The ith position of $\hat{A}$ is either a white-plus square or a mystery square. If the $(i - 1)^{st}$ position of $\hat{A}$ is either a black square or a black-plus square, then $D = \overline{A}$. So, the $(i - 1)^{st}$ position of $\hat{A}$ is a mystery square, and the sash D' is of the form $\hat{A}$.

Case 5: $D = D_1 \blacksquare\!\square D_2$ where $|D_1| = i - 2$, and the $(i - 1)^{st}$ position of $\hat{A}$ is a black square.

Let $D' = D_1 \square\!\square D_2$, so that $D \lessdot D'$. The ith position of $\hat{A}$ is either a white-plus square or a mystery square. Thus, the sash D' is of the form $\hat{A}$. $\qquad\square$

Proposition 2.2.16. *If a sash E is of the form $\hat{B}$, then $E \in [\underline{B}, \overline{B}]$.*

Proof. Suppose that E is of the form $\hat{B}$ and that $E \neq \underline{B}$. We want to show that there exists a sash E' such that $E' \lessdot E$ and E' is of the form $\hat{B}$. If $E \neq \underline{B}$, then for some mystery square of $\hat{B}$, the mystery square is not replaced by a black square on E.

Case 1: The mystery square in the ith position of $\hat{B}$ is replaced by a white square on E.

Subcase 1a: $E = E_1 \square E_2$, where the first tile of E_2 is not a white square.

Let $E' = E_1 \blacksquare E_2$, so that $E' \lessdot E$. The sash E' is of the form $\hat{B}$.

Subcase 1b: $E = E_1 \square\square E_2$.

Let $E' = E_1 \square\!\square E_2$, so that $E' \lessdot E$, and where $i - 1 = |E_1|$. The $(i + 1)^{st}$ position of $\hat{B}$ is either a white square or a mystery square. Either way, E' is of the form $\hat{B}$.

Case 2: The mystery square of $\hat{B}$ is replaced by the left half of a rectangle on E.

Thus the mystery square of $\hat{B}$ is followed by a white square or another mystery square, and

$E = E_1\boxed{}E_2$. Let $E' = E_1\boxed{\blacksquare}E_2$, so that $E \lessdot E'$. The sash E' is of the form $\hat{B}$.

Case 3: The mystery square of $\hat{B}$ is replaced by the right half of a rectangle on E.

Thus the mystery square of $\hat{B}$ is preceded by a black square or another mystery square, and $E = E_1\boxed{}E_2$. Let $E' = E_1\boxed{\blacksquare}E_2$, so that $E \lessdot E'$. The sash E' is of the form $\hat{B}$.

Now, suppose that E is of the form $\hat{B}$ and that $E \neq \overline{B}$. We want to show that there exists a sash E' such that $E \lessdot E'$ and E' is of the form $\hat{B}$. Consider a mystery square in the ith position of $\hat{B}$.

Case 1: $E = E_1\blacksquare E_2$ where $|E_1| = i - 1$, and the $(i+1)^{st}$ position of E is not a white square. Let $E' = E_1\square E_2$, so that $E \lessdot E'$. The sash E' is of the form $\hat{B}$.

Case 2: $E = E_1\boxed{\blacksquare}E_2$ where $|E_1| = i - 1$.

Let $E' = E_1\boxed{}E_2$, so that $E \lessdot E'$. The $(i+1)^{st}$ position of $\hat{B}$ is either a white square or a mystery square. Thus, the sash E' is of the form $\hat{B}$.

Case 3: $E = E_1\boxed{}E_2$ where $|E_1| = i - 1$.

Let $E' = E_1\boxed{}E_2$, so that $E \lessdot E'$. The $(i+1)^{st}$ position of $\hat{B}$ is either a white square or a mystery square. Thus, the sash E' is of the form $\hat{B}$.

Case 4: $E = E_1\boxed{}E_2$ where $|E_1| = i - 2$.

Let $E' = E_1\boxed{}E_2$, so that $E \lessdot E'$. If the $(i-1)^{st}$ position of $\hat{B}$ is a black square, then $E = \overline{B}$. So, the $(i-1)^{st}$ position of $\hat{B}$ is a mystery square. Thus, the sash E' is of the form $\hat{B}$.

Case 5: $E = E_1\boxed{\blacksquare}E_2$ where $|E_1| = i - 2$, and the $(i-1)^{st}$ position of $\hat{B}$ is a black square. Let $E' = E_1\boxed{}E_2$, so that $E \lessdot E'$. Thus, the sash E' is of the form $\hat{B}$. □

Definition 2.2.17. Consider an allowable dotting d of a sash $C \in \Sigma_{n-1}$. Place the numbers 1 through n before, after, and in between each of the $n - 1$ positions of C. Let T be the set of numbers such that either the nearest dotted square to their right is a black square (or the left half of a rectangle), or the nearest dotted square to their left of a white square (or the right half of a rectangle). We say a set T is an *allowable set* for C if it arises in this way from an allowable dotting of C.

$(C)_{\{3,4,5\}}$ = ▢ • ▨ ▢ $(C)_{\{1,2,3\}}$ = ▢ ▢ • ▢ $(C)_{\{1,3,4,5\}}$ = • • ▨ ▢

$(C)_{\{3\}}$ = ▢ • ▨ • $(C)_{\{1,2,3,5\}}$ = ▢ ▢ ▨ • • $(C)_{\{1,3\}}$ = • • ▨ • ▢

$(C)_{\{3,5\}}$ = ▢ • ▨ • • $(C)_{\{1,3,5\}}$ = • • ▨ • •
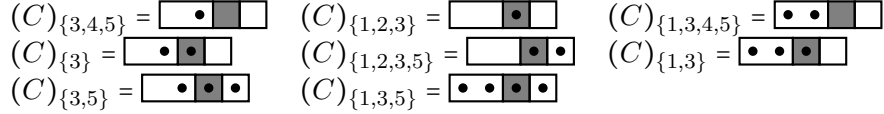
Figure 2.3: The allowable sets and allowable dottings of a sash C

Example 2.2.18. For $d =$ ▨• ▢ • ▢ ▢ ▨• •, the allowable set for C is $T = \{1, 4, 5, 6, 8\}$.

$$\underline{1} \quad 2 \quad 3 \quad \underline{4} \quad \underline{5} \quad \underline{6} \quad 7 \quad \underline{8}$$
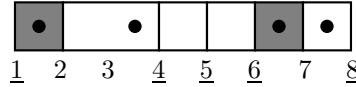
Definition 2.2.19. For T, an allowable set for C, we define $(C)_T$ to be the allowable dotting d of C such that there is a dot in the ith position of d either if $i \in T$ and $i + 1 \notin T$ or if $i + 1 \in T$ and $i \notin T$.

Example 2.2.20. (▨ ▢ ▢ ▢ ▨ ▢)$_{\{1,4,5,6,8\}}$ = ▨• ▢ • ▢ ▢ ▨• •

Figure 2.3 shows the allowable set associated with each of the allowable dottings in Figure 2.2.

We define a map τ from pairs (C, T) where C is a sash and T is an allowable set with respect to C, to permutations. The output is a permutation where the elements of T appear before the elements of T^C, and as we will verify in Proposition 2.2.23, the map σ takes $\tau(C, T)$ to C. The map τ does not necessarily output a Pell permutation.

Definition 2.2.21. Let T be an allowable set for a sash $C \in \Sigma_{n-1}$. First draw a vertical line. We eventually build a permutation by placing all of the elements of T on the left of the vertical line, and all of the elements on T^C on the right of the vertical line, and then removing the line. If $1 \in T$ then place a 1 on the left of the line; otherwise, place the 1 on the right. The guiding principle in defining this map is to place each number i as far right as possible while still making it possible for $\sigma(\tau(C, T))$ to be C and for all entries of T to appear before all entries of T^C. Read the sash from left to right from position 1 to position $n - 1$.

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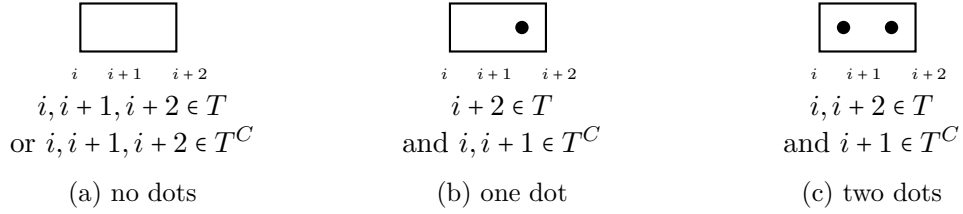
Figure 2.4: Possible rectangle dottings of the ith and $(i+1)^{st}$ positions of C

Suppose C has a black square in the ith position. If $i+1 \in T$, then place $i+1$ immediately to the left of the vertical line. If $i+1 \in T^C$, then place $i+1$ on the far right of the permutation.

Suppose C has a white square in the ith position. If $i, i+1 \in T$ or if $i, i+1 \in T^C$, then place $i+1$ immediately to the left of i. If $i+1 \in T$ and $i \in T^C$, then place $i+1$ immediately to the left of the vertical line. The case where $i \in T$ and $i+1 \in T^C$ is ruled out because if there were a dot on the ith position of C, which is a white square, then $i \in T^C$ and $i+1 \in T$.

Suppose C has a rectangle in the ith and $(i+1)^{st}$ positions. All of the possible dottings of the ith and $(i+1)^{st}$ positions of C are shown in Figure 2.4. If $i \in T$, then $i+2 \in T$. Place $i+2$ immediately to the left of i. There are two possibilities for placing $i+1$. If $i+1 \in T$, then place $i+1$ immediately to the left of the vertical line, and if $i+1 \in T^C$, then place $i+1$ at the far right of the permutation. If $i \in T^C$, then $i+1 \in T^C$. Place $i+1$ at the far right of the permutation. There are two possibilities for placing $i+2$. If $i+2 \in T$, then place $i+2$ immediately to the left of the vertical line, and if $i+2 \in T^C$, then place $i+2$ immediately to the left of i.

Now $\tau(C,T)$ is the permutation that results from ignoring the vertical line.

Example 2.2.22. For C = ▦☐⬛⬛ and $T = \{1,4,5,6,8\}$, the following procedure computes $\tau(C,T) = 16548237$.

$$
\begin{array}{ccccc}
1 \in T & \to & 1 & : & \\
\blacksquare & \to & 1 & : & 2 \\
\square & \to & 14 & : & 23 \\
\square & \to & 154 & : & 23 \\
\square & \to & 1654 & : & 23 \\
\blacksquare & \to & 1654 & : & 237 \\
\square & \to & 16548 & : & 237
\end{array}
$$

Proposition 2.2.23. *If T is an allowable set for $C \in \Sigma_{n-1}$, then $\sigma(\tau(C,T)) = C$*

Proof. Let T be an allowable set for C. Suppose C has a black square in the ith position. Since T is allowable, we cannot have $i \in T^C$ and $i + 1 \in T$. If there is a white square in the $(i+1)^{st}$ position then we cannot have $i, i+1 \in T^C$ and $i+2 \in T$. For every situation $\tau(C,T)$ maps $i+1$ to the right of i and does not map $i+2$ to the left of i, thus $\sigma(\tau(C,T))$ also has a black square in the ith position.

Suppose C has a white square in the ith position. Since T is allowable, we cannot have $i \in T$ and $i+1 \in T^C$. If there is a black square in the $(i-1)^{st}$ position then we cannot have $i-1, i \in T^C$ and $i+1 \in T$. For every situation $\tau(C,T)$ maps $i+1$ to the left of i and, if $i-1$ is to the left of i, does not map $i+1$ to the left of $i-1$, thus $\sigma(\tau(C,T))$ also has a white square in the ith position.

Suppose C has a rectangle in the ith and $(i+1)^{st}$ positions. Considering every possible dotting arrangement as shown in Figure 2.4, we see $\tau(C,T)$ maps $i+1$ to the right of i and $i+2$ to the left of i, thus $\sigma(\tau(C,T))$ also has a rectangle in the ith and $(i+1)^{st}$ positions. $\square$

Proposition 2.2.24. *T is an allowable set for a sash $C \in \Sigma_{n-1}$, $T = [n]$, or $T = \varnothing$ if and only if T is good as described in Section 1.2. That is, the entries of T are the first elements of a permutation $z' \in S_n$ where $\sigma(z') = C$.*

Proof. Suppose $T = [n]$ or $T = \varnothing$, and let $z' = \eta(C)$. The entries of T are the first elements of z' such that $\sigma(z') = C$. Suppose T is an allowable set for a sash C, and let $z' = \tau(C,T) \in S_n$.

The entries of T are the first elements of z' such that $\sigma(z') = C$.

To prove the reverse implication, suppose $z' \in S_n$ such that $\sigma(z') = C$, and let T_j be the set containing the first j entries of z'. If $j = 0$ or $j = n$, then the proposition is true. Assume that $0 < j < n$. Let d be a dotting of C such that there is a dot in the ith position of C either if $i \in T_j$ and $i + 1 \in T_j^C$ or if $i \in T_j^C$ and $i + 1 \in T_j$. We want to show that T_j is an allowable set for C; that is, we want to verify that d is an allowable dotting for C.

Since $0 < j < n$, there exists some i such that $i \in T_j$ and $i + 1 \notin T_j$, so d has at least one dot.

