Modeling and Identification of Material Parameters in Coupled Torsion and Bending [*]

H.T. Banks
and
C.A. Smith[†]

Center for Research in Scientific Computation
North Carolina State University
Raleigh, NC 27659-8205

August 1993

---

[*] Research supported in part by the Air Force Office of Scientific Research under grants AFOSR-90-0091 and AFOSR-F49620-93-1-0198

[†] Current Address: C.S.A. Engineering, Inc., 2850 W. Bayshore Rd., Palo Alto CA 94303-998

1 Introduction

Interest in analysis and control of flexible structures has continued in the past few years with the appearance of new devices (e.g., piezoceramics, shape memory alloys) as a viable means of sensing and actuation in "smart material" structures (see for example [BIW2]). The inherently distributed nature of such control devices have necessitated the development of high fidelity distributed parameter models for such structures. In many instances, torsional as well as bending motions are observed. Coupled torsion and bending is a well known phenomena in pretwisted blading ([Ro]). It is also a common problem in robotics when robot arms with eccentric tip masses undergo the coupled motions during slewing maneuvers. Motivated in part by such an impetus, we have investigated an approximation framework for a parameter identification methodology for structures which exhibit coupled torsion and bending.

In this paper we present a model (Section 2) for coupled torsion and bending in a beam with tip body which we have used with success in experimental studies [BS], [Sm]. We also have used this model to motivate development of an abstract framework for well posedness and convergence of parameter approximation schemes in parameter estimation or inverse problems. This framework is discussed in Sections 3 and 4 below. Throughout our discussions we illustrate use of the framework with the model developed in Section 2. Finally, we present (Section 5) sample computational results from simulation studies with the model in identification of parameters such as linear mass density, stiffness, torsional rigidity and damping (bending and torsional) in vibration experiments.

The model presented in this paper is itself new and generalizes in a practical and important way models used recently by Sakawa and Luo [SL], [L] in control (vibration suppression) studies. The model and abstract framework we present is amenable to incorporation of a variety of different damping assumptions (although here we illustrate with separate Kelvin Voigt damping hypotheses for bending and for torsion) and input or control terms.

2 Model for Coupled Torsion and Bending

We begin by developing a mathematical model for coupled torsion and bending in flexible cantilevered beams with tip bodies. In addition to the standard motions of flexural bending in two orthogonal planes and twisting, the model will also incorporate axial warping. We consider a beam with the material/structural parameters given in Table 1.

It is assumed the beam is homogeneous in composition with constant rectangular cross sections. Because of the symmetry of the cross sections we will assume that the beam twists about its neutral axis. Thus the coupling between torsion and flexural bending results solely from the dynamics of the tip body. We note that this is an important distinction since beams of nonconstant, nonsymmetric cross sections exhibit coupled torsion and bending even in the absence of a tip body (see [C1], [C2], [Ro], [V]).

Let X, Y and Z designate an inertial Cartesian coordinate system where the X axis coincides with the beam's neutral axis in the undeformed state (Figure 1). Let Y and Z span a plane perpendicular to the X axis. The bending of the beam may be decomposed into a lateral displacement $u(t,x)$ in the $X-Y$ plane and a lateral displacement $w(t,x)$ in the $X-Z$ plane.

Figure 1: Beam with Tip Mass

parameter	symbol
Linear mass density	ρ
Nominal Young's Modulus	E
Nominal Shear Modulus	G
Width	w
Thickness	t
Beam length	L

Table 1: Beam Parameters

The twisting of the beam about the X axis is denoted $\phi(t,x)$.

Let Q denote the mass center of the tip body. Let Γ_Q denote the plane passing through Q and perpendicular to the X axis. Let P' denote the tip tangent of the beam and P the orthogonal projection of P' onto Γ_Q. Let c denote the distance between P and P'. Let e denote the length of the vertical component (along the Z-axis) of $\vec{PQ}$ and let d denote the length of the horizontal component (in the Y-direction) of $\vec{PQ}$, all as depicted in Figure 1.

In the spirit of Sakawa and Luo [SL], let X_1, Y_1 and Z_1 designate a second Cartesian coordinate system with origin at the point P (see Figure 2). In the beam's undeformed state the axes X_1, Y_1 and Z_1 are parallel to X, Y and Z respectively. The motion of the rigid tip body can be described in terms of this new coordinate system as it is affixed to the body. We shall work under the assumption (also adopted in [SL]) that the tip body oscillates like a pendulum about P.

The dynamics of the beam-tip body system will be derived using Hamilton's principle. We must therefore find expressions for both the kinetic and potential energies of the system. We shall begin by deriving an expression for the potential energy of the system and then turn to a discussion of the kinetic energy.

A tacit assumption in the elementary linear theory of the torsion of beams is that cross sections of the beam which were parallel to one another and perpendicular to the neutral axis prior to deformation, remain planar and perpendicular to the deformed neutral axis after deformation. We will relax this assumption somewhat by considering a "bulging" or warping of cross sections during deformation. Our model is based upon Barr's communication in [C2]. For the moment we will assume that the beam undergoes a "pure" torsional deformation.

Utilizing the inertial axes X,Y and Z introduced earlier and letting v denote the displacement of a point along the neutral axis in the direction of the X axis, we assume that the displacements are of the form

$$u = -z\ \phi(t,x), \quad w = y\ \phi(t,x), \quad v = \psi(y,z)\ \frac{\partial \phi}{\partial x} \tag{1}$$

where $\phi(t,x)$ is the twist of the beam at time t and position x. We note that in the case of simple torsion (where plane sections remain plane), v is identically zero. The function $\psi(y,z)$ is characteristic of the geometry of the beam's cross section, is commonly referred to as the "warping" function, and is assumed to be known. From Barr's note in [C3] the shear stresses

Figure 2: Non-Inertial Coordinate System

in the plane of the cross sections are found to be

$$\tau_{yx} = G\frac{\partial\phi}{\partial x}\left(\frac{\partial\psi}{\partial y} - z\right), \quad \tau_{zx} = G\frac{\partial\phi}{\partial x}\left(\frac{\partial\psi}{\partial z} + y\right), \tag{2}$$

while the axial stress σ_x accompanying the warping is approximately

$$\begin{aligned}\sigma_x &= E\,\frac{\partial v}{\partial x} \\ &= E\,\psi\,\frac{\partial^2\phi}{\partial x^2}.\end{aligned} \tag{3}$$

It then follows from elementary elasticity theory (see [Sm]) that the total strain energy of a bar undergoing pure torsion can be written

$$V_{tor}(t) = \frac{1}{2}\int_0^L \left\{ C_1\left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + G\tilde{J}\left(\frac{\partial\phi}{\partial x}\right)^2 \right\} dx \tag{4}$$

where

$$C_1 = E\int_A \psi^2\, dydz, \tag{5}$$

$$\tilde{J} = \int\int_A \left(y^2 + z^2\right) dydz + 2\int\int_A \left(\frac{\partial\psi}{\partial z}y - \frac{\partial\psi}{\partial y}z\right) dydz \; + \int_A \left\{\left(\frac{\partial\psi}{\partial z}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right\} dydz, \tag{6}$$

and A denotes a typical beam cross section.

For an infinite rectangular rod ψ is given by $-yz$. Using this as an approximation for the finite beam, we find that

$$C_1 = \frac{Ew^3t^3}{144}. \tag{7}$$

Incorporating these considerations into our model, we may write the total strain energy of the undamped beam undergoing combined torsion and bending as

$$V(t) = \int_0^L \left\{ \frac{C_1}{2}\left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + \frac{G\tilde{J}}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \frac{EI_1}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \frac{EI_2}{2}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 \right\} dx. \tag{8}$$

Let us now focus upon the kinetic energy of the beam. A standard result from the linear theory of beam vibration is that the instantaneous kinetic energy of the undamped beam without tip mass may be written

$$T(t) \;=\; \frac{1}{2}\int_0^L \left\{ \rho\left(\frac{\partial u}{\partial t}\right)^2 + \rho\left(\frac{\partial w}{\partial t}\right)^2 + \rho\kappa^2\left(\frac{\partial\phi}{\partial t}\right)^2 \right\} dx. \tag{9}$$

However, to this we must add terms resulting from the motion of the tip body.

