Energy Gain by Conduction for Concentric Circular Annuli with Uniform Wall Temperature Gradient and Heat Generation

O.A. Arnas

Mechanical Engineering Dept., Louisiana State University, Baton Rouge, Louisiana 70803, U.S.A.

M.A. Ebadian

Mechanical Engineering Dept., Southern University, Baton Rouge, Louisiana 70813, U.S.A.

ABSTRACT

In this paper, the total energy gain by conduction for a concentric circular annulus with uniform wall temperature gradient and heat generation is calculated analytically for various boundary conditions.

The equations for viscous laminar flow in a circular annulus is first developed for an incompressible, constant property fluid where the velocity profile is fully developed. The results are used in the energy equation to determine the temperature field for the case with uniform heat generation. In turn, the results for the heat fluxes from the outer and the inner tubes are determined for all physically interesting and important cases and are presented in analytical closed form.

1. Introduction

In general, energy gain through the walls of heat exchanger tubes is an important technological problem. Shah and London [1] have discussed a great majority of these cases through 1977 and have documented all literature in a compact form including the work of Topakoglu and Arnas [2] which this paper follows closely and extends to include the effect of heat generation in the fluid within the annulus. Recently, Berger, Talbot and Yao [3] have studied in great detail the flow in curved pipes. The methodologies presented there have also been used in this paper to study the flow in annuli.

There are a number of areas where flows of this type are of practical importance. Annuli arise in most piping systems as well as in all double-pipe heat exchangers. It is, therefore, important to know the pressure drop in the developing and the developed parts of the flow and to determine the pumping power required to overcome annulus - induced pressure losses; similarly, the energy gain is required in designing heat exchangers. The effects of the wall temperature gradient and heat generation on the total energy gain by conduction through the walls of a heat exchanger annulus under a number of boundary conditions are important. The only other literature where heat generation in pipes and annuli are considered are [4] and [5]. Experience gained in these papers have been incorporated in the work presented here.

One of the reasons why problems of this nature are not solved analytically is the difficulty faced in formulating the mathematical tools, even for circular annuli. It is this that has caused the absence of many analytical solutions in the literature. Naturally, the momentum solution is needed so that the energy solution can be obtained. The methodologies presented by Topakoglu [6] has been the impetus for all of the ensuing work [2, 4, 5].

Therefore, the total energy gain by conduction for a concentric circular annulus with uniform wall temperature gradient and heat generation is calculated analytically for various boundary conditions.

2. Viscous Flow in Annular Concentric Pipes

The differential equation for the velocity distribution for an incompressible, constant property fluid in laminar flow inside a circular annulus in regions away from the inlet where the velocity profile is considered to be fully developed can be written as

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dW}{dr}\right) = \frac{1}{\mu}\frac{dp}{dz} \tag{1}$$

with a solution of

$$W(r) = \frac{1}{4\mu}\frac{dp}{dz}\left(r^2 + C_1 \log r + C_2\right) \tag{2}$$

where z is the longitudinal coordinate in the direction of flow, $\frac{dp}{dz}$ is the uniform longitudinal pressure gradient, μ is the dynamic viscosity of the fluid and W is the velocity. The boundary conditions to evaluate the constants C_1 and C_2 in eq. (2) are

$$r = W \qquad W = 0 \tag{3}$$

$$r = R \qquad W = 0 \tag{4}$$

which upon substitution into eq. (2) gives

$$W(r) = - \frac{1}{4\mu} \frac{dp}{dz} \{1 - r^2 - [\frac{(1-\omega^2)}{\log(\frac{1}{\omega})}] \log (\frac{1}{r})\} \tag{5}$$

The nondimensional quantities may be defined as

$$w = (\frac{L}{v})W \tag{6}$$

$$\frac{dP}{dZ} = (\frac{L^3}{\mu v}) \frac{dp}{dz} \tag{7}$$

where L is the mean radius of the outer pipe and v is the kinematic viscosity of the fluid.

The mean flow velocity W_m over the tube cross-section is determined from

$$W_m = \frac{1}{\pi L^2} \int_o^L 2\pi r \; W(r) \; dr \tag{8}$$

which reduces to

$$W_m = - \frac{L^2}{8\mu} \frac{dp}{dz} \tag{9}$$

using eq. (5).