If $i \in T_j$ and $i + 1 \in T_j^C$ then d has a dot in the ith position. We know that entries in T_j^C come after entries of T_j in z', so $i + 1$ is to the right of i in z'. Thus the ith dot is on a black square or the left half of a rectangle of C. Similarly, if $i \in T_j^C$ and $i + 1 \in T_j$, then the ith dot of d is on a white square or the right half of a rectangle of C. Thus the first dot of d can be on any object, and dotted positions alternate between a black square (or the left half of a rectangle) and a white square (or the right half of a rectangle).

Suppose that there is a rectangle in the ith and $(i+1)^{st}$ positions of C. If $i \in T_j$ and $i + 1 \in T_j^C$, then $i + 2$ is not an element of T_j^C because $i + 2$ is to the left of i in z'. Thus d has no instances of $\boxed{\bullet}$.

Suppose that there is a black square in the ith position of C and a white square in the $(i + 1)^{st}$ position of C. If $i, i + 1 \in T_j^C$, then $i + 2$ is not an element of T_j because $i + 2$ is to the right of i in z'. Thus d has no instances of $\blacksquare\boxed{\bullet}$.

Therefore we have shown that T_j is an allowable set for C. $\qquad\square$

Let $z \in P_n$ and let $C \in \Sigma_{n-1}$ such that $\sigma(z) = C$. Let I_T and J_T be as defined in Section 1.2. From Theorem 1.2.4 we have that the coproduct on sashes is given by:

$$\Delta_S(C) = \sum_{\substack{T \text{ is allowable,} \\ T=\varnothing, \text{ or } T=[n]}} \sigma(I_T) \otimes \sigma(J_T) \tag{2.7}$$

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Notice $I_\varnothing = \varnothing$, $J_\varnothing = z$, $I_{[n]} = z$, and $J_{[n]} = \varnothing$, so we have:

$$\Delta_S(C) = \varnothing \otimes C + C \otimes \varnothing + \sum_{T \text{ is allowable}} \sigma(I_T) \otimes \sigma(J_T) \tag{2.8}$$

From Proposition 1.2.8 and our discussion of the relationship between the coproduct and the dual product in Section 1.2, we see:

$$\Delta_S(C) = \varnothing \otimes C + C \otimes \varnothing + \sum_{T \text{ is allowable}} \sum_{\substack{D \text{ and } E \text{ such that} \\ \gamma_T(D \otimes E) = C}} D \otimes E \tag{2.9}$$

Proposition 2.2.25. *Given an allowable set T and a sash C such that $(C)_T = d$, $D \otimes E$ is a term of $I_d \otimes J_d$ if and only if $\gamma_T(D \otimes E) = C$.*

Proof. Recall the notation $d = c_1 \bullet_1 c_2 \bullet_2 \cdots c_j \bullet_j c_{j+1}$. Assume that j is even and that $\bullet_1$ is on a black square or the left side of a rectangle, thus $1 \in T$. The cases where j is odd or where $\bullet_1$ is on a white square or the right side of a rectangle are identical, other than some adjustments to indices. Let T_{2i-1} be the ith set of consecutive integers in T and let T_{2i} be the ith set of consecutive integers in T^C. Thus $T = T_1 \cup T_3 \cup \cdots \cup T_{j+1}$ and $T^C = T_2 \cup T_4 \cup \cdots \cup T_j$, where $1 \in T_1 \subseteq T$. Recall $A = c_1 \boxed{?} c_3 \boxed{?} \cdots \boxed{?} c_{j+1}$ and $B = c_2 \boxed{?} c_4 \boxed{?} \cdots \boxed{?} c_j$. Notice that $c_i = \sigma(\eta(C)|_{T_i})$.

If $h, h+1 \in T$, then for some index $2i-1$ we have $h, h+1 \in T_{2i-1}$. Thus, the hth position of C is in c_{2i-1} and we refer to that position on A as the position *corresponding* to h. Similarly, if $h, h+1 \in T_{2i} \subseteq T^C$, then the hth position of C is in c_{2i} and we refer to that position on B as the position *corresponding* to h.

Suppose that $D \otimes E$ is a term of $I_d \otimes J_d$, that is $D \in [\underline{A}, \overline{A}]$ and $E \in [\underline{B}, \overline{B}]$.

As we compute $\gamma_T(D \otimes E)$, we begin by labeling D with the elements of T and labeling E with the elements of T^C. Notice that the hth arrow meets one of the following conditions:

1. If $h, h+1 \in T$, then the hth arrow is in the position of D corresponding to h.

2. If $h, h+1 \in T^C$, then the hth arrow is in the position of E corresponding to h.

3. If $h \in T$ and $h+1 \in T^C$, then the hth arrow is from D to E.

4. If $h \in T^C$ and $h+1 \in T$, then the hth arrow is from E to D.

Case 1: $h, h+1, h+2 \in T$.

Whatever is in the hth position of C is also in the position of D corresponding to h. The map $\gamma_T(D \otimes E)$ places whatever is in the position of D corresponding to h in the hth position of the output.

Case 2: $h, h+1, h+2 \in T^C$.

Whatever is in the hth position of C is also in the position of E corresponding to h. The map $\gamma_T(D \otimes E)$ places whatever is in the position of E corresponding to h in the hth position of the output.

Case 3: $h, h+1 \in T$ and $h+2 \in T^C$.

If C has a rectangle in the hth and $(h+1)^{st}$ positions, then the dotting d would have a dot in the $(h+1)^{st}$ position, which is the right half of a rectangle. The allowable set associated with such a dotting has $h, h+1 \in T^C$ and $h+2 \in T$ which is a contradiction, so C does not have a rectangle in the hth and $(h+1)^{st}$ positions.

If whatever is in the hth position of C is also in the position of D corresponding to h, then $\gamma_T(D \otimes E)$ places whatever is in the position of D corresponding to h in the hth position of the output.

If C has a black square in the hth position, then D may have the left half of a rectangle in the position corresponding to h. The map $\gamma_T(D \otimes E)$ places a black square in the hth position of the output.

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Case 4: $h, h+1 \in T^C$ and $h+2 \in T$.

If C has a black square in the hth position, then the dotting d would have an instance of ▨•, which is a contradiction. Thus, C does not have a black square in the hth position.

If C has a white square or the right half of a rectangle in the hth position, then E has the same object in the position corresponding to h. The map $\gamma_T(D \otimes E)$ places either a white square or the right half of a rectangle respectively in the hth position of the output and a white square in the $(h+1)^{st}$ position.

If C has the left half of a rectangle in the hth position, then E either has a black square or left half of a rectangle in the position corresponding to h. The map $\gamma_T(D \otimes E)$ places a rectangle in the hth and $(h+1)^{st}$ positions of the output.

Case 5: $h \in T$ and $h+1, h+2 \in T^C$.

If C has a rectangle in the hth and $(h+1)^{st}$ positions, then the dotting d would have an instance of $\boxed{\bullet\ \ }$, which is a contradiction. Thus, C does not have a rectangle in the hth and $(h+1)^{st}$ positions and C does have a black square in the hth position. The map $\gamma_T(D \otimes E)$ places a black square in the hth position of the output.

Case 6: $h, h+2 \in T$ and $h+1 \in T^C$.

If C has a black square in the hth position and a white square in the $(h+1)^{st}$ position, then $\underline{A}$ has a black square in the position between the labels h and $h+2$, and $\overline{A}$ has either a black square or the left half of a rectangle in the position between the labels h and $h+2$. Thus, D has either a black square or the left half of a rectangle in the position between the labels h and $h+2$. The map $\gamma_T(D \otimes E)$ places a black square in the hth position and a white square in the $(h+1)^{st}$ position of the output.

If C has a rectangle in the hth and $(h+1)^{st}$ positions, then $\underline{A}$ has a white square in the position between the labels h and $h+2$, and $\overline{A}$ has either a white square or the right half of a rectangle in the position between the labels h and $h+2$. Thus, D has either a white square or the left half of a rectangle in the position between the labels h and $h+2$. The map $\gamma_T(D \otimes E)$ places a rectangle in the hth and $(h+1)^{st}$ positions of the output.

Case 7: $h \in T^C$ and $h+1, h+2 \in T$.

C either has a black square or the left half of a rectangle in the hth position, and whatever is in the $(h+1)^{st}$ position of C is also in the position of D corresponding to $h+1$. We see from cases 4 and 6 that $\gamma_T(D \otimes E)$ places either a black square or the left half of a rectangle, respectively, in the hth position of the output. Also, $\gamma_T(D \otimes E)$ places whatever is in the position of D corresponding to $h+1$ in the $(h+1)^{st}$ position of the output.

Case 8: $h, h+2 \in T^C$ and $h+1 \in T$.

We see from cases 4 and 6 that whatever is in the hth position of C is also in the hth position of $\gamma_T(D \otimes E)$, and we see from cases 5 and 6 that whatever is in the $(h+1)^{st}$ position of C is also in the $(h+1)^{st}$ position of $\gamma_T(D \otimes E)$.

Therefore we have shown that $\gamma_T(D \otimes E) = C$.

Now let us suppose $\gamma_T(D \otimes E) = C$, and we will show that $D \otimes E$ is a term of $I_d \otimes J_d$. It is enough to show that D is of the form $\hat{A}$ and that E is of the form $\hat{B}$ because of Proposition 2.2.15 and Proposition 2.2.16. We refer to the position of D or E that is labeled with the hth arrow as the position of D or E corresponding to h.

Case 1: $h, h+1, h+2 \in T$.

The object in the position of D corresponding to h is the same as the object in the hth position of C, which is also the same as the object in the position of $\hat{A}$ corresponding to h.

Case 2: $h, h+1, h+2 \in T^C$.

The object in the position of E corresponding to h is the same as the object in the hth position of C, which is also the same as the object in the position of $\hat{B}$ corresponding to h.

Case 3: $h, h+1 \in T$ and $h+2 \in T^C$.

As we showed above, C does not have a rectangle in the hth and $(h+1)^{st}$ positions. If D has the left half of a rectangle in the position corresponding to h, then C has a black square in the hth position. Thus, the sash $\hat{A}$ has a black square in the position corresponding to h and a mystery square in the following position. If D has any other object in the position corresponding to h, then C has the same object in the hth position and the sash $\hat{A}$ has the same object as D in the

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position corresponding to h.

Case 4: $h, h + 1 \in T^C$ and $h + 2 \in T$.

As we showed above, C does not have a black square in the hth position. If E has a white square or the right half of a rectangle in the position corresponding to h, then C has the same object as E in the hth position and a white square in the $(h + 1)^{st}$ position. Thus, $\hat{B}$ has the same object as E in the position corresponding to h followed by a mystery square. If E has a black square or the left half of a rectangle in the position corresponding to h, then C has a rectangle in the hth and $(h + 1)^{st}$ positions. Thus, $\hat{B}$ has a black square in the position corresponding to h followed by a mystery square.

Case 5: $h \in T$ and $h + 1, h + 2 \in T^C$.

If there is a black square in the position of E corresponding to $h + 1$, then the $(h + 1)^{st}$ position of C is either a black square or the left half of a rectangle. If $h + 3 \in T$, then C has a rectangle in the $(h + 1)^{st}$ and $(h + 2)^{nd}$ positions, and the sash $\hat{B}$ has a black square in the position corresponding to $h + 1$ followed by a mystery square. If $h + 3 \notin T$, then C has a black square in the $(h + 1)^{st}$ position, and the sash $\hat{B}$ has a black square in the position corresponding to $h + 1$.

If there is the left half of a rectangle in the position of E corresponding to $h + 1$, then the $(h + 1)^{st}$ position of C is the left half of a rectangle. If $h + 3 \in T$, then the sash $\hat{B}$ has a black square in the position corresponding to $h + 1$ followed by a mystery square. If $h + 3 \notin T$, then the sash $\hat{B}$ has a the left half of a rectangle in the position corresponding to $h + 1$.

If there is either a white square or the right half of a rectangle in the position of E corresponding to $h + 1$, then the $(h + 1)^{st}$ position of C is a white square. The sash $\hat{B}$ has a white square in the position corresponding to $h + 1$ preceded by a mystery square.

Case 6: $h, h + 2 \in T$ and $h + 1 \in T^C$.

If D has a black square in the position between the labels h and $h+2$, then C has a black square in the hth position and a white square in the $(h + 1)^{st}$ position. The sash $\hat{A}$ has a black-plus square in the position between the labels h and $h + 2$.

If D has the left half of a rectangle in the position between the labels h and $h+2$, then C has

a black square in the hth position and a white square in the $(h+1)^{st}$ position. If $h+3 \in T$, then the sash $\hat{A}$ has a black-plus square in the position between the labels h and $h+2$ followed by a white square. If $h+3 \notin T$, then the sash $\hat{A}$ has a black-plus square in the position between the labels h and $h+2$ followed by either a white square, a white-plus square, or a mystery square.

If D has a white square in the position between the labels h and $h+2$, then C has a rectangle in the hth and $(h+1)^{st}$ positions. The sash $\hat{A}$ has a white-plus square in the position between the labels h and $h+2$.

If D has the right half of a rectangle in the position between the labels h and $h+2$, then C has a rectangle in the hth and $(h+1)^{st}$ positions. The sash $\hat{A}$ has a white-plus square in the position between the labels h and $h+2$, preceded by either a black square, a black-plus square, or a mystery square..

Case 7: $h \in T^C$ and $h+1 \in T$.

This case has already been fully considered in cases 4 and 6.