An expression for the combined kinetic energy of the beam and the tip body can be found using standard arguments. We refer the interested reader to [L], [Sm] for details. The kinetic energy of the system at any time t is found to be

$$\begin{aligned} T(t) \;=\; & \frac{1}{2}\int_0^L \left\{ \rho\left(\frac{\partial u}{\partial t}\right)^2 + \rho\left(\frac{\partial w}{\partial t}\right)^2 + \rho\kappa^2\left(\frac{\partial \phi}{\partial t}\right)^2 \right\} dx \\ & + \frac{J_Y}{2}\left(\frac{\partial^2 w}{\partial x \partial t}(t,L)\right)^2 + \frac{J_Z}{2}\left(\frac{\partial^2 u}{\partial x \partial t}(t,L)\right)^2 + \frac{J_X}{2}\left(\frac{\partial \phi}{\partial t}(t,L) + c\frac{\partial^2 \phi}{\partial t \partial x}(t,L)\right)^2 \\ & + \frac{1}{2}m\left(\frac{\partial u}{\partial t}(t,L) + c\frac{\partial^2 u}{\partial x \partial t}(t,L) + e\left(\frac{\partial \phi}{\partial t}(t,L) + c\frac{\partial^2 \phi}{\partial t \partial x}(t,L)\right)\right)^2 \\ & + \frac{1}{2}m\left(\frac{\partial w}{\partial t}(t,L) + c\frac{\partial^2 w}{\partial x \partial t}(t,L) + d\left(\frac{\partial \phi}{\partial t}(t,L) + c\frac{\partial^2 \phi}{\partial t \partial x}(t,L)\right)\right)^2 \\ & + \frac{1}{2}m\left(e\frac{\partial^2 w}{\partial x \partial t}(t,L) + d\frac{\partial^2 u}{\partial x \partial t}(t,L)\right)^2 . \end{aligned} \tag{10}$$

The derivation of the equations of motion for the system may now be found from a rather straightforward application of Hamilton's principle of least action which postulates that the variation of the total action of the system considered during any time interval t_0 to t_1 must equal zero. Hence, in the usual notation

$$\delta \int_{t_0}^{t_1} (T - V)\, dt = 0.$$

Here we only summarize the final system describing the motion of the structure, again referring the reader to [Sm] for details of the derivation. Let

$$\xi_1(t,L) \equiv \frac{\partial^2 u}{\partial t^2}(t,L) + e\frac{\partial^2 \phi}{\partial t^2}(t,L) + c\frac{\partial^3 u}{\partial t^2 \partial x}(t,L) + ce\frac{\partial^3 \phi}{\partial t^2 \partial x}(t,L), \tag{11}$$

$$\xi_2(t,L) \equiv \frac{\partial^2 w}{\partial t^2}(t,L) + d\frac{\partial^2 \phi}{\partial t^2}(t,L) + c\frac{\partial^3 w}{\partial t^2 \partial x}(t,L) + cd\frac{\partial^3 \phi}{\partial t^2 \partial x}(t,L), \tag{12}$$

$$\xi_3(t,L) \equiv e\frac{\partial^3 w}{\partial t^2 \partial x}(t,L) \;+\; d\frac{\partial^3 u}{\partial t^2 \partial x}(t,L). \tag{13}$$

These expressions may be intuitively thought of as generalized expressions of linear acceleration for the tip body. The equations of undamped motion are found to be

$$\rho\frac{\partial^2 u}{\partial t^2}(t,x) + \frac{\partial^2}{\partial x^2}\left(M_Z(t,x)\right) \;=\; 0 \tag{14}$$

$$\rho\frac{\partial^2 w}{\partial t^2}(t,x) + \frac{\partial^2}{\partial x^2}\left(M_Y(t,x)\right) = \;0 \tag{15}$$

$$\rho\kappa^2\frac{\partial^2 \phi}{\partial t^2}(t,x) + \frac{\partial^2}{\partial x^2}\left(M_{warp}(t,x)\right) - \frac{\partial}{\partial x}\left(M_X(t,x)\right) \;=\; 0 \tag{16}$$

$$m\ \xi_1(t,L) \ = \ \frac{\partial M_Z}{\partial x}(t,L) \tag{17}$$

$$m\ \xi_2(t,L) \ = \ \frac{\partial M_Y}{\partial x}(t,L) \tag{18}$$

$$me\ \xi_1(t,L) + md\ \xi_2(t,L) + J_X\left(\frac{\partial^2\phi}{\partial t^2}(t,L) + c\frac{\partial^3\phi}{\partial t^2\partial x}(t,L)\right) \ =$$

$$\frac{\partial M_{warp}}{\partial x}(t,L) - M_X(t,L) \tag{19}$$

$$mc\ \xi_1(t,L) + md\ \xi_3(t,L) + J_Z\frac{\partial^3 u}{\partial t^2\partial x}(t,L) \ = \ -M_Z(t,L) \tag{20}$$

$$mc\ \xi_2(t,L) + me\ \xi_3(t,L) + J_Y\frac{\partial^3 w}{\partial t^2\partial x}(t,L) \ = -M_Y(t,L) \tag{21}$$

$$mce\ \xi_1(t,L) + mcd\ \xi_2(t,L) + cJ_X\ \left(\frac{\partial^2\phi}{\partial t^2}(t,L) + c\frac{\partial^3\phi}{\partial t^2\partial x}(t,L)\right) \ = \ -M_{warp}(t,L) \tag{22}$$

where

$$M_X(t,x) = G\tilde{J}\frac{\partial\phi}{\partial x}(t,x), \quad M_{warp}(t,x) = C_1\frac{\partial^2\phi}{\partial x^2}(t,x), \tag{23}$$

and

$$M_Y(t,x) = EI_2\frac{\partial^2 w}{\partial x^2}(t,x), \quad M_Z(t,x) = EI_1\frac{\partial^2 u}{\partial x^2}(t,x), \tag{24}$$

denote the moments about the X, Y and Z axes as computed from standard linear beam theory. In this moment based form we are readily able to introduce internal damping into our model. We illustrate the technique by adopting the Voigt hypothesis which posits that stress is proportional to a combination of strain and strain rate. We may then directly introduce strain rate into the equations of motion through the moments. The altered moments are found to be (see [Sm])

$$M_Z(t,x) = EI_1\frac{\partial^2 u}{\partial x^2}(t,x) + c_u I_1\frac{\partial^3 u}{\partial t\partial x^2}(t,x), \tag{25}$$

$$M_Y(t,x) = EI_2\frac{\partial^2 w}{\partial x^2}(t,x) + c_w I_2\frac{\partial^3 w}{\partial t\partial x^2}(t,x), \tag{26}$$

$$M_{warp}(t,x) = C_1\frac{\partial^2\phi}{\partial x^2}(t,x) + c_{warp}\frac{\partial^3\phi}{\partial t\partial x^2}(t,x), \tag{27}$$

and

$$M_X(t,x) = G\tilde{J}\frac{\partial\phi}{\partial x}(t,x) + c_\phi\tilde{J}\frac{\partial^3 u}{\partial t\partial x^2}(t,x). \tag{28}$$

If we now systematically replace each of the moments in (14)-(24) with these moments which incorporate damping we obtain the governing partial differential equations

$$\rho\frac{\partial^2 u}{\partial t^2}(t,x) + c_u I_1 \frac{\partial^5 u}{\partial t \partial x^4}(t,x) + EI_1 \frac{\partial^4 u}{\partial x^4}(t,x) = 0 \tag{29}$$

$$\rho\frac{\partial^2 w}{\partial t^2}(t,x) + c_w I_2 \frac{\partial^5 w}{\partial t \partial x^4}(t,x) + EI_2 \frac{\partial^4 w}{\partial x^4}(t,x) = 0 \tag{30}$$

$$\rho\kappa^2\frac{\partial^2 \phi}{\partial t^2}(t,x) - c_\phi \tilde{J}\frac{\partial^3 \phi}{\partial t \partial x^2}(t,x) - G\tilde{J}\frac{\partial^2 \phi}{\partial x^2}(t,x) + C_1\frac{\partial^4 \phi}{\partial x^4}(t,x) + c_{warp}\tilde{J}\frac{\partial^5 \phi}{\partial t \partial x^4}(t,x) = 0, \tag{31}$$

along with the natural boundary conditions

$$m\ \xi_1(t,L) = EI_1\frac{\partial^3 u}{\partial x^3}(t,L) + c_u I_1\frac{\partial^4 u}{\partial t \partial x^3}(t,L) \tag{32}$$

$$m\ \xi_2(t,L) = EI_2\frac{\partial^3 w}{\partial x^3}(t,L) + c_w I_2\frac{\partial^4 w}{\partial t \partial x^3}(t,L) \tag{33}$$

$$\begin{aligned} &me\ \xi_1(t,L) + md\ \xi_2(t,L) + J_X\left(\frac{\partial^2 \phi}{\partial t^2}(t,L) + c\frac{\partial^3 \phi}{\partial t^2 \partial x}(t,L)\right) = \\ &\qquad C_1\frac{\partial^3 \phi}{\partial x^3}(t,L) + c_{warp}\frac{\partial^4 \phi}{\partial t \partial x^3}(t,L) - G\tilde{J}\frac{\partial \phi}{\partial x}(t,L) - c_\phi\tilde{J}\frac{\partial^2 \phi}{\partial t \partial x}(t,L) \end{aligned} \tag{34}$$

$$mc\ \xi_1(t,L) + md\ \xi_3(t,L) + J_Z\frac{\partial^3 u}{\partial t^2 \partial x}(t,L) = -EI_1\frac{\partial^2 u}{\partial x^2}(t,L) - c_u I_1\frac{\partial^3 u}{\partial t \partial x^2}(t,L) \tag{35}$$

$$mc\ \xi_2(t,L) + me\ \xi_3(t,L) + J_Y\frac{\partial^3 w}{\partial t^2 \partial x}(t,L) = -EI_2\frac{\partial^2 w}{\partial x^2}(t,L) - c_w I_2\frac{\partial^3 w}{\partial t \partial x^2}(t,L) \tag{36}$$

$$\begin{aligned} &mce\ \xi_1(t,L) + mcd\ \xi_2(t,L) + cJ_X\left(\frac{\partial^2 \phi}{\partial t^2}(t,L) + c\frac{\partial^3 \phi}{\partial t^2 \partial x}(t,L)\right) = \\ &\qquad -C_1\frac{\partial^2 \phi}{\partial x^2}(t,L) - c_{warp}\frac{\partial^3 \phi}{\partial t \partial x^2}(t,L). \end{aligned} \tag{37}$$

These must be used with essential boundary conditions

$$u(t,0) = \frac{\partial u}{\partial x}(t,0) = 0, \tag{38}$$

$$w(t,0) = \frac{\partial w}{\partial x}(t,0) = 0, \tag{39}$$

$$\phi(t,0) = \frac{\partial \phi}{\partial x}(t,0) = 0. \tag{40}$$

The model (29)-(40) has been used to analyze vibration experiments with a structure similar to that depicted in Figure 1 (see [BS], [Sm] for details).

