Structure Analysis Including Kinematical Hardening and Creep

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Résumé

Pour une structure ou un matériau viscoélastique et plastique, nous proposons un modèle avec paramètres internes et paramètres transformés associés. Les constantes introduites sont faciles à identifier expérimentalement. Nous pouvons alors obtenir aisément des bornes "statiques" et "dynamiques" par un calcul élastique , sans suivre l'évolution complète de la structure. De la même façon, pour une période donnée du chargement, nous évaluons des bornes plus précises à partir de la réponse purement viscoélastique de la structure qui est donnée par des équations différentielles du premier ordre. Nous avons de plus une idée de la vitesse de convergence.

Abstract

For a viscoelastic and plastic material or structure, we propose a model, including internal strain parameters and associated transformed parameters, on which experimental identification of the constants can easily be performed. Then we show how to derive easily "static" and "dynamic" bounds for a structure under cyclic loading. They are obtained through an elastic calculus without following the whole evolution of the structure. Then, the linear purely viscoelastic response of the structure is calculated through linear differential equations of first order and more precise bounds, depending on the period are established. Moreover, the speed of convergence towards the limiting state can be evaluated.

1. Introduction

Creep phenomena can no longer be neglected when high temperatures are involved or for some materials like austenitic steel at room temperature. Previously, we proposed a general framework [2], [3]with internal parameters and associated transformed parameters which appeared to be particularly convenient for structures under cyclic loadings. In this model, local stress evolution is the sum of the purely linear elastic evolution under the particular loading and an inelastic part. Elastic or plastic shakedown or ratcheting can be easily predicted and bounds for the limiting state may be derived.

In this paper, we use this framework in a very simplified form for a plastic and viscoelastic material which is just sufficient to find again some recent recommendations by Oak Ridge National Laboratory. Moreover, we indicate how to use it precisely during cyclic loadings. Since one essential feature of the model is that the treatment of the structure and the treatment of the elementary volume are similar, we shall consider only a structure reduced to one elementary volume.

2. Local equations

We take the inelastic strains as internal parameters for the material. Here, they are assumed to be represented only by two tensors $\underline{\alpha}$ and $\underline{\beta}$: $\underline{\alpha}$ describes instantaneous plastic strains and $\underline{\beta}$ accounts for creep strains. These inelastic deformations generate local residual stresses on the elementary inelastic mechanisms, $-\hat{\underline{\alpha}}$ and $-\hat{\underline{\beta}}$. We call $(\hat{\underline{\alpha}}, \hat{\underline{\beta}})$ transformed parameters. We showed that $\hat{\underline{\alpha}}$ and $\hat{\underline{\beta}}$ are linearly dependent on $\underline{\alpha}$ and $\underline{\beta}$ through a symmetric $\boldsymbol{B}$ matrix. Here this matrix is derived from the isotropic linear elastic tensor $\mathbf{L}$:

$$\begin{pmatrix} \hat{\underline{\alpha}} \\ \hat{\underline{\beta}} \end{pmatrix} = \boldsymbol{B} \begin{pmatrix} \underline{\alpha} \\ \underline{\beta} \end{pmatrix} = \begin{pmatrix} \ell\mathbf{L} & n\mathbf{L} \\ n\mathbf{L} & m\mathbf{L} \end{pmatrix} \begin{pmatrix} \underline{\alpha} \\ \underline{\beta} \end{pmatrix} \tag{1}$$

where ℓ, m and n are constant scalars and obey the following inequalities so that $\boldsymbol{B}$ is regular :

$$\ell > 0 \qquad m > 0 \qquad m - n^2 > 0 \tag{2}$$

Usually, since no change of volume occurs during inelastic deformation, $\underline{\alpha}$ and $\underline{\beta}$ will be deviatoric tensors. Their evolution is assumed by the local stresses $\sigma^{(1)}$ and $\sigma^{(2)}$ on the two elementary inelastic mechanisms, which should also be deviatoric.