We have shown, by checking every position, that D is of the form $\hat{A}$ and that E is of the form $\hat{B}$. Therefore, $D \in [\underline{A}, \overline{A}]$, $E \in [\underline{B}, \overline{B}]$, and $D \otimes E$ is a term of $I_d \otimes J_d$. $\qquad\square$

Theorem 2.2.10 follows directly from Equation (2.9) and Proposition 2.2.25.

Chapter 3

The Hopf Algebra of Partial Evaluations

3.1 Limited Descent Avoiders and Partial Evaluations

In this chapter we examine a family of Hopf algebras that are defined by the pattern-avoidance conditions described in Chapter 1.2. The Hopf algebra of sashes is a member of the family described in this chapter. Consider the set $\mathrm{Av}_n[2(31), (k1)234\cdots k-1]$ of permutations of length n that avoid the patterns $2(31)$ and $(k1)234\cdots k-1$. For brevity we call this set "avoiders" and refer to it as Av_n^k.

Proposition 3.1.1. $x \in \mathrm{Av}_n^k$ *if and only if x has no descent larger than $k-2$ and for every descent (j, i) of x, all of the entries $i+1, \ldots, j-1$ are to the right of i in x.*

Proof. Let $x \in \mathrm{Av}_n^k$ and consider (j, i) a descent of x. Because x avoids $2(31)$ all of the entries from $i+1$ to $j-1$ are to the right of the descent. Since x also avoids $(k1)234\cdots k-1$, it does not have $k-2$ entries with values from $i+1$ to $j-1$ to the right of the descent. Thus there are at most $k-3$ entries between $i+1$ and $j-1$. Therefore x has no descent larger than $k-2$, and for each descent the entries between the values of the descent are positioned to the right of the descent.

Suppose x has no descent larger than $k-2$ and for every descent (j,i) of x, all of the entries $i+1,\ldots,j-1$ are to the right of i in x. With no descent larger than $k-2$, the permutation x does not have a $(k1)234\cdots k-1$ pattern. And because for every descent (j,i) of x, all of the entries $i+1,\ldots,j-1$ are to the right of i in x, the permutation x does not have a $2(31)$ pattern. Thus, $x \in \mathrm{Av}_n^k$. $\qquad\square$

If a permutation has no descent larger than $k-2$, then it also has no descent larger than $k-1$, thus $\mathrm{Av}_n^k \subseteq \mathrm{Av}_n^{k+1}$. The number of elements in $\mathrm{Av}_n[2(31)]$ are counted by the Catalan numbers. That is $|\mathrm{Av}_n[2(31)]| = C_n$ where C_n is the nth Catalan number. A formula for these numbers is given by [13, Sequence A000108]:

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

The number of elements in Av_n^k is given by the following proposition.

Proposition 3.1.2.
$$|\mathrm{Av}_n^k| = |\mathrm{Av}_{n-1}^k| + \sum_{i=1}^{k-2} C_{i-1}|\mathrm{Av}_{n-i}^k|$$

Proof. We count the elements of Av_n^k in the following manner. There are $|\mathrm{Av}_{n-1}^k|$ that have n in the last position. Now we consider those that do not have n in the last position. Suppose that n is followed by $n-i$. That is, n is part of a descent of size i. Since these permutations are elements of Av_n^k, each descent must be less than or equal to $k-2$, thus $1 \le i \le k-2$. Suppose there were an entry $j < n-i$ to the right of any entry $\ell \in [n-i+1, n-1]$, then $(n-i)-\ell-j$ would be a $2(31)$ pattern. Thus, the entries $[n-i+1, n-1]$ are at the end of each of these permutations. The entries $[n-i+1, n-1]$ avoid the pattern $2(31)$ so there are C_{i-1} ways for them to be arranged at the end of the permutation. Therefore $C_{i-1}|\mathrm{Av}_{n-i}^k|$ counts the number of elements of Av_n^k such that n is followed by $n-i$. $\qquad\square$

Lemma 3.1.3. *A permutation $x \in S_n$ has a $2(31)$ pattern if and only if x has a $2-3-1$ pattern.*

This fact can be verified easily. See [12]

Proposition 3.1.4. *Any permutation avoiding* 2(31) *is of the form*

$$x_1^1 x_2^1 \cdots x_{j_1}^1 y_1^1 y_2^1 \cdots y_{\ell_1}^1 x_1^2 x_2^2 \cdots x_{j_2}^2 y_1^2 y_2^2 \cdots y_{\ell_2}^2 \cdots x_1^h x_2^h \cdots x_{j_h}^h y_1^h y_2^h \cdots y_{\ell_h}^h$$

with

- $x_{j_1}^1 = 1$

- *the entries* $x_1^i \cdots x_{j_i}^i$ *are in decreasing order*

- *the entries* $x_1^i \cdots y_{\ell_i}^i$ *are all of the values in* $[x_{j_i}^i, x_1^i]$

- x_1^{i+1} *is the leftmost entry of the permutation larger than* x_1^i

- $x_{j_{i+1}}^{i+1} = x_1^i + 1$

Proof. Let $x \in \text{Av}[2(31)]$. Label the entry 1 by $x_{j_1}^1$. All of the entries of x before 1 are in decreasing order or else x would have a 2(31) pattern. Label all of the entries of x before 1 by $x_1^1, x_2^1, \cdots x_{j_1-1}^1$. Label the leftmost entry of x that is larger than x_1^1 by x_1^2, label the entry $x_1^1 + 1$ by $x_{j_2}^2$, and label all of the entries in between by $x_2^2, \cdots x_{j_1-1}^2$. All of the entries labeled x_j^2 are larger than x_1^1 or else $x_1^1 - x_1^2 - x_j^2$ would form a 2(31) pattern. All of the entries labeled x_j^2 are in decreasing order or else $x_\ell^2 - x_j^2 - x_{j_2}^2$ would form a 2(31) pattern when $\ell < j$ and $x_\ell^2 < x_j^2$. By continuing the process we see that it is possible to label the leftmost entry of x that is larger than x_1^i by x_1^{i+1}, and to label $x_1^i + 1$ by $x_{j_{i+1}}^{i+1}$, such that the entries $x_1^i \cdots x_{j_i}^i$ are in decreasing order.

Label the unlabeled entries immediately to the right of $x_{j_i}^i$ by $y_1^i \cdots y_{\ell_i}^i$. We know that x_1^{i+1} is the leftmost entry of x larger than x_1^i, so the entries $y_1^i \cdots y_{\ell_i}^i$ are less than x_1^i. The entries $y_1^i \cdots y_{\ell_i}^i$ are larger than $x_{j_i}^i$, because $x_{j_i}^i$ is either $x_1^{i-1} + 1$ or 1. If $x_{j_i}^i = 1$, then $y_1^i \cdots y_{\ell_i}^i$ are larger than 1. If $x_{j_i}^i = x_1^{i-1} + 1$, then $y_1^i \cdots y_{\ell_i}^i$ are larger than $x_{j_i}^i$ or else $x_1^{i-1} - x_{j_i}^i - y_\ell^i$ would form a 2(31) pattern. For the sake of contradiction, let z be an entry of x with a value between $x_{j_i}^i$ and x_1^i, such that $z \neq x_j^i$ and $z \neq y_\ell^i$. If z is to the left of x_1^i, then $z - x_1^i - x_{j_i}^i$ forms a 2(31) pattern. If z is to the

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right of x_1^{i+1}, then $x_1^i - x_1^{i+1} - z$ forms a 2(31) pattern. Thus such a z does not exist and the entries $x_1^i \cdots y_{\ell_i}^i$ are all of the values in $[x_{j_i}^i, x_1^i]$. $\qquad\qquad\square$

Avoiders are in bijection with a combinatorial object we call partial evaluations. First we will describe a bijection between the set Av[2(31)] and an object called a full evaluation. Then, we will use that bijection to define a bijection between avoiders and partial evaluations.

Definition 3.1.5. A *full evaluation of size n* or a *fully evaluated block of size n* for $n \geq 1$, is an object such that if $n = 1$, then the block is ($\square$) and if $n \geq 2$ the block is a string of n factors $\square$ grouped into pairs by $n - 1$ sets of parentheses.

The number of full evaluations of size n is given by C_{n-1} as seen in [13, Sequence A000108].

Example 3.1.6. The five full evaluations of size 4:

$$\Big(\square(\square(\square\square))\Big) \quad \Big(\square((\square\square)\square)\Big) \quad \Big((\square\square)(\square\square)\Big) \quad \Big((\square(\square\square))\square\Big) \quad \Big(((\square\square)\square)\square\Big)$$

A pair of parentheses is comprised of an open-parenthesis "(" and a closed-parenthesis ")". Each open-parenthesis has a unique closed-parenthesis as the other half of its pair. We say that two pairs of parentheses are *nested* if one pair is contained within the other pair. Two pairs of parentheses are *non-nested* if the set of squares in one pair of parentheses is disjoint from the set of squares in the other. Any two pairs of parentheses are either nested or non-nested.

Definition 3.1.7. A *sub-block* of a full evaluation X is a full evaluation contained in and including a pair of parentheses in X.

Fully evaluated blocks extend the definition of a full evaluation to blocks of size 1.

Definition 3.1.8. A *partial evaluation of length n* is a sequence consisting of fully evaluated blocks and unevaluated blocks of size 1, *i.e.* $\square$. Let Σ_n be the set of partial evaluations where the total of the sizes of the evaluated blocks, plus the number of unevaluated blocks, is n.

We think of the squares as "factors" to be operated on by a binary operation. Thus, this object is called a "partial evaluation" because the parentheses indicate in what order to evaluate

$$\square\square\square \qquad (\square)\square\square \qquad \square(\square)\square \qquad \square\square(\square)$$
$$(\square)(\square)\square \quad (\square)\square(\square) \quad \square(\square)(\square) \quad (\square)(\square)(\square)$$
$$(\square\square)\square \qquad \square(\square\square) \qquad (\square\square)(\square) \qquad (\square)(\square\square)$$
$$\big(\square(\square\square)\big) \qquad \big((\square\square)\square\big)$$

Figure 3.1: The fourteen partial evaluations of length 3, *i.e.* the elements of Σ_3

the operation, but only partially. For example, some squares are left unevaluated, hence the unevaluated squares.

The symbol $\|$ denotes the partial evaluation of length 0 that is the unique element of Σ_0. For $X \in \Sigma_n$, we use the notation $|X| = n$ to describe the length of a partial evaluation. The set of partial evaluations of length n with evaluated blocks of size less than or equal to $k-2$ is called Σ_n^k. Fully evaluated blocks of size 1 are called *evaluated squares* and unevaluated blocks of size 1 are called *unevaluated squares*.

Proposition 3.1.9.
$$|\Sigma_n^k| = |\Sigma_{n-1}^k| + \sum_{i=1}^{k-2} C_{i-1}|\Sigma_{n-i}^k|$$

Proof. We count the elements of Σ_n^k in the following manner. There are $|\Sigma_{n-1}^k|$ partial evaluations that have an unevaluated square in the last position. Now we consider those that do not have an unevaluated square in the last position. Suppose that the last block of a partial evaluation is of size i. Since these partial evaluations are elements of Σ_n^k, each block must be less than or equal to $k-2$, thus $1 \leq i \leq k-2$. There are C_{i-1} blocks that could be in the last position. Therefore $C_{i-1}|\Sigma_{n-i}^k|$ counts the number of elements of Σ_n^k such that the last evaluated block of a partial evaluation is of size i. $\qquad \square$

Before we define the bijection between avoiders and partial evaluations, we define a map τ from S_n to full evaluations of size $n+1$. Variations of this map have appeared in many sources including $[3, 5, 7, 9, 11]$. For $x = x_1 x_2 \cdots x_n \in S_n$, begin with $n+1$ squares and label the n spaces

between the squares:

$$\square 1 \square 2 \square \cdots \square n \square$$

Then, read x from **right** to **left** and evaluate in the x_i^{th} position:

$$\square 1 \square 2 \square \cdots (\square x_n \square) \cdots \square n \square$$

Continue to x_1 and then remove the labels. We define $\tau(\varnothing) = (\square)$.

Example 3.1.10. Computation of $\tau(3214) = (((\square\square)\square)(\square\square))$:

$$
\begin{array}{rcl}
\tau(3214) & \Rightarrow & \square 1 \square 2 \square 3 \square 4 \square \\
3214 & \Rightarrow & \square 1 \square 2 \square 3 (\square 4 \square) \\
3214 & \Rightarrow & (\square 1 \square) 2 \square 3 (\square 4 \square) \\
3\mathbf{2}14 & \Rightarrow & ((\square 1 \square) 2 \square) 3 (\square 4 \square) \\
\mathbf{3}214 & \Rightarrow & (((\square 1 \square) 2 \square) 3 (\square 4 \square)) \\
& \Rightarrow & (((\square\square)\square)(\square\square))
\end{array}
$$

Let M be a set of n integers and let x be a permutation of M. We define $\tau(x) = \tau(\mathrm{st}(x))$.