3 Well Posedness and Approximation

In this section we turn to issues of well posedness and computational approximation for abstract second order systems motivated by the model of coupled torsion and bending presented in the previous section. Our objective in doing so is to provide a general computational foundation that will allow for the investigation of questions related to various internal damping mechanisms and regularity of solutions as well as input (control) terms. Thus the framework will also provide a firm foundation for future efforts in control. We note in particular the growing body of literature currently involving the development of control strategies using piezoceramics (e.g. [BIW2], [BWIS], [BSm], [BSS]) which result in unbounded control operators.

We focus our attention upon second order systems of the form

$$\begin{aligned} &\ddot{z}(t) + A_2\dot{z}(t) + A_1 z(t) = f(t) \\ &z(0) = z_0 \\ &\dot{z}(0) = v_0 \end{aligned} \tag{41}$$

where A_1 is a generalized stiffness operator and A_2 is a generalized damping operator. The precise sense in which we consider solutions will be detailed below. Traditionally, the abstract system is studied in the context of linear operator theory. We, however, consider an equivalent formulation for the system based upon a variational or weak formulation. The power of such a framework lies in its concise well posedness results and amenability to computational approximation.

To facilitate our discussions we shall need concepts related to sesquilinear forms defined in the context of a Gelfand triple $V \hookrightarrow H \hookrightarrow V^*$ of Hilbert spaces. For details of sesquilinear forms and Gelfand triples we refer the reader to [Wl], [Sh]; here we summarize pertinent ideas. Given a complex Hilbert space V, $\sigma : V \times V \to \mathbf{C}$ is a *sesquilinear form* on V if $\phi \to \sigma(\phi,\theta)$ is linear and $\theta \to \sigma(\phi,\theta)$ is conjugate linear. The form is *symmetric* if $\sigma(\phi,\theta) = \overline{\sigma(\theta,\phi)}$ for all ϕ,θ in V; the form is *V-bounded* or *V-continuous* if there exists a constant k such that for all ϕ,θ in V

$$|\sigma(\phi,\theta)| \le k\,|\phi|_V\,|\theta|_V\,.$$

The space V is usually contained in a larger (state) space H which is also a Hilbert space. If V is continuously and densely embedded in H, we write $V \hookrightarrow H$. Since H is a Hilbert space we may (through the Riesz map) identify it with its topological dual H^*. It will then follow that $H = H^* \hookrightarrow V^*$ and hence one writes $V \hookrightarrow H = H^* \hookrightarrow V^*$ and says that V, H, V^* form a Gelfand triple with pivot space H. In the context of a Gelfand triple we say that a sesquilinear form σ on V is *V-coercive* if there exists constants $c > 0$ and λ_0 such that for all $\phi \in V$ we have

$$Re\ \sigma(\phi,\phi) + \lambda_0\,|\phi|_H^2 \ge c\,|\phi|_V^2\,.$$

To discuss damped systems, it is convenient to introduce a third Hilbert space V_2 that satisfies $V \hookrightarrow V_2 \hookrightarrow H$ (with equality with either V or H a possibility) and to extend the concept of a Gelfand triple to

$$V \hookrightarrow V_2 \hookrightarrow H \hookrightarrow V_2^* \hookrightarrow V^* \tag{42}$$

or a "Gelfand quintuple". The stiffness and damping operators A_1 and A_2 of (41) are associated with sesquilinear forms σ_1 and σ_2 defined on V and V_2, respectively, through the "duality products", $\langle\cdot,\cdot\rangle_{V^*,V}$ and $\langle\cdot,\cdot\rangle_{V_2^*,V_2}$, which are extensions by continuity of the inner product $\langle\cdot,\cdot\rangle_H$ on $H\times V$ and $H\times V_2$ to $V^*\times V$ and $V_2^*\times V_2$, respectively (see [Wl] for a more complete discussion of duality products).

To be more precise, we assume we have a stiffness sesquilinear form σ_1 on V that is symmetric, V-continuous and V-coercive. There exists $A_1\in\mathcal{L}(V,V^*)$ such that $\sigma_1(\phi,\theta)=\langle A_1\phi,\theta\rangle_{V^*,V}$. Moreover, assume we have a damping sesquilinear form σ_2 on V_2 that is V_2-continuous and V_2-coercive. It can also be shown that there exists $A_2\in\mathcal{L}(V_2,V_2^*)$ such that $\sigma_2(\phi,\theta)=\langle A_2\phi,\theta\rangle_{V_2^*,V_2}$. Within this framework, we may define an abstract system (41) that includes (29)-(40) as a special case. We assume that the input or control term f satisfies $f\in L_2((0,T),V_2^*)$ and consider the weak or variational evolution system

$$(W1)\qquad \begin{aligned}\langle\ddot z(t),\phi\rangle_{V^*,V}+\sigma_2(\dot z(t),\phi)+\sigma_1(z(t),\phi)&=\langle f(t),\phi\rangle_{V_2^*,V_2},\\ z(0)=z_0,\quad \dot z(0)&=v_0,\end{aligned}$$

for solutions $z(t)\in V$. The system $(W1)$ is formally equivalent to the operator evolution system on V^*

$$(W2)\qquad \begin{aligned}\ddot z(t)+A_2\dot z(t)+A_1 z(t)&=f(t)\\ z(0)=z_0,\quad \dot z(0)&=v_0.\end{aligned}$$

In this form one can establish well posedness and regularity properties of solutions. The theorem we state along with its proof may be found in [BIW1].

Theorem 1 *Suppose the sequilinear form σ_1 is a symmetric, V-continuous and V-coercive, the sesquilinear form σ_2 is V_2-continuous and V_2-coercive, $f\in L_2((0,T),V_2^*)$ and $z_0\in V$, $v_0\in H$. Then there exists a unique solution z of (W1) (equivalently (W2)) with $z\in L_2((0,T),V)$,$\dot z\in L_2((0,T),V_2)$ and $\ddot z\in L_2((0,T),V^*)$. Moreover, solutions of (W1) (equivalently (W2)) depend continuously upon the data (z_0,v_0,f) in the sense that the map*

$$(z_0,z_1,f)\to(z,\dot z) \tag{43}$$

is continuous from $V\times H\times L_2((0,T),V_2^*)$ to $L_2((0,T),V)\times L_2((0,T),V_2)$.

To illustrate the concepts introduced above and the use of this theorem, we return to the coupled torsion and bending system (29)-(40) and show how it can be formulated in a weak form. We begin by introducing a state variable $z(t)$ in the state space

$$H=L^2(0,L)\times L^2(0,L)\times L^2(0,L)\times\mathbf{R}^6. \tag{44}$$

For $\Theta,\ \Psi\ \in\ H$ where

$$\begin{aligned}\Theta&=(\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6,\theta_7,\theta_8,\theta_9)^T,\\ \Psi&=(\psi_1,\psi_2,\psi_3,\psi_4,\psi_5,\psi_6,\psi_7,\psi_8,\psi_9)^T,\end{aligned} \tag{45}$$

we recall the standard inner product

$$\langle \Theta, \Psi \rangle_H = \langle \theta_1, \psi_1 \rangle + \langle \theta_2, \psi_2 \rangle + \langle \theta_3, \psi_3 \rangle + \sum_{i=4}^{9} \theta_i \psi_i \tag{46}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual L^2 inner product on $(0, L)$.