Defining the Reynolds number as $Re_L = \frac{2W_m L}{v}$ and using eq. (9)

$$RE_L = - \frac{L^3}{4\mu v} \frac{dp}{dz} \tag{10}$$

which is further reduced to

$$RE_L = - \frac{1}{4} \frac{dP}{dZ} \tag{11}$$

using eq. (7). Substituting equations (6) and (11) into eq. (5), the non-dimensional velocity distribution for a concentric circular annulus becomes [7]

$$w = RE_L[(1-r^2) - (1-\omega^2) \frac{\log r}{\log \omega}] \tag{12}$$

3. Temperature Distribution

The temperature distribution for the flow in the annulus is determined by solving the energy equation

$$W \frac{\partial T}{\partial Z} = \alpha[\frac{1}{r} \frac{d}{dr} (r \frac{dE}{dr})] + \frac{\alpha}{k} Q_{GEN} \tag{13}$$

where $\frac{\partial T}{\partial Z}$ is the longitudinal temperature gradient C, α is the thermal diffusivity, E is an excess temperature, k is the thermal conductivity and Q_{GEN} is the heat generation density.

Letting q to be a characteristic heat flux, such as an averaged heat flux from the outer and inner walls, the non-dimensional variables may be defined as

$$q_{GEN} = Q_{GEN} \frac{L}{q} \tag{14}$$

$$c = C \frac{k}{q} \tag{15}$$

$$e = \frac{Ek}{Lq} \tag{16}$$

Using equations (5) and (14)-(16), eq. (13) reduces to

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{de}{dr}\right) = c\ w\ PR - q_{GEN} \tag{17}$$

where PR is the Prandtl number, $\frac{v}{\alpha}$.

Substituting from eq. (12) into eq. (13) and integrating, the excess temperature distribution becomes

$$\begin{aligned} e = {} & e_{in} \frac{\log r}{\log \omega} + \frac{1}{4} q_{GEN} (1 - r^2 + C_1 \log r) \\ & + c\ RE_L\ PR \left(\frac{r^2}{4} - \frac{r^4}{16} + \frac{1}{4} C_1 r^2 \log r \right. \\ & - \frac{1}{4} C_1 r^2 - \frac{3}{16} - \frac{3}{16} C_1 \log r - \frac{3}{16} C_1 \omega^2 \log r \\ & \left. + \frac{1}{4} C_1^2 \log r + \frac{1}{4} C_1 \right) \end{aligned} \tag{14}$$

where the boundary conditions

$$r = 1 \qquad e = 0 \tag{15}$$

$$r = \omega \qquad e = e_{in} \tag{16}$$

have been used and where

$$C_1 = - \frac{(1-\omega^2)}{\log \omega} \tag{17}$$

4. Heat Fluxes Through the Walls

The element of heat flux, du, measured in the positive direction of r through an elemental area of r=constant cylindrical surface is

$$du = - k \frac{dE}{dN} L\ ds \tag{18}$$

where

$$\frac{d}{dN} = \frac{1}{L} \frac{\partial}{\partial r} \tag{19}$$

$$dS = r\ dr \tag{20}$$

Using eq. (16), eq. (19) becomes

$$du = - Lqr \frac{\partial e}{\partial r} dr \tag{21}$$

The heat gain rates per unit length of inner and outer pipes, u_i and u_o, which are taken to be positive when heat flows into the fluid, are expressed as

$$u_i = -Lq\omega \int_o^{2\pi} \left.\frac{\partial e}{\partial r}\right)_{r=\omega} dr \tag{22}$$

$$u_o = Lq\omega \int_o^{2\pi} \left.\frac{\partial e}{\partial r}\right)_{r=1} dr \tag{23}$$

Defining a second non-dimensional inner wall temperature as

$$\beta = \frac{E_{in}}{RE_L \ PR \ L \ C} \tag{24}$$

and using equations (15) and (16), eq. (26) reduces to

$$e_{in} = \beta \ G \tag{25}$$

where

$$G = c \ RE_L \ PR \tag{26}$$

Upon substitution from equations (14) and (25), equations (22) and (23) become, respectively,

$$u_i = - 2\pi Lq \{\beta G \frac{1}{\log \omega} + \frac{1}{4} q_{GEN} (C_1 - 2\omega^2) + G[- \frac{3}{16} C_1 + \frac{1}{16} C_1\omega^2 + \frac{1}{2} \omega^2 - \frac{1}{4} \omega^4 + \frac{1}{4} C_1^2 + \frac{1}{2} C_1\omega^2 \log \omega]\} \tag{27}$$

$$u_o = 2\pi Lq\{\beta G \frac{1}{\log \omega} + \frac{1}{4} q_{GEN} (C_1 - 2) + G[\frac{1}{16} C_1 - \frac{3}{16} C_1\omega^2 + \frac{1}{4} + \frac{1}{4} C_1^2]\} \tag{28}$$