Example 3.1.11. $\tau(6539) = \tau(\mathrm{st}(6539)) = \tau(3214) = (((\square\square)\square)(\square\square))$

Lemma 3.1.12. *If $x \in \mathrm{Av}_n[2(31)]$ and $y \in S_n$ such that y is obtained from x by a series of $\pi^{\uparrow}$-moves, then $\tau(x) = \tau(y)$.*

Proof. It is enough to show that the statement is true when y is obtained from x by a single $\pi^{\uparrow}$-move. Let $x = \cdots x_j \cdots x_i x_{i+1} \cdots$ and let $y = \cdots x_j \cdots x_{i+1} x_i \cdots$ such that x and y only vary in the ith and $(i+1)^{st}$ positions and $x_i < x_j < x_{i+1}$. The blocks $\tau(x)$ and $\tau(y)$ are equivalent except for the order that the x_i^{th} and x_{i+1}^{st} positions are evaluated. Since x_j is to the left of x_i and x_{i+1}, and because $x_i < x_j < x_{i+1}$, we see that the parentheses placed when evaluating the x_i^{th} position and the parentheses placed when evaluating the x_{i+1}^{st} position are non-nested. Thus the same result is achieved regardless of which position is evaluated first. $\square$

Lemma 3.1.13. *Let $x \in \mathrm{Av}_n[2(31)]$ and write x in the form described in Proposition 3.1.4. If*
$$\hat{x} = x_1^1 \cdots x_{j_1}^1 x_1^2 \cdots x_{j_2}^2 \cdots x_1^h \cdots x_{j_h}^h y_1^1 \cdots y_{\ell_1}^1 y_1^2 \cdots y_{\ell_2}^2 \cdots y_1^h \cdots y_{\ell_h}^h, \text{ then } \tau(x) = \tau(\hat{x}).$$

Proof. Let $i_1 < i_2$. The sub sequence of x: $x_1^{i_1} - y_\ell^{i_1} - x_j^{i_2}$ is a $2-1-3$ pattern. Since $x_1^{i_1}$ is to the left of $y_\ell^{i_1}$ for any ℓ, swapping adjacent entries of the form $y_\ell^{i_1}$ and $x_j^{i_2}$ is a $\pi^\uparrow$-move. Thus $\hat{x}$ is obtained from x by a series of $\pi^\uparrow$-moves, and from Lemma 3.1.12 we have $\tau(x) = \tau(\hat{x})$. $\square$

Define ρ to be the map from fully evaluated blocks of size $n+1$ to $\mathrm{Av}_n[2(31)]$ described as follows:

Begin with a full evaluation of size $n+1$ and insert the numbers 1 to n in between each square. Between the ith and $(i+1)^{st}$ squares there may be some close-parentheses, possibly followed by some open-parentheses. Place i after the close-parentheses and before the open-parentheses.

An evaluated block of size $n+1$ has n pairs of parentheses. The map ρ builds a permutation by removing a pair of parentheses from the block and placing an entry of the permutation at each step. When we remove the outermost pair of parentheses from a fully evaluated block, we are left with two sub-blocks or squares. Find the leftmost fully evaluated block, remove the outermost pair of parentheses, and place the number in between the revealed sub-blocks or squares as the next entry of the permutation in the output. Continue until all n numbers have been written in a permutation. Define $\rho[(\square)] = \varnothing$.

Example 3.1.14. We use the map ρ to compute $\rho[(((\square\square)\square)(\square\square))] = 3214$:

$$\Rightarrow \quad (((\square 1\square)2\square)3(\square 4\square))$$
$$\Big(((\square 1\square)2\square)3(\square 4\square)\Big) \quad \Rightarrow \quad 3$$
$$\Big((\square 1\square)2\square\Big)3(\square 4\square) \quad \Rightarrow \quad 32$$
$$\Big(\square 1\square\Big)2\square 3(\square 4\square) \quad \Rightarrow \quad 321$$
$$\square 1\square 2\square 3\Big(\square 4\square\Big) \quad \Rightarrow \quad 3214$$

We prove by induction that the output of ρ is a permutation that avoids $2(31)$. The only evaluated block of size 1 is $(\square)$, and $\rho[(\square)] = \varnothing \in S_0$ which does not contain a $2(31)$ pattern. Also, the only evaluated block of size 2 is $(\square\square)$, and $\rho[(\square\square)] = 1$ which does not contain a

2(31) pattern. Assume that if the input to ρ is an evaluated block of size less than or equal to i, then the output of ρ does not contain a 2(31) pattern. Let X be an evaluated block of size $i+1$. Consider $\rho(X)$. When we remove the outermost pair of parentheses, we are left with two blocks or squares, X_1 and X_2, of size less than or equal to i. Let X_2 begin in the $(j+1)^{st}$ position of X. The output of ρ is of the form: $j \cdot \rho(X_1) \cdot (\rho(X_2))_{[j+1,i]}$. Notice that j is greater than all of the entries of $\rho(X_1)$ and less than all of the entries of $(\rho(X_2))_{[j+1,i]}$. There are no 2(31) patterns where j serves as a "2". By induction, there are no 2(31) patterns within $\rho(X_1)$ and there are no 2(31) patterns within $(\rho(X_2))_{[j+1,i]}$. Also there are no 2(31) patterns where an entry of $\rho(X_1)$ serves as a "2", because all of the entries of $(\rho(X_2))_{[j+1,i]}$ are larger than all of the entries of $\rho(X_1)$. Therefore the output of ρ does not contain a 2(31) pattern.

Proposition 3.1.15. *The restriction of τ to 2(31)-avoiding permutations is a bijection to fully evaluated blocks with inverse ρ.*

Proof. Both $\mathrm{Av}_n[2(31)]$ and fully evaluated blocks are counted by the Catalan numbers. So it is enough to show that the map τ is onto. Notice that $\tau(\rho(X)) = X$, because the map ρ unravels an evaluated block from the outside in, and the map τ builds an evaluated block from the inside out. Therefore $\tau^{-1} = \rho$. □

We are now ready to define a map σ from S_n to Σ_{n-1}^k. Given a permutation, start with the entry 1 which we call a. Look for an entry of x that we call b, that is the smallest entry of x larger and to the left of a such that $b - a \le k - 2$. If no such entry exists, then output $\square$ and move on to the entry $a + 1$ which we call the new a. If there is an entry b, then we examine $x|_{[a+1,b-1]}$ and output $\tau(x|_{[a+1,b-1]})$. We call b the new a and repeat the process until $a = n$.

Example 3.1.16. k = 6, $\sigma(421395867) = (\square)(\square\square)\square((\square(\square\square))\square)$

$$
\begin{array}{llll}
a = 1,\ b = 2 & 4\mathbf{21}395867 & \Rightarrow & \tau(\varnothing) & = & (\square) \\
a = 2,\ b = 4 & \mathbf{4}21395867 & \Rightarrow & \tau(3) & = & (\square\square) \\
a = 4,\ \text{no } b & 421395867 & \Rightarrow & & & \square \\
a = 5,\ b = 9 & 421\mathbf{3}95867 & \Rightarrow & \tau(867) & = & ((\square(\square\square))\square)
\end{array}
$$

Example 3.1.17. k = 6, $\sigma(941257836) = (\square(\square\square))\square((\square\square)(\square\square))$

$$
\begin{array}{llll}
a = 1,\ b = 4 & 9\mathbf{4}1257836 & \Rightarrow & \tau(23) & = & (\square(\square\square)) \\
a = 4,\ \text{no } b & 9\mathbf{4}1257836 & \Rightarrow & & & \square \\
a = 5,\ b = 9 & \mathbf{9}41\mathbf{2}57836 & \Rightarrow & \tau(786) & = & ((\square\square)(\square\square))
\end{array}
$$

Remark 3.1.18. Each element $x \in \mathrm{Av}_n^k$ is of the form described in Proposition 3.1.4. The entries of x that serve as a when computing $\sigma(x)$ are the entries x_j^i. The entries of x that are operated on by τ are the entries y_ℓ^i.

Let M be a set of n integers and let x be a permutation of the entries of M. Define $\sigma(x) = \sigma(\mathrm{st}(x))$.

Example 3.1.19. $\sigma(14 \cdot 12 \cdot 11 \cdot 13 \cdot 19 \cdot 15 \cdot 18 \cdot 16 \cdot 17) = \sigma(\mathrm{st}(14 \cdot 12 \cdot 11 \cdot 13 \cdot 19 \cdot 15 \cdot 18 \cdot 16 \cdot 17)) = \sigma(421395867) = (\square)(\square\square)\square((\square(\square\square))\square)$

Define a map μ from Σ_{n-1}^k to Av_n^k as follows: First place the number 1 which we call a. Given $X \in \Sigma_{n-1}^k$, examine the first block X_1. Let j be the size of X_1. We place the numbers $a+1$ through $a+j$ in the next step. If X_1 is an unevaluated square, place $a+j$ which equals $a+1$ at the far right of the permutation. If X_1 is a fully evaluated block, then place $a+j$ immediately to the left of a, and place $(\rho(X_1))_{[a+1,a+j-1]}$ at the far right of the permutation. Continue this process with the next block letting the old $a+j$ serve as the new a.

Example 3.1.20. Computation of $\mu[(\square)(\square\square)\square((\square(\square\square))\square)] = 421395867$

$$
\begin{array}{lllll}
& & & \Rightarrow & 1 \\
(\square) & \Rightarrow & (\rho[(\square)])_\varnothing = \varnothing & \Rightarrow & 21 \\
(\square\square) & \Rightarrow & (\rho[(\square\square)])_{[3,3]} = 3 & \Rightarrow & 4213 \\
\square & & & \Rightarrow & 42135 \\
((\square(\square\square))\square) & \Rightarrow & (\rho[((\square(\square\square))\square)])_{[6,8]} = 867 & \Rightarrow & 421395867
\end{array}
$$

We need to verify that the output of the map μ is an element of Av_n^k. We consider each step of computing $\mu(X)$ for some partial evaluation X. If X has a block of size 1, then at most a

descent of size 1 is created. If X has a block of size $j > 1$, then a descent of size j is created. The largest j can be is $k - 2$ and the entries placed to the far right of the permutation also can not have any descents larger than $k - 2$. Every time a descent is created, the entries in between the values of the descent are placed to the right of the descent. Also, the output of the map ρ avoids $2(31)$ patterns. Therefore the output of the map μ is an element of Av_n^k.

Proposition 3.1.21. *The restriction of σ to Av_n^k is a bijection to Σ_{n-1}^k with inverse μ.*

Proof. Let $X \in \Sigma_{n-1}^k$ and we consider $\sigma(\mu(X))$. Let X_i be the ith block of X such that X_i begins in the jth position of X. If $X_i = \square$, the map μ places $j + 1$ to the right of j and nothing larger than j is placed to the left of j. When computing the ith block of $\sigma(\mu(X))$, the entry j serves as a and there are no entries of $\mu(X)$ larger and to the left of j, so σ outputs $\square$. If X_i is a fully evaluated block of size h, then μ places $j + h$ immediately to the left of j, and places $(\rho(X_i))_{[j+1,j+h-1]}$ to the right of j. Thus $j + h$ is the smallest thing left of and larger than j in $\mu(X)$, so $\sigma(\mu(X))$ outputs $\tau((\rho(X_i))_{[a+1,a+j-1]}) = \tau(\rho(X_i)) = X_i$. Therefore $\sigma(\mu(X)) = X$.

Let $x \in \mathrm{Av}_n^k$ and consider $\mu(\sigma(x))$. Let us examine the output of one step in the process to compute $\sigma(x)$. Recall that x is of the form described in Proposition 3.1.4. We begin with x_j^i, an entry of x.

Case 1: $j \neq 1$.

x_{j-1}^i is the smallest entry of x larger and to the left of x_j^i such that $x_{j-1}^i - x_j^i \leq k - 2$. Consider the entries $[x_j^i + 1, x_{j-1}^i - 1]$. They are to the right of the descent (x_{j-1}^i, x_j^i). Let $\ell \in [x_j^i + 1, x_{j-1}^i - 1]$. There are no values h smaller than x_j^i after any of the entries $[x_j^i + 1, x_{j-1}^i - 1]$, because $x_j^i - \ell - h$ is a $2(31)$ pattern. There are no values h larger than x_{j-1}^i before any of the entries $[x_j^i + 1, x_{j-1}^i - 1]$ and after x_{j-1}^i, because $x_{j-1}^i - h - \ell$ is a $2(31)$ pattern. Thus the entries $[x_j^i + 1, x_{j-1}^i - 1]$ are an adjacent substring of x. The map σ outputs $\tau(x|_{[x_j^i+1,x_{j-1}^i-1]})$. And μ acting on this block places x_{j-1}^i immediately to the left of x_j^i and places $\rho(\tau(x|_{[x_j^i+1,x_{j-1}^i-1]}))$ to the right of the permutation such that there are no entries smaller than x_j^i after them, and there are no entries larger than x_{j-1}^i between x_{j-1}^i and the entries $[x_j^i + 1, x_{j-1}^i - 1]$.

Case 2: $j = 1$.

There is no entry of x called b such that $b - x_1^i \leq k - 2$. The map σ outputs an unevaluated square. The map μ acting on the unevaluated square places $x_1^i + 1$ to the right of the permutation and no entries larger than x_1^i are placed to the left of x_1^i.