Let

$$z_1(t, \cdot) = (u(t, \cdot), w(t, \cdot), \phi(t, \cdot)) \tag{47}$$

and

$$z_2(t, L) = (u(t, L), Du(t, L), w(t, L), Dw(t, L), \phi(t, L), D\phi(t, L)) \tag{48}$$

where for consistency throughout we will use the notation

$$Du \equiv \frac{\partial u}{\partial x} \ , \ \dot{u} \equiv \frac{\partial u}{\partial t}.$$

Formally we may write that

$$z(t) = (z_1(t, \cdot), z_2(t, L)). \tag{49}$$

We next define the matrix

$$\Lambda \ = \begin{pmatrix} \Lambda_1 & \Lambda_2 & \Lambda_3 \\ \Lambda_2^T & \Lambda_4 & \Lambda_5 \\ \Lambda_3^T & \Lambda_5^T & \Lambda_6 \end{pmatrix}, \tag{50}$$

which is composed of the blocks

$$\Lambda_1 = \begin{pmatrix} m & mc \\ mc & J_Z + mc^2 + md^2 \end{pmatrix}, \quad \Lambda_2 \ = \begin{pmatrix} 0 & 0 \\ 0 & mde \end{pmatrix}, \tag{51}$$

$$\Lambda_3 = \begin{pmatrix} me & mce \\ mce & mc^2e \end{pmatrix}, \quad \Lambda_4 = \begin{pmatrix} m & mc \\ mc & J_Y + mc^2 + me^2 \end{pmatrix}, \tag{52}$$

$$\Lambda_5 \ = \begin{pmatrix} md & mdc \\ mcd & mc^2d \end{pmatrix}, \quad \Lambda_6 = \begin{pmatrix} J_X + m\{e^2 + d^2\} & cJ_X + mc\{e^2 + d^2\} \\ cJ_X + mc\{e^2 + d^2\} & c^2\{J_X + me^2 + md^2\} \end{pmatrix}. \tag{53}$$

Finally let M represent the linear operator with matrix representation

$$M = \begin{pmatrix} \rho & 0 & 0 & \mathbf{0}_{1\times 6} \\ 0 & \rho & 0 & \mathbf{0}_{1\times 6} \\ 0 & 0 & \rho\kappa^2 & \mathbf{0}_{1\times 6} \\ \mathbf{0}_{6\times 1} & \mathbf{0}_{6\times 1} & \mathbf{0}_{6\times 1} & \Lambda \end{pmatrix}. \tag{54}$$

We now define an equivalent inner product on H. The equivalence of the two norms is a direct consequence of the strict positivity of each of the material parameters ρ, $\rho\kappa^2$, m, c and the nonnegativity of the remaining constants. The positive definiteness of M follows from its symmetry. Letting $\tilde{\Theta} = (\theta_4, ..., \theta_9)$, $\tilde{\Psi} = (\psi_4, ..., \psi_9)$, we define the inner product by

$$\begin{aligned} \langle \Theta, \Psi \rangle_M &= \langle M\Theta, \Psi \rangle_H \\ &= \rho\langle \theta_1, \psi_1 \rangle + \rho\langle \theta_2, \psi_2 \rangle + \rho\kappa^2 \langle \theta_3, \psi_3 \rangle + \left\langle \Lambda\tilde{\Theta}, \tilde{\Psi} \right\rangle_{R^6} . \end{aligned} \tag{55}$$

Let

$$H_L^2(0, L) = \{\theta \in H^2(0, L) \mid \theta(0) = 0, D\theta(0) = 0\} \tag{56}$$

where $H^2(0, L)$ is the usual Sobolev space of $L_2(0, L)$ functions with first and second derivatives in $L_2(0, L)$ (see [Wl]). Define the Hilbert space V as

$$V = \left\{ \begin{array}{c} \Theta \in H \mid \, : \, \theta_1 \in H_L^2(0, L), \; \theta_2 \in H_L^2(0, L), \; \theta_3 \in H_L^2(0, L) \\ \theta_4 \; = \; \theta_1(L), \; \theta_5 \; = \; D\theta_1(L), \; \theta_6 \; = \; \theta_2(L), \; \theta_7 \; = \; D\theta_2(L), \\ \theta_8 \; = \; \theta_3(L), \; \theta_9 \; = \; D\theta_3(L). \end{array} \right\} \tag{57}$$

with inner product

$$\langle \Theta, \Psi \rangle_V = \langle D^2\theta_1, D^2\psi_1 \rangle + \langle D^2\theta_2, D^2\psi_2 \rangle + \langle D^2\theta_3, D^2\psi_3 \rangle. \tag{58}$$

One can establish (using a Sobolev embedding result [Wl]) that the associated V norm is equivalent to the standard norm on $H_L^2(0, L) \times H_L^2(0, L) \times H_L^2(0, L) \times \mathbf{R}^6$. The motivation for choosing such an inner product will become apparent after consideration of the stiffness sesquilinear form of our system. Standard density and embedding results of Sobolev guarantee that we have a Gelfand triple associated with the spaces V, H, and V^* defined above.

For $\Psi = \left(\psi_1(t, \cdot), \psi_2(t, \cdot), \psi_3(t, \cdot), \tilde{\Psi}\right) \in V$, we may now formally develop the variational form of the system. We begin by multiplying each of the governing equations (29)-(31) by respective test functions and integrating along the length of the beam. We obtain a system in weak form

$$\langle \rho\ddot{u} + c_u I_1 D^4 \dot{u} + EI_1 D^4 u, \; \psi_1 \rangle \; = \; 0 \tag{59}$$

$$\langle \rho\ddot{w} + c_w I_2 D^4 \dot{w} + EI_2 D^4 w, \; \psi_2 \rangle \; = \; 0 \tag{60}$$

$$\langle \rho\kappa^2 \ddot{\phi} - c_\phi \tilde{J} D^2 \dot{\phi} - G\tilde{J} D^2 \phi + c_{warp} D^4 \dot{\phi} + C_1 D^4 \phi, \psi_3 \rangle \; = \; 0. \tag{61}$$

Multiplying each of the equations of the natural boundary conditions (32)-(37) by a test function ψ_i, $i = 4, ..., 9$, we obtain an additional system of the form

$$\tilde{\Psi}^T \Lambda \begin{pmatrix} \ddot{u}(t,L) \\ D\ddot{u}(t,L) \\ \ddot{w}(t,L) \\ D\ddot{w}(t,L) \\ \ddot{\phi}(t,L) \\ D\ddot{\phi}(t,L) \end{pmatrix} + \tilde{\Psi}^T \begin{pmatrix} -c_u I_1 D^3 \dot{u}(t,L) \\ c_u I_1 D^2 \dot{u}(t,L) \\ -c_w I_2 D^3 \dot{w}(t,L) \\ c_w D^2 \dot{w}(t,L) \\ -c_{warp} D^3 \dot{\phi}(t,L) + c_\phi \tilde{J} D \dot{\phi}(t,L) \\ c_{warp} D^2 \dot{\phi}(t,L) \end{pmatrix}$$

$$+ \tilde{\Psi}^T \begin{pmatrix} -EI_1 D^3 u(t,L) \\ EI_1 D^2 u(t,L) \\ -EI_2 D^3 w(t,L) \\ EI_2 D^2 w(t,L) \\ -C_1 D^3 \phi(t,L) + G\tilde{J} D\phi(t,L) \\ C_1 D^2 \phi(t,L) \end{pmatrix} = 0. \tag{62}$$

The variational form of our system is then given by requiring that (59)-(62) to hold for all $\Psi = (\psi_1, \psi_2, \psi_3, \tilde{\Psi}) \in V$.

Motivated by the system (59)-(62) we introduce two sesquilinear forms. For $\Theta,\ \Psi\ \in V$ we define the "stiffness" sesquilinear form

$$\begin{aligned} \sigma_1(\Theta, \Psi) \ &= \ \langle EI_1 D^2\theta_1, D^2\psi_1 \rangle \ + \ \langle EI_2 D^2\theta_2, D^2\psi_2 \rangle \\ &\quad + \ \langle C_1 D^2\theta_3, D^2\psi_3 \rangle \ + \ \langle G\tilde{J} D\theta_3, D\psi_3 \rangle, \end{aligned} \tag{63}$$

and the "damping" sesquilinear form

$$\begin{aligned} \sigma_2(\Theta, \Psi) \ &= \ \langle c_u I_1 D^2\theta_1, D^2\psi_1 \rangle \ + \ \langle c_w I_2 D^2\theta_2, D^2\psi_2 \rangle \\ &\quad + \ \langle c_{warp} D^2\theta_3, D^2\psi_3 \rangle \ + \ \langle c_\phi \tilde{J} D\theta_3, D\psi_3 \rangle. \end{aligned} \tag{64}$$

In the case of Kelvin-Voigt damping as in this example, we may choose $V_2 = V$. Other choices for V_2 are more appropriate for different damping models (see [BIW1][W] for detailed examples). The system represented in (59)-(62) is formally equivalent to a weak system written in terms of the sesquilinear forms σ_1 and σ_2. Integration by parts twice in the inner products in (59)-(62) produces a series of boundary conditions which when combined with the equalities of (62) and the inertial terms associated with the tip mass results in the system

$$\begin{aligned} &\langle \ddot{z}(t), \Psi \rangle_M \ + \ \sigma_2(\dot{z}(t), \Psi) \ + \ \sigma_1(z(t), \Psi) \ = \ 0, \quad \Psi \in V, \\ &z(0) = z_0, \quad \dot{z}(0) = v_0. \end{aligned} \tag{65}$$

The system (65) will be referred to as the weak second order form of the coupled torsion and bending model. We may proceed to establish properties of σ_1 and σ_2 in order that we may apply the well posedness results of Theorem 1.

Theorem 2 *The form σ_1 as defined in (63) is a symmetric, bounded, and V-coercive sesquilinear on V.*

Proof: Symmetry of the form follows clearly from the definition. From physical considerations we may assume that there exists a μ sufficiently large and an $\alpha > 0$ such that α and μ are lower and upper bounds for each of EI_1, EI_2, C_1, $G\tilde{J}$.