4.1 Special Cases of β

4.1.1 Insulated Outer wall, $u_o = o$

Equation (28) reduces to give

$$\beta = \beta_i$$

$$= - \frac{1}{4G} q_{GEN} (C_1 - 2) \log \omega$$

$$+ (\frac{C_1}{16} - \frac{3C_1}{16} \omega^2 + \frac{1}{4} + \frac{1}{4} C_1^2) \log \omega \tag{29}$$

which would give for eq. (28)

$$u_o = 2\pi Lq \; G \frac{(\beta - \beta_i)}{\log \omega} \tag{30}$$

4.1.2 Insulated Inner Wall, $u_i = o$

Equation (27) reduces to give

$$\beta = \beta_o$$

$$= - \frac{1}{4G} q_{GEN} (C_1 - 2\omega^2) \log \omega$$

$$+ \log \omega(- \frac{3}{16} C_1 + \frac{1}{16} C_1\omega^2 + \frac{1}{2}\omega^2$$

$$- \frac{1}{4}\omega^4 + \frac{1}{4} C_1^2 + \frac{1}{2} C_1\omega^2 \log \omega) \tag{31}$$

which would give for eq. (27)

$$u_i = 2\pi Lq \; G \frac{(\beta_o - \beta)}{\log \omega} \tag{32}$$

The ratio of eq. (32) to eq. (30) gives

$$\frac{u_i}{u_o} = \frac{(\beta - \beta_o)}{(\beta_i - \beta)} \tag{33}$$

Letting the heat gain rate from the outer wall to that from both walls per unit length of the pipe be denoted by λ and given as

$$\lambda = \frac{(\beta - \beta_i)}{(\beta_o - \beta_i)} \tag{34}$$

and

$$\lambda_o = \frac{\beta_i}{(\beta_i - \beta_o)}$$

$$\Delta = \frac{\lambda}{\lambda_o} \tag{36}$$

then

$$\beta = \beta_i(1 - \Delta) \tag{37}$$

Substitution of eq. (37) into eq. (27) gives

$$u_i = -2\pi Lq\{\frac{1}{2} q_{GEN}(1-\omega^2) + G[- \frac{1}{4} C_1(1-\omega^2)$$

$$- \frac{1}{4}(1+\omega^4) + \frac{1}{2} \omega^2 + \frac{1}{2}C_1\omega^2 \log \omega]$$

$$+ \Delta[\frac{1}{4} q_{GEN} (C_1-2) + G(\frac{1}{16} C_1$$

$$- \frac{3}{16} C_1\omega^2 + \frac{1}{4} + \frac{1}{4} C_1^2)]\} \tag{38}$$

Equation (37) into eq. (28) yields,

$$u_o = 2\pi Lq\{\Delta[\frac{1}{4} q_{GEN} (C_1-2) + G(\frac{1}{16} C_1$$

$$- \frac{3}{16} C_1\omega^2 + \frac{1}{4} + \frac{1}{4} C_1^2)]\} \tag{39}$$

Finally, the three special cases become:

1. Insulated outer wall
   $u_o = 0$, $\beta = \beta_i$, $\lambda = 0$, $\Delta = 0$
2. Equal wall temperatures
   $T_o = T_i$, $\beta = 0$, $\lambda = \lambda_o$, $\Delta = 1$
3. Insulated inner wall
   $u_i = 0$, $\beta = \beta_o$, $\lambda = 1$, $\Delta = \Delta_o = \frac{1}{\lambda_o}$

5. Conclusions

Equations (38) and (39) give the total energy gain by conduction from the inner and outer tubes, respectively, for a concentric circular annulus with uniform wall temperature gradient and heat generation. Numerical calculations are easily obtainable; the choice is left to the reader and to his computer language and facilities. It is seen that the inner and outer tube energy gains are interrelated and u_i is calculated in terms of u_o.

Other boundary conditions can also be used; this is left to the reader's needs.

References

[1] SHAH, R. K., LONDON, A. L., Laminar Flow Forced Convection in Ducts, Academic Press, New York, (1978).

[2] TOPAKOGLU, H. C., ARNAS, O. A., "Convective Heat Transfer for Steady Laminar Flow Between Two Confocal Elliptic Pipes with Longitudinal Uniform Wall Temperature Gradient", Int. J. Heat Mass Transfer, 17, 1487-1498 (1974).

[3] BERGER, S. A., TALBOT, L., YAO, L. S., "Flow in Curved Pipes", Ann. Rev. Fluid Mech., 15, 461-512 (1983).

[4] ARNAS, O. A., ATAM, M. K., TOPAKOGLU, H. C., "Heat Transfer for Laminar Uniform Heat Generating Fluid Flow in a Curved Pipe", ASME Paper 79-HT-93.

[5] ARNAS, O. A., EBADIAN, M. A., TOPAKOGLU, H. C., "Temperature Field for Flow in a Circular Annulus with Variable Heat Generation," Proceedings of the 17th Midwestern Mechanics Conference, 42-44 (1981).

[6] TOPAKOGLU, H. C., "Steady Laminar Flows of an Incompressible Viscous Fluid in Curved Pipes," J. Math. Mech. 16, 1321-1338 (1967).

[7] MILNE-THOMSON, L. M., Theoretical Hydrodynamics, MacMillan, New York, (1960).