Therefore $\mu(\sigma(x)) = x$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Proposition 3.1.22. *$x, y \in S_n$ are equivalent if and only if $\sigma(x) = \sigma(y)$.*

Proof. From section 1.2 we know that the permutations $x = x_1 \cdots x_n$ and $y = y_1 \cdots y_n$ are equivalent if and only if $\pi_\downarrow(x) = \pi_\downarrow(y)$. Thus to prove the forward direction of the proposition, it is enough to consider the case where y is obtained from x by a single $\pi_\downarrow$-move. Let's consider a $\pi_\downarrow$-move in the ith position of x. First we will suppose that $x_i - x_{i+1} \geq k - 1$. The relative position of x_i with regard to x_{i+1} is irrelevant to the map σ, thus $\sigma(x) = \sigma(y)$. Now we suppose that $2 \leq x_i - x_{i+1} \leq k - 2$. There can only be a $\pi_\downarrow$-move in the ith position of x if an entry with a value in $[x_{i+1} + 1, x_i - 1]$ is to the left of x_i. In this case, both $\sigma(x)$ and $\sigma(y)$ have a block corresponding to the entries from x_{i+1} to the smallest such entry in $[x_{i+1} + 1, x_i - 1]$ that is to the left of x_i. Therefore $\sigma(x) = \sigma(y)$.

To prove the reverse implication suppose that x and y are not equivalent, that is $\pi_\downarrow(x) \neq \pi_\downarrow(y)$. Since $\pi_\downarrow(x)$ and $\pi_\downarrow(y)$ are avoiders, and avoiders are in bijection with partial evaluations, $\sigma(\pi_\downarrow(x)) \neq \sigma(\pi_\downarrow(y))$. By the previous paragraph, $\sigma(x) = \sigma(\pi_\downarrow(x)) \neq \sigma(\pi_\downarrow(y)) = \sigma(y)$. $\qquad$ □

Fundamentally, a partial evaluation is made of up two types of positions that we call "black" positions and "white" positions as shown in Figure 3.2. Unevaluated squares are black positions and evaluated squares are white positions. Within a fully evaluated block with more than one square, any square immediately to the right of an open-parenthesis is a black position, and any square immediately to the left of a close-parenthesis is a white position.

The reason behind this categorization of squares is seen when we relate a partial evaluation X to any permutation x such that $\sigma(x) = X$.

	Blocks of size 1	Squares within a block of size greater than 1
Black positions:	□	(□
White positions:	(□)	□)

Figure 3.2: A summary of black positions and white positions

Lemma 3.1.23. *Suppose* $\sigma(x) = X$. *Then position* h *of* X *is black if and only if* $h + 1$ *is to the right of* h *in* x *and white if and only if* $h + 1$ *is to the left of* h *in* x.

Proof. Suppose $h + 1$ is to the right of h in x. If X has an unevaluated square at position h, then position h is a black position. If a fully evaluated block of X begins at h, then since $h + 1$ is to the right of h in x, the fully evaluated block is not of size 1. So again h is a black position. Otherwise the hth position of X is either the middle or end of a block. There exist some entries ℓ_1 and ℓ_2 of x such that the block that spans the hth position of X is $\tau(x|_{[\ell_1+1,\ell_2-1]})$. Note that $h \neq \ell_2 - 1$ because $h + 1$ is to the right of h, and ℓ_2 is to the left of $\ell_2 - 1$. Thus $h, h + 1 \in [\ell_1 + 1, \ell_2 - 1]$. When τ is acting on $x|_{[\ell_1+1,\ell_2-1]}$ it reads the permutation from right to left, so the $(h + 1 - \ell_1)^{th}$ position is evaluated before the $(h - \ell_1)^{th}$ position. Thus the $(h - \ell_1)^{th}$ position of the block, which is the hth position of X, is a black position.

Suppose that position h of X is a black position. If position h of X is an unevaluated square, then the permutation x has $h + 1$ to the right of h. If the hth position is the first position of the block of size ≥ 2 that contains position h of X, then $h + 1$ is to the right of h in x. Otherwise there would be an evaluated square in the hth position of X. If the hth position is not the first position of the block that spans position h of X, then the block spans the hth and $h + 1^{st}$ positions of X. When computing $\sigma(x)$, the map τ will evaluate the $h + 1^{st}$ position of X before the hth, thus $h + 1$ is to the right of h in x. $\square$

We now describe the partial order on partial evaluations. From Proposition 1.2.2, the cover relations in Σ_n^k are exactly the relations $\sigma(y) \lessdot \sigma(x)$ where $x \in \mathrm{Av}_n^k$ and y is covered by x in S_n.

Proposition 3.1.24. *The cover relations on partial evaluations are:*

1. *For partial evaluations A and D, and for full evaluations B and C such that $|B|+|C| \leq k-2$:*

$$A(BC)D \lessdot ABCD$$

   (where $((\square)C) = (\square C)$, $(B(\square)) = (B\square)$, and $((\square)(\square)) = (\square\square)$)

2. *For partial evaluations A and C, and for either a full evaluation B such that $|B| = k - 2$ or for $B = \square$:*

$$A\square BC \lessdot A(\square)BC \text{ and } A\square \lessdot A(\square)$$

3. *For partial evaluations A, B_1, and C, and for a full evaluation B_2 of size ≥ 2, where B_1 is the partial evaluation obtained by removing every pair of parentheses from B_2 that include the first square of B_2, removing the first square from B_2, and evaluating any remaining unevaluated squares:*

$$A\square B_1 C \lessdot AB_2 C$$

4. *For partial evaluations A and C, and for full evaluations or unevaluated squares U, V, and W:*

$$A(U(VW))C \lessdot A((UV)W)C$$

Proof. Let $x \in \mathrm{Av}_n^k$ and $y \in S_n$ such that y is covered by x in the weak order. That is, $x = x_1 \cdots x_h x_{h+1} \cdots x_n$ and $y = x_1 \cdots x_{h+1} x_h \cdots x_n$, where $x_{h+1} < x_h$. We consider $\sigma(x)$ and $\sigma(y)$. Recall that x is of the form described in Proposition 3.1.4. We consider the cases where either $x_{h+1} = x_j^i$ for $j \geq 2$ or $x_{h+1} = y_\ell^i$ for $\ell \geq 2$, because if $x_{h+1} = x_1^i$ or if $x_{h+1} = y_1^i$ then $x_h < x_{h+1}$.

Case 1: For $j > 2$, $x_{h+1} = x_j^i$ and $x_{h-1} - x_{h+1} \leq k - 2$.

It follows that $x_h = x_{j-1}^i$ and $x_{h-1} = x_{j-2}^i$. Let $A = \sigma(y|_{[1,x_{h+1}]}) = \sigma(x|_{[1,x_{h+1}]})$, let $B = \sigma(x|_{[x_{h+1},x_h]})$, let $C = \sigma(x|_{[x_h,x_{h-1}]})$, and let $D = \sigma(y|_{[x_{h-1},n]}) = \sigma(x|_{[x_{h-1},n]})$. Notice that $y|_{[x_{h+1},x_{h-1}]} = x_{h-1} \cdot x_{h+1} \cdot x_h \cdot (y|_{[x_{h+1}+1,x_h-1]})(y|_{[x_h+1,x_{h-1}-1]})$. Since x_h is the farthest left of

all the entries in y with values between x_{h+1} and x_{h-1}, when computing this block of $\sigma(y)$, the position labeled x_h will be the last to be evaluated. Thus, we see that this block is $((\tau(y|_{[x_{h+1}+1, x_h-1]}))(\tau(y|_{[x_h+1, x_{h-1}-1]}))) = (BC)$. If $x_h = x_{h+1}+1$, then $B = (\square)$ and $y|_{[x_{h+1}, x_{h-1}]} = (\square C)$. If $x_{h-1} = x_h + 1$, then $C = (\square)$ and $y|_{[x_{h+1}, x_{h-1}]} = (B\square)$. If both $x_h = x_{h+1} + 1$ and $x_{h-1} = x_h + 1$, then $y|_{[x_{h+1}, x_{h-1}]} = (\square\square)$. Therefore, we have shown that the first partial order relation of the proposition holds:

$$A(BC)D = \sigma(y) < \sigma(x) = ABCD$$

Case 2: For $j > 2$, $x_{h+1} = x_j^i$, $x_h = x_{j-1}^i = x_{h+1} + 1$ and $x_{h-1} = x_{j-2}^i = x_{h+1} + k - 1$.

Let $A = \sigma(y|_{[1, x_{h+1}]}) = \sigma(x|_{[1, x_{h+1}]})$, let $B = \sigma(y|_{[x_h, x_{h-1}]}) = \sigma(x|_{[x_h, x_{h-1}]})$, and let $C = \sigma(y|_{[x_{h-1}, n]}) = \sigma(x|_{[x_{h-1}, n]})$. Notice that $x_{h-1} - x_h = k - 2$; thus the block B is of size $k - 2$. We see $\sigma(y|_{[x_{h+1}, x_h]}) = \square$ and $\sigma(x|_{[x_{h+1}, x_h]}) = (\square)$. Therefore, we have shown that the second partial order relation of the proposition holds for B such that $|B| = k - 2$:

$$A\square BC = \sigma(y) < \sigma(x) = A(\square)BC$$

Case 3: $x_{h+1} = x_2^i$ and $x_h = x_1^i = x_{h+1} + 1$.

Suppose that $x_h < n$. Thus $x_h + 1 = x_{j_{i+1}}^{i+1}$. Let $A = \sigma(y|_{[1, x_{h+1}]}) = \sigma(x|_{[1, x_{h+1}]})$ and let $C = \sigma(y|_{[x_h+1, n]}) = \sigma(x|_{[x_h+1, n]})$. We see $\sigma(y|_{[x_{h+1}, x_h]}) = \square$, $\sigma(x|_{[x_{h+1}, x_h]}) = (\square)$, and $\sigma(y|_{[x_h, x_h+1]}) = \sigma(x|_{[x_h, x_h+1]}) = \square$. Therefore, we have shown that the second partial order relation of the proposition holds for B such that $B = \square$:

$$A\square\square C = \sigma(y) < \sigma(x) = A(\square)\square C$$

If $x_h = n$, then let $A = \sigma(y|_{[1, x_{h+1}]}) = \sigma(x|_{[1, x_{h+1}]})$. We see:

$$A\square = \sigma(y) < \sigma(x) = A(\square)$$

Case 4: For $j \geq 2$, $x_{h+1} = x_j^i$, $x_h = x_{j-1}^i \neq x_{h+1} + 1$, and if x_{j-2}^i exists then $x_{j-2}^i - x_{h+1} > k - 2$.

We see that $\sigma(y) = \sigma(y|_{[1,x_{h+1}]}) \square \sigma(y|_{[x_{h+1}+1,x_h]}) \sigma(y|_{[x_h,n]})$. Let $A = \sigma(y|_{[1,x_{h+1}]}) = \sigma(x|_{[1,x_{h+1}]})$, let $B_1 = \sigma(y|_{[x_{h+1}+1,x_h]})$, let $B_2 = \sigma(x|_{[x_{h+1},x_h]})$ and let $C = \sigma(y|_{[x_h,n]}) = \sigma(x|_{[x_h,n]})$.

We now consider B_1 in more detail. Let $y|_{[x_{h+1}+1,x_h]} = x_h \cdots x_{\ell_m} \cdots x_{\ell_2} \cdots x_{\ell_1} \cdots (x_{h+1}+1) \cdots$, where x_{ℓ_1} is the smallest thing left of and larger than $x_{h+1} + 1$ such that $x_{\ell_1} - (x_{h+1} + 1) \leq k - 2$, the entry x_{ℓ_2} is the smallest thing left of and larger than x_{ℓ_1} such that $x_{\ell_2} - x_{\ell_1} \leq k - 2$, and so on. It is possible for $x_h = x_{\ell_1}$. We know that $x_h - (x_{h+1} + 1) \leq k - 2$ because $x_h - x_{h+1} \leq k - 2$, so for $x_{h+1} + 1$ and for each x_ℓ there exists an entry of $y|_{[x_{h+1}+1,x_h]}$ that is the smallest thing left of and larger than it such that their difference is less than $k - 2$. Thus B_1 is a partial evaluation made up of $m + 1$ blocks of size $x_{\ell_1} - (x_{h+1} + 1)$, $x_{\ell_2} - x_{\ell_1}$, and so on, respectively. Notice that B_1 does not have any unevaluated squares.

Now we consider B_2. The fully evaluated block B_2 is such that $|B_2| = |B_1| + 1$. Note $B_2 = \sigma(x|_{[x_{h+1},x_h]}) = \sigma(x_h x_{h+1} \cdots x_{\ell_m} \cdots x_{\ell_2} \cdots x_{\ell_1} \cdots (x_{h+1}+1) \cdots) = \tau(\cdots x_{\ell_m} \cdots x_{\ell_2} \cdots x_{\ell_1} \cdots (x_{h+1}+1) \cdots)$. Let $\hat{B}_1$ be the partial evaluation obtained by changing all of the evaluated squares of B_1 to unevaluated squares. To compute B_2 we begin with an unevaluated square followed by $\hat{B}_1$. All that is left to evaluate are the positions $(x_{h+1}+1)$, ℓ_1, $\ldots$, ℓ_m in increasing order. Each of these additional evaluations will add a pair of parentheses to B_2 with an open parenthesis before the initial square.

Therefore B_1 is the partial evaluation obtained by removing every pair of parentheses from B_2 that include the first square of B_2, removing the first square from B_2, and evaluating any remaining unevaluated squares, and we have shown that the third partial order relation of the proposition holds:

$$A \square B_1 C = \sigma(y) \lessdot \sigma(x) = A B_2 C$$

Case 5: For $j \geq 2$, $x_{h+1} = y_j^i$.