We may argue

$$\begin{aligned}
\sigma_1(\Theta, \Theta) &= EI_1 \left|D^2\theta_1\right|^2 + EI_2 \left|D^2\theta_2\right|^2 + C_1 \left|D^2\theta_3\right|^2 + G\tilde{J} \left|D\theta_3\right|^2 \\
&\geq \alpha \left(\left|D^2\theta_1\right|^2 + \left|D^2\theta_2\right|^2 + \left|D^2\theta_3\right|^2 \right) + G\tilde{J} \left|D\theta_3\right|^2 \\
&\geq \alpha \left|\Theta\right|_V^2 + \ G\tilde{J} \left|D\theta_3\right|^2 \\
&\geq \alpha \left|\Theta\right|_V^2 , \qquad (66)
\end{aligned}$$

which establishes V-coercivity.

Boundedness also follows from use of standard L^2 inequalities and Sobolev estimates in the inequalities

$$\begin{aligned}
\left|\sigma_1(\Theta, \Psi)\right|^2 &= \Big(\langle EI_1 D^2\theta_1, D^2\psi_1 \rangle \ + \ \langle EI_2 D^2\theta_2, D^2\psi_2 \rangle \\
&\qquad + \ \langle C_1 D^2\theta_3, D^2\psi_3 \rangle \ + \ \langle G\tilde{J} D\theta_3, D\psi_3 \rangle \Big)^2 \\
&\leq \Big(EI_1 \left|D^2\theta_1\right| \left|D^2\psi_1\right| \ + \ EI_2 \left|D^2\theta_2\right| \left|D^2\psi_2\right| \\
&\qquad + \ C_1 \left|D^2\theta_3\right| \left|D^2\psi_3\right| \ + \ G\tilde{J} \left|D\theta_3\right| \left|D\psi_3\right| \Big)^2 \\
&\leq \mu^2 \left(\left|D^2\theta_1\right|^2 + \left|D^2\theta_2\right|^2 + \left|D^2\theta_3\right|^2 + \left|D\theta_3\right|^2 \right) \times \\
&\qquad \left(\left|D^2\psi_1\right|^2 + \left|D^2\psi_2\right|^2 + \left|D^2\psi_3\right|^2 + \left|D\psi_3\right|^2 \right) \\
&\leq k \left|\Theta\right|_V^2 \left|\Psi\right|_V^2 . \qquad (67)
\end{aligned}$$

Since the form σ_2 is the same as σ_1 except for differing coefficients, the same arguments (under boundedness assumptions on $c_u I_1$, $c_w I_2$, $c_\phi \tilde{J}$, c_{warp}) can be used to establish that σ_2 is V-coercive and bounded. Hence Theorem 1 guarantees existence, regularity and continuous dependence of solutions to the torsion-bending model system (29)-(40).

4 Estimation and Approximation

Having established that typical torsion/bending models for structures can be formulated in an abstract framework for general second order systems $(W1)$, in this section we turn to parameter estimation or identification techniques for these models and systems. First we will provide a mathematical approximation framework in which one can investigate estimation problems arising in the study of coupled torsion and bending. Then we shall show that these results are applicable to the system presented in Section 2 and present a viable approximation scheme for computational studies. Our approximation framework presentation relies heavily on previous abstract results in [BK], [BWIS] and [BR].

The parameter estimation problem which we shall consider may be abstractly formulated as follows: Find $\overline{q} \in Q$ that minimizes the least squares functional

$$J(q) = \sum_{i=1}^{M} |\mathcal{C}z(t_i;q) - z_i|^2 \tag{68}$$

where $z(t;q)$ are the parameter dependent weak solutions of $(W1)$ and z_i are observations of the state $z(t;q)$ which lie in an observation space Z. The set Q is some admissable parameter set contained in a metric space (Q,d) and the operator $\mathcal{C}$ is a mapping from the states to the observations. As a concrete example, one might consider estimation of the stiffness, torsional rigidity and damping in the example of Section 2 using accelerometer data. That is, one might wish to choose a best value of

$$q = (\rho, EI_1, EI_2, C_1, G\tilde{J}, c_u I_1, c_w I_2, c_{warp}, c_\phi \tilde{J}) \tag{69}$$

from some set Q of (spatially varying) functional parameters, e.g. $Q \subset L^\infty(0,L;R^9)$, based on observations $\{(u_i, w_i, \phi_i)\}_{i=1}^M$ of the accelerations $\left\{(\frac{\partial^2 u}{\partial t^2}(t_i,\tilde{x}), \frac{\partial^2 w}{\partial t^2}(t_i,\tilde{x}), \frac{\partial^2 \phi}{\partial t^2}(t_i,\tilde{x}))\right\}_{i=1}^M$ at some point $\tilde{x}$ on the beam.

The inverse problem outlined above involves an infinite dimensional state $z(t)$ and an unbounded operator $\mathcal{C}$. Thus in order to develop computational techniques to handle the minimization, we must make finite dimensional approximations to the state spaces H and V. The family of approximating subspaces, denoted H^N, can be developed using a number of different schemes. For the example discussed below, we have utilized a class of schemes based upon choosing H^N as subspaces generated by cubic splines. Other examples (involving spectral approximations in structural acoustics systems) may be found in [BSS] and [BSm].

We shall give a brief summary of the pertinent ideas, referring the reader to [BK], [BWIS], [BR] and [Sm] for more details. For ease in exposition here, we consider only constant parameters. Hence the parameter set Q is a compact subset of R^9. To this end a family of approximating estimation problems with finite dimensional state spaces can be formulated by seeking $\bar{q}^N \in Q$ which minimizes

$$J^N(q) = \sum_i \left|\mathcal{C}z^N(t_i;q) - z_i\right|^2, \tag{70}$$

where $z^N(t;q)$ is the solution to the finite dimensional approximation of $(W1)$ defined by

$$\begin{aligned} &\langle \ddot{z}^N(t), \theta^N \rangle \;+\; \sigma_1(z^N(t), \theta^N) \;+\; \sigma_2(\dot{z}^N(t), \theta^N) \;=\; \langle P^N f(t), \theta^N \rangle\,, \quad \theta^N \in H^N, \\ &z^N(0) = P^N z_0, \quad \dot{z}^N(0) = P^N v_0. \end{aligned} \tag{71}$$

The operator P^N is the orthogonal projection of H onto H^N.

One desires, of course, to guarantee that if the approximating families H^N are chosen properly, then solutions $\overline{q}^N$ of minimizing (70) converge in some sense to a parameter $\overline{q}$ that minimizes the original criterion (68). To this end, we make certain fundamental assumptions on how the family H^N approximates H and V. We assume:

(H1) The finite dimensional subspaces H^N satisfy $H^N \subset V$.

(H2) For each $\psi \in V$, $|\psi - P^N \psi|_V \to 0 \quad$ as $N \to \infty$.

(H3) For each $\theta \in V_2$, $|\theta - P^N \theta|_{V_2} \to 0 \quad$ as $N \to \infty$.

We note that condition (H2) also implies that $\left|\psi - P^N \psi\right|_H \to 0$ for each $\psi \in H$.

From the definition of the sesquilinear forms introduced earlier for our torsion/bending example, it is clear that in typical applications these forms will be parameter dependent. To allow for this dependence some regularity is needed. Therefore we introduce continuity with respect to parameter concepts for the sesquilinear forms. It will be assumed that the sesquilinear forms $\sigma_1 = \sigma_1(q)$ and $\sigma_2 = \sigma_2(q)$ are defined on Q and satisfy the following type of continuity condition.

(H4) The parameter dependent sesquilinear form $\sigma : Q \times V \times V \to \mathbf{C}$ is V q-continuous in the sense that there exists a constant γ_1 depending only upon Q such that if for all $\phi, \psi \in V$, and $q, \tilde{q} \in Q$ we have that

$$|\sigma(q)(\phi, \psi) - \sigma(\tilde{q})(\phi, \psi)| \leq \gamma_1 \ d(q, \tilde{q}) \ \ |\phi|_V \, |\psi|_V$$

for all $\phi, \psi \in V$, and $q, \tilde{q} \in Q$.

We are now able to state a theorem (see [BWIS]) of considerable importance related to the convergence and continuous dependance (with respect to data) of solutions of the approximate optimization problems involving (70) and (71). In fact, in many cases one may conclude that solutions $\bar{q}^N$ of the approximate problems converge to solutions $\bar{q}$ for the original infinite dimensional optimization problem involving (65) and (68) (see [Ba], [BK] for a more complete discussion).

Theorem 3 *Suppose that H^N satisfies (H1)-(H3) and that the sesquilinear form $\sigma_1(q)$ is a V q-continuous, symmetric, V-bounded, and V-coercive sesquilinear form and $\sigma_2(q)$ is a V_2 q-continuous, V_2-bounded, V_2-coercive sesquilinear form. Furthermore, assume that*

$$q \to f(t; q) \textit{ is continuous from } Q \textit{ to } L_2((0,T), V_2^*). \tag{72}$$

Let q^N be arbitrary in Q such that $q^N \to q$ in Q; then for $t > 0$ as $N \to \infty$ we have

$$z^N(t; q^N) \to z(t; q) \quad \textit{in } V \textit{ norm}, \tag{73}$$

$$\dot{z}^N(t; q^N) \to \dot{z}(t; q) \quad \textit{in } V_2 \textit{ norm}, \tag{74}$$

where z^N is the solution to (71) corresponding to q^N and z is the solution to $(W1)$ corresponding to q.

We note that the above "state" convergence (a type of "uniform in q" convergence of $z^N(t; q) \to z(t; q)$, $\dot{z}^N(t; q) \to \dot{z}(t; q)$) can be used to guarantee convergence (at least subsequential - see [Ba], [BK]) of optimal parameters $\overline{q}^N$ to $\overline{q}$ in the event that observations of displacement and/or velocity are used in (68) and (70). Convergence in the case of accelerometer data requires greater regularity in f or a stronger damping form. In this case we may cite the following result from [BR] which establishes sufficient convergence arguments for systems exhibiting Kelvin Voigt like damping.

Theorem 4 *Suppose that all of the hypotheses of Theorem 3 hold with the additional hypothesis that $V_2 = V$. Let q^N be arbitrary in Q such that $q^N \to q$ in Q; then for $t > 0$ as $N \to \infty$ we have*

$$\frac{d^{(m)}}{dt^{(m)}} z^N(t; q^N) \to \frac{d^{(m)}}{dt^{(m)}} z(t; q) \quad \textit{in } V \textit{ norm}, \tag{75}$$

for all $m \geq 0$.

We now return to the coupled torsion bending system presented in Section 2 and argue that all the assumptions of Theorem 4 are satisfied for this example. To illustrate we assume that we wish to estimate the material parameters $\rho, EI_1, EI_2, G\tilde{J}, C_1, c_u I_1, c_w I_2, c_\phi \tilde{J}, c_{warp}$ based upon accelerometer data. In accordance with physical considerations we will assume then that Q represents a compact subset of the Euclidean space R^9. Let us also assume that our laboratory data consists of observations $\{u_i\}$, $\{w_i\}$ and $\{\phi_i\}$ of acceleration recorded at one point $\tilde{x}$ along the span of the beam for $X - Y$ bending, $X - Z$ bending and torsion, respectively. The cost functional of the associated estimation problem would then have the form

$$J(q) = \sum_i \left| \frac{\partial^2 u}{\partial t^2}(t_i, \tilde{x}; q) - u_i \right|^2 + \left| \frac{\partial^2 w}{\partial t^2}(t_i, \tilde{x}; q) - w_i \right|^2 + \left| \frac{\partial^2 \phi}{\partial t^2}(t_i, \tilde{x}; q) - \phi_i \right|^2 \tag{76}$$

subject to (65).

We begin by directing our attention to continuity with respect to parameters of the sesquilinear forms $\sigma_1(q)$ and $\sigma_2(q)$. Recall that for our example, the sesquilinear forms are given in (63) and (64) for q defined in (69). The continuity arguments are the same for both sesquilinear forms with the exception of differing constants; hence, we consider only $\sigma_1(q)$.

Theorem 5 *$\sigma_1(q)$ is a Vq-continuous sesquilinear form.*

Proof: Let q, $\tilde{q} \in Q$. From the definition of σ_1 we have that

$$\begin{aligned} \sigma_1(q)(\Theta, \Psi) \;&=\; \langle q_2 D^2\theta_1, D^2\psi_1 \rangle \;+\; \langle q_3 D^2\theta_2, D^2\psi_2 \rangle \\ &\qquad +\; \langle q_4 D^2\theta_3, D^2\psi_3 \rangle \;+\; \langle q_5 D\theta_3, D\psi_3 \rangle. \end{aligned} \tag{77}$$

Hence

$$\begin{aligned} |\sigma_1(q)(\Theta, \Psi) - \sigma_1(\tilde{q})(\Theta, \Psi)|^2 \;&=\; \Big(\{q_2 - \tilde{q}_2\} \left\langle D^2\theta_1, D^2\psi_1 \right\rangle \;+\; \{q_3 - \tilde{q}_3\} \left\langle D^2\theta_2, D^2\psi_2 \right\rangle \\ &\qquad +\; \{\{q_4 - \tilde{q}_4\} \left\langle D^2\theta_3, D^2\psi_3 \right\rangle \;+\; \{\{q_5 - \tilde{q}_5\} \langle D\theta_3, D\psi_3 \rangle \Big)^2 \\ &\leq\; \Big(\{q_2 - \tilde{q}_2\} \left|D^2\theta_1\right| \left|D^2\psi_1\right| \;+\; \{q_3 - \tilde{q}_3\} \left|D^2\theta_2\right| \left|D^2\psi_2\right| \\ &\qquad +\; \{\{q_4 - \tilde{q}_4\} \left|D^2\theta_3\right| \left|D^2\psi_3\right| \;+\; \{\{q_5 - \tilde{q}_5\} |D\theta_3| \, |D\psi_3| \Big)^2 \\ &\leq\; \gamma \sum_{i=1}^{9} |q_i - \tilde{q}_i|^2 \left(\left|D^2\theta_1\right|^2 + \left|D^2\theta_2\right|^2 + \left|D^2\theta_3\right|^2 + |D\theta_3|^2 \right) \times \\ &\qquad \left(\left|D^2\psi_1\right|^2 + \left|D^2\psi_2\right|^2 + \left|D^2\psi_3\right|^2 + |D\psi_3|^2 \right) \\ &\leq\; \gamma \, |q - \tilde{q}|^2_{\mathbf{R}^9} \, |\Theta|^2_V \, |\Psi|^2_V \end{aligned} \tag{78}$$

where γ is some positive constant. This yields the desired continuity.

For the example we now construct a family of approximate problems; we present the details of our construction. Central to our approach is a cubic spline based Galerkin approximation to the initial value problem (65). For each $N = 1, 2...$, let π^N denote a uniform partition of $[0, L]$, $\pi^N : 0 = 0 < x_1 < x_2 < \cdots < x_N = L$. Let $\{B_i^N(x)\}_{i=-1}^{N+1}$ denote the standard cubic spline basis elements on π^N (e.g., see [Sch]). In short, each B_i^N is a C^2 function on $[0, L]$ which is a cubic polynomial on each subinterval $[(i-1)\frac{L}{N}, i\frac{L}{N}]$, $i = 1, ..., N$. The support of B_i^N is $[(i-2)\frac{L}{N}, (i+2)\frac{L}{N}] \cap [0, L]$ with $B_i^N(i\frac{L}{N}) = 4$, $DB_i^N(i\frac{L}{N}) = 0$, $B_i^N((i \pm 1)\frac{L}{N}) = 1$, and $DB_i^N((i \pm 1)\frac{L}{N}) = \mp\frac{N}{L}$. Recall that for our example the space V is a subset of $H_L^2(0, L) \times H_L^2(0, L) \times H_L^2(0, L) \times R^6$. In order that the subspaces H^N satisfy (H1), the first three coordinates of the approximating elements must satisfy the zero displacement, zero slope condition. Hence, linear combinations of the spline elements $\{B_i^N(x)\}$ are utilized (see [BK] for a typical example). For the problem at hand, three different sets of elements,

$$\{\beta_i^N(\cdot)\}_{i=1}^{N+1},\ \{\gamma_i^N(\cdot)\}_{i=1}^{N+1},\ \{\tau_i^N(\cdot)\}_{i=1}^{N+1} \tag{79}$$

result from this procedure, the first two related to bending in the $X - Y$ and $X - Z$ planes, respectively, and the last to torsion. For example we would define $\{\beta_i^N\}_{i=1}^{N+1}$ by $\beta_1^N = B_0^N - 2B_1^N - 2B_{-1}^N$ and $\beta_i^N = B_i^N$, $i = 2, ..., N+1$. Note that as required $\beta_i^N(0) = 0$ and $D\beta_i^N(0) = 0$.

Define the basis elements $\theta_i^N, i = 1, ..., 3N+3$ by

$$\theta_i^N = (\beta_i^N(\cdot), 0, 0, \beta_i^N(L), D\beta_i^N(L), 0, 0, 0, 0) \quad i = 1, ..., N+1, \tag{80}$$

$$\theta_{N+1+i}^N = (0, \gamma_i^N(\cdot), 0, 0, 0, \gamma_i^N(L), D\gamma_i^N(L), 0, 0) \quad i = 1, ..., N+1, \tag{81}$$

$$\theta_{2N+2+i}^N = (0, 0, \tau_i^N(\cdot), 0, 0, 0, 0, \tau_i^N(L), D\tau_i^N(L)), \quad i = 1, ..., N+1. \tag{82}$$

Let H_u^N, H_w^N and H_ϕ^N denote the spans of the sets $\{\theta_i^N\}$, $\{\theta_{N+1+i}^N\}$ and $\{\theta_{2N+2+i}^N\}$, respectively. We then define H^N by the product $H^N = H_u^N \times H_w^N \times H_\phi^N$ and observe that $\dim H^N = 3N+3$. Recalling the definition of V (see (57)), we immediately see that $H^N \subset V$. Condition (H2) and (H3) are verified using standard interpolatory spline estimates (see [Sch]). Similar arguments may be found in [BK] and [Sm].