Let ℓ be the smallest integer such that the entry $x_\ell^i < x_{h+1}$. Thus, $x_\ell^i < x_{h+1} < x_h < x_{\ell-1}^i$. Let $A = \sigma(y|_{[1,x_\ell^i]}) = \sigma(x|_{[1,x_\ell^i]})$ and let $C = \sigma(y|_{[x_{\ell-1}^i,n]}) = \sigma(x|_{[x_{\ell-1}^i,n]})$. Let $B_1 = \sigma(y|_{[x_\ell^i,x_{\ell-1}^i]}) = \tau(y|_{[x_\ell^i+1,x_{\ell-1}^i-1]})$ and let $B_2 = \sigma(x|_{[x_\ell^i,x_{\ell-1}^i]}) = \tau(x|_{[x_\ell^i+1,x_{\ell-1}^i-1]})$. B_1 and B_2 are fully evaluated

blocks, and A and C are partial evaluations. Since $y|_{[x_\ell^i+1, x_{\ell-1}^i-1]}$ and $x|_{[x_\ell^i+1, x_{\ell-1}^i-1]}$ differ by switching x_h and x_{h+1}, B_2 covers B_1 in the Tamari lattice. That is, B_1 can be represented as $(U(VW))$ and B_2 can be represented as $((UV)W)$ for full evaluations or unevaluated squares U, V, and W. Therefore:

$$A(U(VW))C = \sigma(y) \lessdot \sigma(x) = A((UV)W)C$$

$\square$

3.2 The Hopf Algebra of Partial Evaluations

In this section, we describe the product on partial evaluations and we give a partial description of the coproduct on partial evaluations. From section 1.2, we know that for $U = \{2(31), (k1)23\dots k-1\}$, the Hopf algebra of avoiders is given by $(\mathbb{K}[\mathrm{Av}_\infty], \bullet_{\mathrm{Av}}, \Delta_{\mathrm{Av}})$. Thus for $x \in \mathrm{Av}_p^k$ and $y \in \mathrm{Av}_q^k$, such that $y' = (y)_{[p+1,p+q]}$, $\sigma(x) = X$, and $\sigma(y) = Y$, the product of avoiders is:

$$x \bullet_{\mathrm{Av}} y = [x \cdot y', \pi_\downarrow(y' \cdot x)]$$

and the product of partial evaluations is:

$$X \bullet_{\mathrm{PE}} Y = \sigma(x) \bullet_{\mathrm{PE}} \sigma(y) = [\sigma(x \cdot y'), \sigma(y' \cdot x)]$$

Definition 3.2.1. We define a map δ from partial evaluations whose first block is an unevaluated square to full evaluations. To compute $\delta[X]$, mark all of the unevaluated squares and unevaluate all of the evaluated squares. Then begin with the rightmost marked square and evaluate it with the block immediately following it. Continue evaluating this block with the block immediately following it until you reach the far right of the partial evaluation. Repeat this process with the rightmost marked square until you have a fully evaluated block.

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Example 3.2.2. $\delta\big[\square(\square(\square(\square\square)))\square\square(\square)(\square)\big] = ((\square(\square(\square(\square\square))))(\square((\square\square)\square)))$

$$\square(\square(\square\square))\square\square(\square)(\square)$$

Mark unevaluated squares:	$\boxtimes(\square(\square\square))\boxtimes\boxtimes(\square)(\square)$
Unevaluate evaluated squares:	$\boxtimes(\square(\square\square))\boxtimes\boxtimes\square\square$
Start with rightmost marked square and evaluate to right:	$\boxtimes(\square(\square\square))\boxtimes\Big(\big(\boxtimes\square\big)\square\Big)$
Using the next rightmost marked square, evaluate to right:	$\boxtimes(\square(\square\square))\Big(\boxtimes((\square\square)\square)\Big)$
Using the next rightmost marked square, evaluate to right:	$\Big(\big(\boxtimes(\square(\square\square))\big)(\square((\square\square)\square))\Big)$

$$((\square(\square(\square\square)))(\square((\square\square)\square)))$$

Definition 3.2.3. Let $X \in \Sigma_n^k$. We define partial evaluations X_1 and X_2 such that X is the concatenation of the partial evaluations X_1 and X_2. The partial evaluation X_2 has an unevaluated square in the first position and $|X_2|$ is as large as possible such that $|X_2| \le k - 3$.

Theorem 3.2.4.

$$X \bullet_{\mathrm{PE}} Y = [X\square Y, X_1\delta[X_2(\square)]Y]$$

Example 3.2.5. $X = (\square)\square(\square(\square\square))\square\square(\square)$, $Y = ((\square(\square\square)))$, and $k = 11$. Since $k - 3 = 8$, $X_1 = (\square)$ and $X_2 = \square(\square(\square\square))\square\square(\square)$. Recall $\delta[X_2(\square)] = \delta[\square(\square(\square\square))\square\square(\square)(\square)] = ((\square(\square(\square\square)))(\square((\square\square)\square)))$.

$X \bullet Y = [X\square Y, X_1\delta[X_2(\square)]Y]$

$\quad = [(\square)\square(\square(\square\square))\square\square(\square)\square(\square(\square\square)), (\square)((\square(\square(\square\square)))(\square((\square\square)\square)))(\square(\square\square))]$

Proof of Theorem 3.2.4. Let $x \in \mathrm{Av}_p^k$ such that $\sigma(x) = X$ and let $y \in \mathrm{Av}_q^k$ such that $\sigma(y) = Y$. Let $y' = (y)_{[p+1,n]}$, where $n = p + q$. We want to show that $\sigma(x \cdot y') = X\square Y$ and $\sigma(y' \cdot x) = X_1\delta[X_2(\square)]Y$.

Consider $\sigma(x \cdot y')$. None of the entries of y' are left of the entries of x, and all of the entries of y' are larger than the entries of x. Thus $\sigma(x \cdot y')$ outputs X as it evaluates the first p entries of $x \cdot y'$. Then it outputs $\square$ because there is nothing larger than p to the left of it, and $p + 1$ is an entry of y' to the right. Lastly $\sigma(x \cdot y')$ outputs Y as it evaluates the last q entries of $x \cdot y'$. Therefore, $\sigma(x \cdot y') = X \square Y$.

Consider $\sigma(y' \cdot x)$. We know $p+1$ is the smallest entry of y' that is larger and to the left of any of the entries of x. Recall that the permutation x is of the form described in Proposition 3.1.4. Let $x_h = x_1^i$ for the smallest i such that $p + 1 - x_1^i \leq k - 2$. Consider $\sigma(x|_{[1,x_h]})$ and $\sigma(x|_{[x_h,p]})$. Because $x_h = x_1^i$, the first block of $\sigma(x|_{[x_h,p]})$ is an unevaluated square. The partial evaluation $\sigma(x|_{[x_h,p]})$ is of size $p - x_h$. Since $x_h = x_1^i$ for the smallest i such that $p + 1 - x_h \leq k - 2$, we know $|\sigma(x|_{[x_h,p]})|$ is as large as possible such that $|\sigma(x|_{[x_h,p]})| \leq k - 3$. Thus, $X_1 = \sigma(x|_{[1,x_h]})$ and $X_2 = \sigma(x|_{[x_h,p]})$. Therefore $\sigma(y' \cdot x) = \sigma(x|_{[1,x_h]}) \sigma(y' \cdot x|_{[x_h,p+1]}) \sigma(y'|_{[p+1,p+q]}) = X_1 \sigma(y' \cdot x|_{[x_h,p+1]}) Y$, and it is left to show that $\sigma(y' \cdot x|_{[x_h,p+1]}) = \delta[X_2(\square)]$.

The size of the block $\sigma(y' \cdot x|_{[x_h,p+1]})$ is $p + 1 - x_h$ and $\sigma(y' \cdot x|_{[x_h,p+1]}) = \sigma\big((p+1) \cdot x_h \cdot x|_{[x_h+1,p]}\big) = \tau(x|_{[x_h+1,p]})$. Let $\hat{x} = \text{st}(x|_{[x_h+1,p]})$. The permutation $\hat{x} = \hat{x}_1^1 \cdots \hat{x}_{j_1}^1 \hat{y}_1^1 \cdots \hat{y}_{\ell_1}^1 \cdots \hat{x}_1^h \cdots \hat{x}_{j_h}^h \hat{y}_1^h \cdots \hat{y}_{\ell_h}^h$ avoids the pattern $2(31)$ so it is of the form described in Proposition 3.1.4. Let us consider $\tau(\hat{x})$. Recall the the map τ begins with $p - x_{h+1} + 2$ squares and then evaluates positions of a block as dictated by reading a permutation from right to left. Lemma 3.1.13 shows that we get the same result if we think of τ as a two-step process: first acting on entries of the form $\hat{y}_j^i$, and then acting on entries of the form $\hat{x}_j^i$. As we read $\hat{x}$ from right to left, entries of the form $\hat{y}_j^i$ give us all of the fully evaluated blocks of X_2 that are of size ≥ 2. Now let us consider entries of the form $\hat{x}_j^i$. For a particular i, the entries of the form $\hat{x}_j^i$ are in descending order, and if $i_1 < i_2$ then, $\hat{x}_j^{i_1} < \hat{x}_\ell^{i_2}$. Mark each unevaluated square in the output that is in the $\hat{x}_{j_i}^i$ position. As we read $\hat{x}$ from right to left, entries of the form $\hat{x}_j^i$ evaluate the first position after the rightmost marked square and evaluate each of the unevaluated positions to the end, then they evaluate the first position after the new rightmost marked square and evaluate each of the unevaluated positions to the end, and so on. This process is precisely $\delta[X_2(\square)]$. Therefore $\sigma(y' \cdot x) = X_1 \delta[X_2(\square)] Y$. $\qquad\square$

no dots:	$(((\square\square)\square)(\square\square))$
one dot on the last square:	$(((\square\square)\square)(\square\boxdot))$

non-nested parentheses:

$$\Big(((\boxdot\square)\square)(\square\boxdot)\Big) \quad \Big(\big((\boxdot\square)\boxdot\big)(\square\square)\Big)$$

$$\Big(\big((\square\square)\boxdot\big)\big(\boxdot\square\big)\Big) \quad \Big(\big(\big(\boxdot\square\big)\square\big)(\square\square)\Big)$$

$$\Big(\big(\big(\boxdot\boxdot\big)\square\big)\big(\boxdot\square\big)\Big) \quad \Big(((\square\square)\square)\big(\boxdot\square\big)\Big)$$

Figure 3.3: Allowable dottings of $\sigma(614325)$

We now give a partial description of the coproduct of partial evaluations. First, we need to develop a combinatorial understanding of what a good set as defined in Section 1.2 looks like in terms of avoiders and partial evaluations.

Definition 3.2.6. For a partial or full evaluation X, a *dotting of X* is X with some set of positions dotted. An *allowable dotting of a full evaluation X* is a dotting that meets one of the following conditions:

1. no dots

2. one dot on the last square

3. for a set of non-nested pairs of parentheses, dots on the first and last square within each pair of parentheses

Definition 3.2.7. An *allowable dotting of a partial evaluation X* is a dotting that meets *all* of the following conditions:

1. dots alternate between black and white positions of X (there is at least one dot and the first dot can be on either a black or a white position)

2. each fully evaluated block meets one of the conditions for an allowable dotting of a full evaluation

3. if the first dot to the right of an unevaluated square that has not been dotted is on a white position, then that dotted white position is $k-2$ or more positions to the right of the unevaluated square

Example 3.2.8. Some allowable dottings of $\sigma(142385679)$:

$$\boxdot(\square\boxdot)\square\big(\boxdot(\square\boxdot)\big)\boxdot \quad \text{is allowable for } k \geq 5$$

$$\boxdot(\square\square)\square\big(\square(\square\boxdot)\big)\boxdot \quad \text{is allowable for } k = 5, \text{ but not for } k > 5$$

Definition 3.2.9. Consider an allowable dotting of a partial evaluation $X \in \Sigma_{n-1}^k$. Place the numbers 1 through n before, after, and in between each of the $n-1$ squares of X. Let T be the set of numbers such that either the nearest dotted position to their right is a black position, or the nearest dotted position to their left is a white position. We define such a T to be an *allowable set* for X.

Example 3.2.10. $d = (((\boxdot\square)\boxdot)(\boxdot\boxdot)) \Rightarrow (((1\boxdot2\square)3\boxdot)4(\boxdot5\boxdot6)) \Rightarrow T = \{1,4,6\}$

Each allowable dotting of X has a unique allowable set. Given a partial evaluation X and an allowable set T for X, we can compute the associated allowable dotting by placing a dot in each position i of X such that either $i \in T$ and $i+1 \notin T$ or $i \notin T$ and $i+1 \in T$. Thus, each allowable set of X has a unique allowable dotting of X. For an allowable set of X, we denote the corresponding allowable dotting of X by $[X]_T$.

Example 3.2.11. $[(((\square\square)\square)(\square\square))]_{\{1,4,6\}} = (((\boxdot\square)\boxdot)(\boxdot\boxdot))$

We now define a map $\hat{\mu}$ from pairs (X,T) where X is a partial evaluation and T is an allowable set with respect to X, to permutations. The output is not necessarily an avoider.

Definition 3.2.12. Let T be an allowable set for a partial evaluation $X \in \Sigma_{n-1}^k$. Let T^C be all of the integers in $[n]$ that are not contained in T. We define a map $\hat{\mu}$ by the following process.