The Galerkin equations in H^N corresponding to (65) for $z^N(t) \in H^N$ are given by

$$\begin{gathered} \langle \ddot{z}^N(t), \theta_i^N \rangle_M \ + \ \sigma_1(z^N(t), \theta_i^N) \ + \ \sigma_2(\dot{z}^N(t), \theta_i^N) \ = \ \langle P^N f(t), \theta_i^N \rangle \ , \quad \theta_i^N \in H^N, \\ z^N(0) = P^N z_0, \quad \dot{z}^N(0) = P^N v_0. \end{gathered} \tag{83}$$

Setting

$$z^N(t) = \sum_{i=1}^{3N+3} \xi_i^N(t)\ \theta_i^N. \tag{84}$$

the resulting initial value problem for (65) is equivalent to the linear, homogeneous, second order (3N + 3)-dimensional system given by

$$M^N \frac{d^2\xi^N}{dt^2}(t) + C^N \frac{d\xi^N}{dt}(t) + K^N \xi^N(t) = F^N(t),$$
$$\xi^N(0) = \xi_0^N, \quad \frac{d\xi^N}{dt}(0) = \zeta_0^N, \tag{85}$$

where $\xi^N(t) = (\xi_1^N(t), \xi_1^N(t), ..., \xi_{3N+3}^N(t))^T$ and ξ_0^N, ζ_0^N are obtained from expansion for $P_{z_0}^N, P_{v_0}^N$, respectively. The entries in the $(3N+3) \times (3N+3)$ matrices M^N, C^N, and K^N are given by

$$M_{i,j}^N = \langle \theta_i^N, \theta_j^N \rangle_M, \tag{86}$$

$$C_{i,j}^N = \sigma_2(\theta_i^N, \theta_j^N), \tag{87}$$

$$K_{i,j}^N = \sigma_1(\theta_i^N, \theta_j^N). \tag{88}$$

For each $t > 0$, the components of the $3N + 3$ vector are given by $F_i^N(t) = \langle f(t), \theta_i^N \rangle_H$.

At this stage we note that our approximation results in a second order system of ordinary differential equations which are at least in principle routine to solve. We now consider the corresponding minimization problem involving (70). For our system, a sequence of finite dimensional estimation problems result which consist of finding a $\overline{q}^N \in Q$ which minimizes

$$J(q) = \sum_i \left| \frac{\partial^2 u^N}{\partial t^2}(t_i, \tilde{x}; q) - u_i \right|^2 + \left| \frac{\partial^2 w^N}{\partial t^2}(t_i, \tilde{x}; q) - w_i \right|^2 + \left| \frac{\partial^2 \phi^N}{\partial t^2}(t_i, \tilde{x}; q) - \phi_i \right|^2 \tag{89}$$

where for each $q \in Q$,

$$\frac{\partial^2 u^N}{\partial t^2}(t; q) = \sum_{i=1}^N \xi_i^N(t; q)\ \beta_i,$$
$$\frac{\partial^2 w^N}{\partial t^2}(t; q) = \sum_{i=1}^N \xi_{N+1+i}^N(t; q)\ \gamma_i,$$
$$\frac{\partial^2 \phi^N}{\partial t^2}(t; q) = \sum_{i=1}^N \xi_{2N+2+i}^N(t; q)\ \tau_i,$$

and $\xi^N(t; q)$ is the unique solution of (85) corresponding to $q \in Q$.

5 Numerical Results

To demonstrate the efficacy of the approximation methods for identification of parameters outlined above, we have developed software packages and tested them on examples with simulated data from typical vibration experiments. The software packages involved integration of the approximate system (85) along with an iterative optimization algorithm to minimize the associated approximation to (76) (see (68)). All calculations were performed on the IBM RISC System/6000 at the Center for Research in Scientific Computation, North Carolina State University. The forward integration of (85) was performed using Gear's method for stiff systems (IMSL routine DGEAR). The second time derivative of ξ^N was subsequently computed using a second-order centered difference on the generalized displacement,

$$\frac{d^2\xi^N}{dt^2}(t) = \frac{\xi^N(t+\Delta) - 2\xi^N(t) + \xi^N(t-\Delta)}{\Delta^2}. \tag{90}$$

The approximating finite-dimensional least squares minimization problems for (89) could then be solved using a modification of the IMSL implementation of the Levenberg Marquardt algorithm (routine ZXSSQ), an iterative quasi-Newton method utilizing a model-trust region approach. Finally, all FFTs were generated using the MATLAB implementation of the discrete fast Fourier transform (function FFT).

To illustrate results, we report here on one example involving only bending in the $X-Y$ plane and torsion. That is in the system (29)-(40), we omit the equation (30) for bending in the $X-Z$ plane along with all related w terms in the boundary and initial conditions. In this example we attempted to estimate the parameters ρ, EI_1, $G\tilde{J}$, $c_u I_1$ and $c_\phi \tilde{J}$ from "simulated observation data". To generate simulated "data" (with and without observation noise), we followed techniques that are now standard practice in the parameter estimation literature (e.g. see the rather detailed discussions in [BK]). Briefly, numerical tests were carried out as follows: "data" in our inverse algorithms were generated from functions selected so that all of the boundary conditions of the problem were satisfied. "True" parameter values ρ^*, EI_1^*, $G\tilde{J}^*$, $c_u I_1^*$, $c_\phi \tilde{J}^*$ were then chosen. Substituting these choices of functions and parameters into the equations of coupled torsion and bending, we obtain forcing functions so that the differential equations of u and ϕ are satisfied. For observation points we used equally spaced measurements in time, $t_i = \frac{1}{256}$ seconds, $i = 1, ..., 4096 \equiv M$; and in space we assumed we observed at the tip of the beam, $\tilde{x} = L$. In our simulated experiment we assumed observations of $X-Y$ bending acceleration $\{u_i\}$ and torsion acceleration $\{\phi_i\}$, are accessible. Thus the cost functional under consideration is

$$J^N(q) = \sum_i \left\{ \left| \frac{\partial^2 u^N}{\partial t^2}(t_i; q) - u_i \right|^2 + \left| \frac{\partial^2 \phi^N}{\partial t^2}(t_i; q) - \phi_i \right|^2 \right\}. \tag{91}$$

The solutions of our system were chosen to be

$$u^*(t,x) = e^{-.09t}\left\{-sinx + \frac{sin3x}{3}\right\}\left\{\frac{sin2\pi t}{4\pi^2} + \frac{sin10\pi t}{100\pi^2} + \frac{sin50\pi t}{2500\pi^2}\right\}$$

$$\phi^*(t,x) = e^{-.08t}\left\{-\frac{sin5x}{5} + \frac{sin8x}{8}\right\}\left\{\frac{sin2\pi t}{4\pi^2} + \frac{sin10\pi t}{100\pi^2} + \frac{sin50\pi t}{2500\pi^2}\right\}.$$

Motivated by actual laboratory experiments, the following physically realistic parameter (see [Bs], [Sm]) values are utilized in our simulations.

Mass Polar Moment	κ^2	5.06×10^{-4}
warping constant	C_1	.00756
Kelvin Voigt Warping	c_{warp}	7.56×10^{-5}
tip body mass	m	.436
Moment of Inertia (X_1 axis)	J_X	.00355
Moment of Inertia (Z_1 axis)	J_Z	.00355
Horizontal Offset	c	.0127
Vertical Offset	e	.0245

Table 2: System Parameters

Linear Mass Density	ρ	.659
Mass Polar Moment	$\rho\kappa^2$	3.33×10^{-4}
Stiffness	EI	14.9
Torsional Rigidity	GJ	22.1
Kelvin Voigt Bending	$c_u I_1$	1.49×10^{-1}
Kelvin Voigt Torsion	$c_\phi \tilde{J}$	4.42×10^{-1}

Table 3: True Parameters

The "true" parameter values to be identified are shown in Table 3. As before these parameters are chosen so that they represent actual typical beam parameters.

The results of our identification studies were most satisfactory. The table below indicates the start up parameter values $q^{(0)}$ in our iterative scheme, the final estimates $q^{(f)}$ and the relative error $q^* - q^{(f)}$ as compared to the "true" estimates.

q	$q^{(0)}$	$q^{(f)}$	$q^* - q^{(f)}$	Relative Error
EI	14.1	14.8	.1	.67%
GJ	20.1	21.8	.3	1.38%
ρ	.759	.633	.026	3.95%
$C_u I$	.300	.165	.016	10.74%
$C_t J$	.200	.434	.008	1.81%

Table 4: Computational Results, No Noise

We also recorded the reduction in the associated cost functional. Letting J^0 denote the cost functional associated with the start up values and J^f denote the cost functional for the converged estimates, we found $J^0 = 71.3$ while $J^f = .572$.