First draw a vertical line. We will build a permutation by placing all of the elements of T on the left of the vertical line, and all of the elements on T^C on the right of the vertical line, and then removing the line. If $1 \in T$ then place a 1 on the left of the line; otherwise, place the 1 on

the right. The numbers 2 through n are each placed as far right as possible while still making it possible for $\sigma(\hat{\mu}(X,T)) = X$. We will consider each block of X from left to right and place the numbers 2 through n in the following manner.

Suppose X has an unevaluated square in the ith position. If $i + 1 \in T$, then place $i + 1$ immediately left of the vertical line. If $i + 1 \in T^C$, then place $i + 1$ on the far right of the permutation.

Suppose X has a fully evaluated block X_i of size j beginning in the ith position. We will examine the possible assignments of the entries $[i, i + j]$ to the sets T and T^C, based on the possible allowable dottings associated with T.

Case 1: $[X]_T$ has no dots on the block X_i.
If $[i, i+j] \subseteq T$, then place $i+j$ immediately to the left of i and place $(\rho(X_i))_{[i+1,i+j-1]}$ immediately left of the vertical line. If $[i, i + j] \subseteq T^C$, then place $i + j$ immediately to the left of i and place $(\rho(X_i))_{[i+1,i+j-1]}$ to the far right of the permutation.

Case 2: $[X]_T$ only has a dot in the last position of X_i, which is a white position.
$i + j \subseteq T$ and $[i, i + j - 1]\ T^C$. Place $i + j$ immediately left of the vertical line, and place $(\rho(X_i))_{[i+1,i+j-1]}$ to the far right of the permutation.

Case 3: $[X]_T$ has dots on the first and last square within pairs of non-nested parentheses. Since the first dot of $[X]_T$ is in a black position, $i \in T$. Since the last dot of $[X]_T$ is in a white position, $i + j \in T$. Place $i + j$ immediately to the left of i, place $\rho(X)|_{[i+1,i+j-1]\cap T}$ immediately left of the vertical line and place $\rho(X)|_{[i+1,i+j-1]\cap T^C}$ to the far right of the permutation.

The output of $\hat{\mu}(X,T)$ is the permutation that results from ignoring the vertical line.

Example 3.2.13. For $X = \square(\square\square)(\square)(\square(\square\square))$ and $T = \{1, 5, 6, 8\}$, the computation $\hat{\mu}(X,T) = 18564237$.

$$
\begin{array}{lccc}
1 \in T & \rightarrow & 1 & \vdots \\
\square & \rightarrow & 1 & \vdots \quad 2 \\
(\square\square) & \rightarrow & 1 & \vdots \quad 423 \\
(\square) & \rightarrow & 15 & \vdots \quad 423 \\
(\square(\square\square)) & \rightarrow & 1856 & \vdots \quad 4237
\end{array}
$$

Proposition 3.2.14. *For a partial evaluation X and for T an allowable set of X,*

$$\sigma(\hat{\mu}(X,T)) = X$$

Proof. Suppose that we are in the process of computing $\hat{\mu}(X,T)$, and we are now examining a block of X that we call X_i. This block begins in the ith position of X. Assume that up to this point, $\hat{\mu}(X,T)$ is computed in such a way that $\sigma(\hat{\mu}(X,T)) = X$.

Case 1: X_i is an unevaluated square.

The map $\hat{\mu}(X,T)$ maps $i+1$ to the right of i. If $i \in T$, then $\hat{\mu}(X,T)$ does not map any entry larger than i to the left of i. If $i \in T^C$, then $\hat{\mu}(X,T)$ maps the smallest entry of T that is larger than i to the left of i. For $i \in T^C$, the smallest entry of T that is larger than i is one plus the position of the first dot on $[X]_T$ after the ith position. The position of the first dot on $[X]_T$ after the ith position must be larger than $i+k-2$, because $[X]_T$ is an allowable dotting. So the smallest number that is larger than i and to the left of i in $\hat{\mu}(X,T)$ is larger than $i+k-1$. Note $i+k-1-i > k-2$. Thus regardless of whether $i \in T$ or $i \in T^C$, $\sigma(\hat{\mu}(X,T))$ has an unevaluated square in the ith position.

Case 2: X_i is a fully evaluated block of size j.

The map $\hat{\mu}(X,T)$ maps $i+j$ to the left of i, such that $i+j$ is the smallest thing larger than and left of i. Thus, $\sigma(\hat{\mu}(X,T))$ has a block of size j in the ith position. It is just left to show that the block outputted is X_i. We have $\hat{\mu}(X,T)|_{[i,i+j]} = (i+j) \cdot i \cdot \rho(X)|_{[i+1,i+j-1] \cap T} \cdot \rho(X)|_{[i+1,i+j-1] \cap T^C}$. So $\sigma(\hat{\mu}(X,T)|_{[i,i+j]}) = \tau(\rho(X)|_{[i+1,i+j-1] \cap T} \cdot \rho(X)|_{[i+1,i+j-1] \cap T^C})$.

Suppose for sake of contradiction that $X_i \neq \tau(\rho(X)|_{[i+1,i+j-1] \cap T} \cdot \rho(X)|_{[i+1,i+j-1] \cap T^C})$, that is, there is some entry $h \in [i+1,i+j-1] \cap T$ and an entry $\ell \in [i+1,i+j-1] \cap T^C$ such that the position $h-i$ of $\tau(\text{st}(\rho(X)|_{[i+1,i+j-1] \cap T} \cdot \rho(X)|_{[i+1,i+j-1] \cap T^C}))$ needs to be evaluated before the position $\ell-i$ to obtain X_i. Because $h \in T$, in the dotting $[X]_T$ the nearest dotted position to the right of h is a black position or the nearest dotted position to the left of h is a white position. Similarly, because $\ell \in T^C$, in the dotting $[X]_T$ the nearest dotted position to the left

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of ℓ is a black position or the nearest dotted position to the right of ℓ is a white position.

Subcase 2a: For some fully evaluated blocks or unevaluated squares U, V, and W, there is a sub-block of X_i that can be represented as $((UV)W)$ where V begins in the hth position of X and W begins in the ℓth position of X.

In order to meet the conditions above $[X]_T$ has a dot in a black position within the sub-block of X_i represented by V. However, every dot in a black position within the sub-block of X_i represented by V is followed by a dot in a white position within the sub-block of X_i represented by V. This is a contradiction because ℓ is right of a dot in a black position.

Subcase 2b: For some fully evaluated blocks or unevaluated squares U, V, and W, there is a sub-block of X_i that can be represented as $(U(VW))$ where V begins in the ℓth position of X and W begins in the hth position of X.

In order to meet the conditions above $[X]_T$ has a dot in a white position within the sub-block of X_i represented by V. However, every dot in a white position within the sub-block of X_i represented by V is preceded by a dot in a black position within the sub-block of X_i represented by V. This is a contradiction because ℓ is left of a dot in a white position.

Thus we have shown $\sigma(\hat{\mu}(X,T))$ has X_i in the ith position. $\qquad \square$

Proposition 3.2.15. *Let $X \in \Sigma_{n-1}^k$ and $x \in \mathrm{Av}_n^k$ such that $\sigma(x) = X$. T is an allowable set for X, $T = \varnothing$, or $T = [n]$ if and only if T is good with respect to x.*

Proof. Recall from Section 1.2 that a set T is good if the entries of T are the first elements of a permutation $x' \in S_n$ where $\sigma(x') = X$. Suppose that T is an allowable set for $X \in \Sigma_{n-1}^k$, and let $\hat{\mu}(X,T) = x' \in S_n$. Observe that the entries of T are the first elements of x' and that $\sigma(x') = X$. If $T = \varnothing$ or if $T = [n]$, then the entries of T are the first elements of x.

Now, suppose $x' \in S_n$ such that $\sigma(x') = X$, and let T_h be the set containing the first h entries of x'. If $h = 0$ or $h = n$, then $T_h = \varnothing$ or $T = [n]$ respectively. For $1 \le h \le n - 1$, we want to show that T_h is an allowable set for X, that is, we want to show that $[X]_{T_h}$ is an allowable dotting for X.

If $i \in T_h$ and $i + 1 \in T_h^C$ then $[X]_{T_h}$ has a dot in the ith position. We know that entries in T_h^C come after entries of T_h in x', so $i + 1$ is to the right of i in x'. Thus the ith dot is on a black position of X. Similarly, if $i \in T_h^C$ and $i + 1 \in T_h$, then the ith dot of $[X]_{T_h}$ is on a white position of X. Thus the dots of $[X]_{T_h}$ alternate between black positions and white positions of X.

Suppose that there is a fully evaluated block X_i of size j beginning in the ith position of X. If $[i, i + j] \subseteq T_h$ or if $[i, i + j] \subseteq T_h^C$, then there are no dots on X_i in the dotting $[X]_{T_h}$. If $j + 1 \in T_h$ and $[i, i + j - 1] \subseteq T_h^C$, then the dotting $[X]_{T_h}$ has one dot in the last position of X_i. If $i, i + j \in T_h$ and the entries $[i + 1, i + j - 1]$ are either all in T_h or all in T_h^C, then let us consider $\sigma(x'|_{[i,i+j]}) = \tau(x'|_{[i+1,i+j-1]})$. As the map τ reads the permutation $x'|_{[i+1,i+j-1]}$ from right to left, it will first evaluate the positions corresponding to entries in $[i + 1, i + j - 1] \cap T_h^C$. At this step in the process, the resulting block will look like fully evaluated sub-blocks separated by any number of unevaluated squares. The dotting $[X]_{T_h}$ has dots in the first and last position of each of these fully evaluated sub-blocks. Thus, for a set of non-nested pairs of parentheses, the dotting $[X]_{T_h}$ has a dot in the first and last square in each pair of parentheses. Therefore we have shown that each block of $[X]_{T_h}$ meets one of the conditions for an allowable dotting of a full evaluation.

Suppose that $[X]_{T_h}$ has an unevaluated square in the ith position that has not been dotted, and that the first dot to the right of that square is in a white position j positions to the right of the unevaluated square. Thus $[i, i + j] \subseteq T_h^C$ and $i + j + 1 \in T_h$. That is, $i + j + 1$ is the smallest entry larger than and to the left of i in x'. If $i + j + 1 - i \leq k - 2$, then X would have a block of size $j + 1$ in the ith position instead of an unevaluated square. Thus $i + j + 1 - i > k - 2$, that is, $j > k - 3$, or equivalently $j \geq k - 2$, because j and k are integers. $\square$

Thus we conclude the general discussion of the coproduct of partial evaluations. Now we describe the coproduct of partial evaluations for $k = 3$, and then we will give a partial description of the coproduct of partial evaluations that is isomorphic to the coproduct for avoiders of the set $U = \{2(31)\}$. A dual version of this coproduct is found in [2, Chapter 17.3]. Aguiar and Mahajan's map was instrumental in the understanding of the dual product and coproduct of

sashes, as well as the coproduct of the tilings that we mention here.

The Hopf algebra of partial evaluations for $k = 3$ has already been introduced in Chapter 1.1. Consider the tilings of black squares and white squares to be tilings of unevaluated squares and evaluated squares respectively. Of the cover relations defined in Proposition 3.1.24, only (2) applies to the $k = 3$ case. Thus, the cover relation for tilings is given by $D \square E \lessdot D(\square)E$, where D and E are tilings of evaluated and unevaluated squares. From Definition 3.2.7 we see that allowable dottings of these objects are dottings that alternate between evaluated and unevaluated squares. That is, conditions (2) and (3) of Definition 3.2.7 do not create any additional restrictions for allowable dottings.

Given an allowable dotting d for a tiling $C \in \Upsilon_n$, we define two objects A and B that are similar to tilings, but they contain an additional object called a mystery square $\boxed{?}$. Let $d = c_1 \bullet_1 c_2 \bullet_2 \cdots c_j \bullet_j c_{j+1}$, where each c_i is a sub tiling of C without any dots.

If $\bullet_1$ is on an unevaluated square, then let A be the concatenation of the *odd* c_i with a mystery square in between each c_i (where i is odd), and let B be the concatenation of the *even* c_i with a mystery square in between each c_i (where i is even). If $\bullet_1$ is on an evaluated square, then let A be the concatenation of the *even* c_i with a mystery square in between each c_i, and let B be the concatenation of the *odd* c_i with a mystery square in between each c_i.

We use the objects A and B to define four tilings $\underline{A}$, $\overline{A}$, $\underline{B}$, and $\overline{B}$. The tilings $\underline{A}$ and $\underline{B}$ are obtained by replacing all of the mystery squares on A or B respectively with unevaluated squares. The tilings $\overline{A}$ and $\overline{B}$ are obtained by replacing all of the mystery squares on A or B respectively with evaluated squares.

Let $I_d = \sum \left[\underline{A}, \overline{A} \right]$ and $J_d = \sum \left[\underline{B}, \overline{B} \right]$. Thus $I_d \otimes J_d$ denotes $\sum_{\substack{D \in [\underline{A}, \overline{A}] \\ E \in [\underline{B}, \overline{B}]}} D \otimes E$. The coproduct of tilings is given by:

$$\Delta_\Upsilon(C) = \varnothing \otimes C + C \otimes \varnothing + \sum_{\substack{\text{allowable} \\ \text{dottings} \\ d \text{ of } C}} I_d \otimes J_d \tag{3.1}$$

Now we will give a partial description of the coproduct on partial evaluations with no

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restriction on the size of the fully evaluated blocks. Let Av_n^∞ denote the set $\mathrm{Av}_n[\{2(31)\}]$. Also, Σ_{n-1}^∞ is the set of partial evaluations of length $n-1$ with no restriction on the size of the fully evaluated blocks.

Lemma 3.2.16. *Let* $z \in \mathrm{Av}_n^\infty$, *and let* T *be good with respect to* z. *Recall* z_{min} *and* z_{max} *from Section 1.2.*

$$z_{min}|_T = z|_T$$

$$z_{min}|_{T^C} = z|_{T^C}$$

$$z_{max}|_T = \pi^\uparrow(z)|_T$$

$$z_{max}|_{T^C} = \pi^\uparrow(z)|_{T^C}$$

Proof. Given $z \in \mathrm{Av}_n^\infty$ and a good set T, consider the $\pi^\uparrow$-moves required to transition from z to z_{min}. Beginning with the permutation z, each entry of T that is right of an entry of T^C in z moves to the left until all of the entries of T are to the left of the entries of T^C. This permutation is z_{min}. Because none of the entries of T changed their relative order, we can conclude the first two equations of the Lemma.

We make a similar argument for the last two equations by considering the $\pi_\downarrow$-moves required to transition from $\pi^\uparrow(z)$ to z_{max}. $\qquad\square$

Lemma 3.2.17. *Let* $z \in \mathrm{Av}_n^\infty$, *and let* T *be good with respect to* z *such that* $|T| = p$.

$$\sigma(z|_T) = \sigma(\pi^\uparrow(z)|_T)$$

Proof. Since $z \in \mathrm{Av}_n^\infty$, the permutation $\mathrm{st}(z|_T) \in \mathrm{Av}_p^\infty$. So, $\mathrm{st}(z|_T)$ is of the form $\mathrm{st}(z|_T) = x_1^1 \cdots x_{j_1}^1 y_1^1 \cdots y_{\ell_1}^1 \cdots x_1^h \cdots x_{j_h}^\ell y_1^h \cdots y_{\ell_h}^h$. Assume that computations of $\sigma(z|_T)$ and $\sigma(\pi^\uparrow(z)|_T)$ produce identical output when evaluating the values from 1 to x, and we will consider their output for the following block.

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Case 1: $x = x_1^i$.

The output of $\sigma(z|_T)$ is an unevaluated square, because there is nothing larger than x_1^i to the left of x_1^i in $\text{st}(z|_T)$. Thus, there is nothing larger than x_1^i to the left of x_1^i in $\text{st}(\pi^\uparrow(z)|_T)$, so the output of $\sigma(\pi^\uparrow(z)|_T)$ is an unevaluated square.

Case 2: $x = x_j^i$ for $j \geq 2$.

The output of $\sigma(z|_T)$ is a fully evaluated block of size $x_{j-1}^i - x_j^i$, because x_{j-1}^i is the smallest thing larger and to the left of x_j^i in $\text{st}(z|_T)$. Particularly, the output is the block $\tau((z|_T)_{[x_j^i+1, x_{j-1}^i-1]})$. Since all of the values in $[x_j^i+1, x_{j-1}^i-1]$ are to the right of x_j^i in $\text{st}(z|_T)$, the smallest thing larger and to the left of x_j^i in $\text{st}(\pi^\uparrow(z)|_T)$ is also x_{j-1}^i. Thus, the output of $\sigma(\pi^\uparrow(z)|_T)$ is a block of size $x_{j-1}^i - x_j^i$, and is specifically $\tau((\pi^\uparrow(z)|_T)_{[x_j^i+1, x_{j-1}^i-1]})$. By Lemma 3.1.12, $\tau((z|_T)_{[x_j^i+1, x_{j-1}^i-1]}) = \tau((\pi^\uparrow(z)|_T)_{[x_j^i+1, x_{j-1}^i-1]})$. $\qquad\square$

From the previous two Lemmas and from Section 1.2, we now have:

$$\Delta_{\text{PE}}(Z) = \varnothing \otimes Z + Z \otimes \varnothing + \sum_{\substack{\text{allowable} \\ \text{dottings } d}} \sigma(I_T) \otimes \sigma(J_T)$$

$$= \varnothing \otimes Z + Z \otimes \varnothing + \sum_{\substack{\text{allowable} \\ \text{dottings } d}} [\sigma(z_{min}|_T), \sigma(z_{max}|_T)] \otimes [\sigma(z_{min}|_{T^C}), \sigma(z_{max}|_{T^C})]$$

$$= \varnothing \otimes Z + Z \otimes \varnothing + \sum_{\substack{\text{allowable} \\ \text{dottings } d}} [\sigma(z|_T), \sigma(\pi^\uparrow(z)|_T)] \otimes [\sigma(z|_{T^C}), \sigma(\pi^\uparrow(z)|_{T^C})]$$

$$= \varnothing \otimes Z + Z \otimes \varnothing + \sum_{\substack{\text{allowable} \\ \text{dottings } d}} \sigma(z|_T) \otimes [\sigma(z|_{T^C}), \sigma(\pi^\uparrow(z)|_{T^C})]$$

Consider an allowable dotting d of a partial evaluation $Z \in \Sigma_{n-1}^\infty$, where $Z = \sigma(z)$ for some $z \in \text{Av}_n^\infty$. We now define a sash A that is obtained from d. In Proposition 3.2.19 we will show that $A = \sigma(z|_T)$ for the set T associated with d. If the first dot of d is in a white position, then delete everything in d before the first dot and delete the position with the first dot. If the last dot of d is in a black position, then delete everything in d after the last dot and delete the position with the last dot. Note that if the first dot of d is in a white position, then it is either

on an evaluated square or in the last position of a fully evaluated block, and if the last dot of d is in a black position, then it is on an unevaluated square. Thus we are left with a dotting $\hat{d}$ such that if it has any dots, they begin on a black position and end on a white position.

If $\hat{d}$ does not have any dots, then $A = \hat{d}$. If $\hat{d}$ does have dots, then examine the dots from left to right and execute the following procedure: If a dot in a black position is on an unevaluated square, then delete all of the positions of the partial evaluation in between and including the black position and the following dotted white position. Replace the deleted portion of the partial evaluation with an unevaluated square.

If a dot in a black position is not on an unevaluated square, then consider the innermost pair of parentheses around both the dotted black position and the following dotted white position. If these dots are in the first and last positions of a fully evaluated block, then replace the entire block with an evaluated square. Otherwise, replace these parentheses and everything inside of them with a square.

Example 3.2.18.

$$d = (\square)(\blacksquare)\blacksquare(\square\blacksquare)\big(\blacksquare(\square\blacksquare)\big)\square\Big(\big(\square(\blacksquare(\square\blacksquare))\big)\square\Big)(\square)\blacksquare(\square\square)$$

$$\hat{d} = \blacksquare(\square\square\blacksquare)\Big(\blacksquare(\square\square\blacksquare)\Big)\square\Big(\big(\square\big(\blacksquare(\square\square\blacksquare)\big)\big)\square\Big)(\square)$$

$$A = \square(\square)\square\Big(\big((\square\square)\square\big)\Big)(\square)$$

Proposition 3.2.19. *Let $z \in \mathrm{Av}_n^\infty$ and let T be good with respect to z such that $\sigma(z) = Z \in \Sigma_{n-1}^\infty$ and such that $[Z]_T = d$. If A is as defined above, then $A = \sigma(z|_T)$.*

Proof. Let $T = \{t_1, t_2, \ldots, t_p\}$ for $t_1 < t_2 < \cdots < t_p$. If the first dot of d is in the hth position of Z, which is a white position, then the values $[1, h]$ are in T^C and have no barring on $z|_T$. Similarly, if the last dot of d is in the ℓth position of Z, which is a black position, then the values $[\ell + 1, n]$ are in T^C and have no barring on $z|_T$. Thus $(\rho(\hat{d}))_{[h+1,\ell]}|_T = z|_T$, where $h = 0$ if the first dot of d is in a black position and $\ell = n$ if the last dot of d is in a white position.

If $\hat{d}$ has a dot on an unevaluated square in the ith position of Z, followed by a dot in the jth position of Z, which is a white position, then $i, j + 1 \in T$ such that $i = t_r$ and $j + 1 = t_{r+1}$ for some r. In z, the entry $j + 1$ is to the right of i, and nothing larger than i is to the left of i. Thus in $z|_T$, the entry t_{r+1} is to the right of t_r, and nothing larger than t_r is to the left of t_r. The sash A has an unevaluated square in the rth position, so $(\rho(A))_T$ has the entry t_{r+1} to the right of t_r, and nothing larger than t_r is to the left of t_r.

If $\hat{d}$ has a dot on the first position of a block in the ith position of Z, followed by a dot on the last position of that block in the jth position of Z, which is a white position, then $i, j+1 \in T$ such that $i = t_r$ and $j + 1 = t_{r+1}$ for some r. In z, the entry $j + 1$ is to the left of i. Thus in $z|_T$, the entry t_{r+1} is the smallest thing larger and to the left of t_r. The sash A has an evaluated square in the rth position, so in $(\rho(A))_T$ the entry t_{r+1} is the smallest thing larger and to the left of t_r.

Suppose $\hat{d}$ has dots in the first and last position of some set of non-nested parentheses within a single block of Z. Let the block be size j and begin in the ith position of Z. Let all of the dots on black positions within the block be in the i_1^{st}, i_2^{nd}, $\ldots$, and i_s^{th} positions of Z. Similarly, let all of the dots on white positions within the block be in the j_1^{st}, j_2^{nd}, $\ldots$, and j_s^{th} positions of Z. Thus the set T contains the entries: $[i, i_1] \cup [j_1 + 1, i_2] \cup [j_2 + 1, i_3] \cup \cdots \cup [j_s + 1, i + j]$. Let $i = t_{r_1}$ and $i + j = t_{r_2}$. Notice that in both $z|_T$ and $(\rho(A))_T$, the entry $i + j$ is the smallest thing larger and to the left of i. Thus, $\sigma(z|_{[t_{r_1}, t_{r_2}] \cap T}) = \tau(z|_{[t_{r_1+1}, t_{r_2-1}] \cap T})$, which is precisely $\sigma((\rho(A))|_{[t_{r_1}, t_{r_2}] \cap T})$. That is $\sigma(z|_{[t_{r_1}, t_{r_2}] \cap T})$ is the same as the block obtained by replacing each set of non-nested dotted parentheses in the dotted block of $\hat{d}$ and everything inside of them with a square.

Since $\sigma((\rho(A))_T) = A$ we have shown that $A = \sigma(z|_T)$. $\qquad\qquad\square$

Therefore, for a partial evaluation Z:

$$\Delta_{\mathrm{PE}}(Z) = \varnothing \otimes Z + Z \otimes \varnothing + \sum_{\substack{\text{allowable} \\ \text{dottings } d}} A \otimes [\sigma(z|_{T^C}), \sigma(\pi^{\uparrow}(z)|_{T^C})]$$

REFERENCES

[1] Marcelo Aguiar, Nantel Bergeron, and Frank Sottile. Combinatorial Hopf algebras and generalized Dehn-Sommerville relations. *Compos. Math.*, 142(1):1–30, 2006.

[2] Marcelo Aguiar and Swapneel Mahajan. *Monoidal functors, species and Hopf algebras*, volume 29 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 2010. With forewords by Kenneth Brown and Stephen Chase and André Joyal.

[3] Marcelo Aguiar and Frank Sottile. Structure of the Loday-Ronco Hopf algebra of trees. *J. Algebra*, 295(2):473–511, 2006.

[4] Eric Babson and Einar Steingrímsson. Generalized permutation patterns and a classification of the Mahonian statistics. *Sém. Lothar. Combin.*, 44:Art. B44b, 18 pp. (electronic), 2000.

[5] L. Foissy. Les algèbres de Hopf des arbres enracinés décorés. I. *Bull. Sci. Math.*, 126(3):193–239, 2002.

[6] Israel M. Gelfand, Daniel Krob, Alain Lascoux, Bernard Leclerc, Vladimir S. Retakh, and Jean-Yves Thibon. Noncommutative symmetric functions. *Adv. Math.*, 112(2):218–348, 1995.

[7] Ralf Holtkamp. Comparison of Hopf algebras on trees. *Arch. Math. (Basel)*, 80(4):368–383, 2003.

[8] Shirley Law and Nathan Reading. The Hopf algebra of diagonal rectangulations. *J. Combin. Theory Ser. A*, 119(3):788–824, 2012.

[9] Jean-Louis Loday and María O. Ronco. Hopf algebra of the planar binary trees. *Adv. Math.*, 139(2):293–309, 1998.

[10] Claudia Malvenuto. Produits et coproduits des fonctions quasi-symétriques et de l'algèbre des descentes. *(LACIM)*, 16, 1994.

[11] P. Palacios. Una generalización a operads de la construcción de Hopf de Connes-Kreimer. *Tesis de licenciatura*, 2002.

[12] Nathan Reading. Lattice congruences, fans and Hopf algebras. *J. Combin. Theory Ser. A*, 110(2):237–273, 2005.

[13] N. J. A. Sloane. The on-line encyclopedia of integer sequences. `http://oeis.org`. 2013.

[14] Moss E. Sweedler. *Hopf Algebras.* W. A. Benjamin, Inc., New York, NY, 1969.