To further simulate laboratory conditions, noise was then added to the simulation data. This was done in the following way. A sequence of normally distributed random numbers $\{\epsilon_i\}_{i=1}^{M}$ of mean zero and standard deviation $\beta/2$ was generated and then added to the data using the formula:

$$\tilde{u}(t_i, 1) = (1 + \epsilon_i)\, u^*(t_i, 1) \tag{92}$$

$$\tilde{\phi}(t_i, 1) = (1 + \epsilon_i)\, \phi^*(t_i, 1). \tag{93}$$

The noise is added at intensity levels of 5%, 10%, 20% and 40% relative error. The results using these data sets with our algorithm are depicted in the following series of tables. As one can readily see, the algorithm performed quite well even in the presence of noise in the data.

q	$q^{(0)}$	$q^{(f)}$	$q^* - q^{(f)}$	Relative Error
EI	14.1	14.8	.1	.67%
GJ	20.1	21.9	.2	.90%
ρ	.759	.633	.026	3.95%
$C_u I$	.300	.164	.015	10.7%
$C_T J$	.200	.437	.005	1.13%

Table 5: 5% noise or $\beta = .05$; $J^0 = 72.5$, $J^f = 1.56$

q	$q^{(0)}$	$q^{(f)}$	$q^* - q^{(f)}$	Relative Error
EI	14.1	14.8	.1	.67%
GJ	20.1	21.8	.3	1.36%
ρ	.759	.628	.026	4.7%
$C_u I$	.300	.163	.014	9.4%
$C_T J$	.200	.437	.005	1.1%

Table 6: 10% noise or $\beta = .10$; $J^0 = 75.6$, $J^f = 4.36$

q	$q^{(0)}$	$q^{(f)}$	$q^* - q^{(f)}$	Relative Error
EI	14.1	14.8	.1	.67%
GJ	20.1	22.0	.1	.0045%
ρ	.759	.634	.025	3.79%
C_uI	.300	.169	.02	13.42%
C_TJ	.200	.434	.008	1.81%

Table 7: 20% noise or $\beta = .20$; $J^0 = 85.3$, $J^f = 15.1$

q	$q^{(0)}$	$q^{(f)}$	$q^* - q^{(f)}$	Relative Error
EI	14.1	14.8	.1	.67%
GJ	20.1	21.5	.6	2.7%
ρ	.759	.629	.030	4.55%
C_uI	.300	.169	.031	15.5%
C_TJ	.200	.434	.034	8.5%

Table 8: 40% noise or $\beta = .40$; $J^0 = 134$, $J^f = 64.1$

6 Concluding Remarks

We have presented a model for coupled torsion and bending to describe vibrations in a complex articulated structure. The model presented is an important extension of recent studies as it includes axial warping and incorporates a more general internal damping mechanism than that presented in [SL] and [L]. In actual fact, the model has been used successfully used with experimental data (see [BS] and [Sm]). In addition, we have shown that the model fits into a general framework of abstract second order systems for which well posedness and regularity results have been established and for which computational methods for both simulation and parameter estimation algorithms can be studied. All abstract mathematical results are illustrated in the context of our model. We have presented one particular example of an approximation scheme and the subsequent implementation of our parameter estimation methodologies. Moreover, we demonstrate via a test example, the computational performance one might expect in similar estimation studies. In particular, the example suggests how the algorithm might behave in the presence of noisy data.

We encourage the reader to note the generality of the theoretical and computational framework within which the current model is studied. There is considerable latitude in these discussions for subsequent studies in control as well as parameter identification and estimation. Future studies might for example concentrate upon the effects of warping or the incorporation of more sophisticated damping mechanisms. We note in particular the need for the development of high fidelity models for similar structures utilizing "smart" materials as both sensors and actuators (see [BIW2], [BWIS] and [BSm]).

7 Acknowlegments

The authors would like to gratefully acknowledge Dr. Yun Wang of North Carolina State University for fruitful discussions. The authors would also like to gratefully acknowledge Prof. D.J. Inman of Virginia Polytechnic and State University and Dr. Jeff Umland of J.P.L. for discussions on experiments and data (collected at the Mechanical Systems Laboratory, SUNY at Buffalo) which were instrumental in motivating the study reported here.

8 References

[Ba] H.T. Banks, *On a variational approach to some parameter estimation problems*, in Distributed Parameter Systems (ed. by F. Kappel, et al), Springer Lecture Notes in Control and Info. Sci., 75 (1985), pp. 1-23.

[BG] H.T. Banks, S.S. Gates, I.G. Rosen, Y. Wang, *Identification of a distributed parameter model*, SIAM J. Control and Optimization, 26 (1988), pp. 743-762

[BK] H.T. Banks and K. Kunisch, *Estimation Techniques for Distributed Parameter Systems*, Boston, Birkhauser, 1989.

[BI] H.T. Banks and D.J. Inman, *On damping mechanisms in beams*, CAMS Report #89-3, University of Southern California, September 1989; ASME J. Applied Mechanics, 58 (1991), pp. 716-723.

[BIU] H.T. Banks, D.J. Inman, J.W. Umland, *Damping in coupled bending torsion - an experiment*, CAMS Report #91-18, University of Southern California, August 1991; in Proc. 1991 Amer. Control Conf. (June 26-28, Boston), pp. 2994-2999.

[BIW1] H.T. Banks, K. Ito, Y. Wang, *Well posedness for damped second order systems with unbounded input operators*, Center for Research in Scientific Computation Technical Report, CRSC-TR93-10, North Carolina State University; Differential and Integral Equations, submitted.

[BIW2] H.T. Banks, K. Ito, Y. Wang, *Computational methods for identification and control in structures with piezoceramic actuators and sensors*, Center for Research in Scientific Computation Technical Report, CRSC-TR92-2, North Carolina State University; Journal of Intelligent Material Systems and Structures, to appear.

[BWIS] H.T. Banks, Y. Wang, D.J. Inman, and J.C. Slater, *Approximation and parameter identification for damped second order systems with unbounded input operators*, Center for Research in Scientific Computation Technical Report, CRSC-TR93-9, North Carolina State University; Control: Theory and Advanced Technologies, submitted.

[BR] H.T. Banks and D.A. Rebnord, *Analytic semigroups: applications to inverse problems for flexible structures*, in Differential Equations with Applications in Biology, Physics, and Engineering (J. Goldstein and F. Kappel eds), Marcel Dekker, (1991), pp. 21-35.

[BS] H.T. Banks and C. A. Smith, *Modeling of coupled bending and torsion in elastic structures*, Center for Research in Scientific Computation Technical Report, CRSC-TR92-23, North Carolina State University; to appear in Proceedings of the 14th ASME Biennial Conference on Mechanical Vibrations and Noise (September 19-22, Albuquerque, NM).

[BSm] H.T. Banks and R.C. Smith, *Active control of acoustic pressure fields using smart material technologies*, ICASE Tech Rep 93-31, NASA Langley Res Ctr., Hampton, VA, May, 1993; IMA Proceedings, Workshop in Flow Control (Univ. Minnesota, Nov. 16-20, 1992, Minneapolis), to appear.

[BSS] H.T. Banks, R.J. Silcox and R.C. Smith, *The modeling and control of acoustic/structure interaction problems via piezoceramic actuators: 2-D numerical examples*, Center for Research in Scientific Computation Technical Report, CRSC-TR92-1, North Carolina State University; Journal of Vibration and Acoustics, to appear.

[Barr] A.D.S. Barr, *Torsional waves in uniform bars of non-circular cross section*, Proc Inst Mech Eng, London, 176 (1962), pp. 315-322.

[C1] W. Carnegie, *Static bending of pre-twisted cantilevered blading*, Proc Inst Mech Eng, London, 171 (1957), pp. 873-894.

[C2] W. Carnegie, *Vibrations of pre-twisted cantilevered blading*, Proc Inst Mech Eng, London, 173 (1959), pp. 343-374.

[C3] W. Carnegie, *Vibrations of pre-twisted cantilevered blading: an additional effect due to torsion*, Proc Inst Mech Eng, London, 176 (1962), pp. 315-322.

[CP] R.W. Clough and J. Penzien, *Dynamics of Structures*, New York, McGraw-Hill, 1975.

[L] Z. H. Luo, *Dynamics and Control of Flexible Structures*, PhD Thesis, Osaka University, January 1990.

[R] E. Reissner, *On non-uniform torsion of cylidrical rods*, J Math Phys, 31 (1952), pp. 214-221.

[Ro] A. Rosen, *Structural and dynamic behavior of pretwisted rods and beams*, Applied Mechanics Review, 49 (1991), pp. 483-514.

[Sch] M.H. Schultz, *Spline Analysis*, Englewood Cliffs: Prentice Hall, 1973.

[Sh] R.E. Showalter, *Hilbert Space Methods for Partial Differential Equations*, San Francisco, Pitman, 1979.

[SL] Y. Sakawa and Z.H. Luo, *Modeling and control of coupled bending and torsional vibrations of flexible beams*, IEEE Transactions on Automatic Control, 34 (1989), pp. 970-977.

[Sm] C.A. Smith, *Modeling and Analysis of Coupled Torsion and Bending in Flexible Structures*, PhD Thesis, University of Southern California, May 1993.

[W] Y. Wang, *Damping Modeling and Parameter Estimation in Timoshenko Beams*, PhD Thesis, Brown University, May 1991.

[We] R. Weinstock, *Calculus of Variations*, New York, Dover, 1987.

[Wl] J. Wloka, *Partial Differential Equations*, Cambridge University Press, 1987.

[V] M. Vajta, M. Tiernego, P. Lohnberg, J. van Dijk, *Coupled bending and torsional vibrations: how it works in flexible robot arms*, 8-th World Congress on the Theory of Machines and Mechanisms, August 26-31, 1991, Prague, Czechoslovakia.