$$\begin{aligned} \underline{\sigma}^{(1)} &= \mathbf{A}^{(1)}\underline{\sigma} - \hat{\underline{\alpha}} = \operatorname{dev} \underline{\sigma} - \ell\mathbf{L}\underline{\alpha} - n\mathbf{L}\underline{\beta} = \underline{s} - 2\mu\ell\underline{\alpha} - 2\mu n\underline{\beta} \\ \underline{\sigma}^{(2)} &= \mathbf{A}^{(2)}\underline{\sigma} - \hat{\underline{\beta}} = k \operatorname{dev} \underline{\sigma} - n\mathbf{L}\underline{\alpha} - m\mathbf{L}\underline{\beta} = k\underline{s} - 2\mu n\underline{\alpha} - 2\mu m\underline{\beta} \end{aligned} \tag{3}$$

where $\hat{\underline{\alpha}}$ and $\hat{\underline{\beta}}$ are deviatoric as $\boldsymbol{B}$ is isotropic ; $\underline{\sigma}$ is the actual loading with its deviator $\underline{s}$; $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ are the elastic localization tensors on the two mechanisms.

The global elastic and inelastic strains ε^e and ε^p are here thus written :

$$\begin{aligned} \underline{\varepsilon}^p &= \mathbf{A}^{(1)T}\underline{\alpha} + \mathbf{A}^{(2)T}\underline{\beta} = \underline{\alpha} + k\underline{\beta} \\ \underline{\varepsilon}^e &= \mathbf{L}^{-1}\underline{\sigma} = \frac{1+\nu}{E}\underline{\sigma} - \frac{\nu}{E}\operatorname{tr}(\underline{\sigma})\underline{1} = \frac{1}{2\mu}\underline{\sigma} - \frac{\nu}{E}\operatorname{tr}(\underline{\sigma})\underline{1} \end{aligned} \tag{4}$$

The evolutions laws are given by :

$$\begin{aligned} &\dot{\underline{\alpha}} \in \partial \Psi_{c(\underline{\sigma})}(\underline{\sigma}^{(1)}) \Leftrightarrow \dot{\underline{\alpha}} \in -\partial \Psi_{\hat{c}(\hat{\underline{\alpha}})}(\hat{\underline{\alpha}}) \\ &\dot{\underline{\beta}} = \underline{\sigma}^{(2)} / \eta = (\mathbf{A}^{(2)}\underline{\sigma} - \hat{\underline{\beta}}) / \eta \end{aligned} \tag{5}$$

where $\hat{C}(\underline{\alpha})$ is obtained from the convex of elasticity C :

$$\sigma^{(1)} \in C \Leftrightarrow \frac{1}{2}\,\sigma^{(1)}.\,\sigma^{(1)} - \sigma_0^2 \leqslant 0 \qquad (6)$$

through a translation of vector $\mathbf{A}^{(1)}\underline{\alpha}$ in the $\hat{\underline{\alpha}}$ space.

3. "Static" and "dynamic" bounds

3.1 "Dynamic" bound

For a very rapid loading, no viscous effect can appear. We have purely kinematical hardening evolution with actual $E,\nu,h = 3\mu\ell$ (hardening modulus) and σ_0 (yield criterion). J. Zarka and J. Casier [1] showed how to solve this elastoplastic problem. Experimental data will give E,ν,ℓ and σ_0 . Viscoelastic relaxation has no time to develope, so we have a lower bound for inelastic strain $\underline{\varepsilon}^p$.

3.2 "Static" bound

If the loading is very slow, no stresses appear on the dashpots modelling the creep ; therefore, $\sigma^{(1)}$ is zero and the creep strain tensor is given by :

$$\underline{\dot{\varepsilon}} = \frac{1}{m}\left(\frac{k}{2\mu}\,\underline{\dot{s}} - n\underline{\alpha}\right) \qquad (7)$$

As in § 3.1, we come back to study a structure with kinematical hardening only, but new apparent constants are involved :

$$\underline{\alpha}^{(1)S} = \underline{s} - 2\mu\left[(\ell - n^2/m) \,/\, (1/kn/m)\right]\underline{\alpha} = \underline{s} - h_s\underline{\alpha}$$

$$|\underline{\alpha}^{(1)s}| \leqslant \sigma_0{}^s = \sigma_0 \,/\, |1 - kn/m| \qquad (8)$$

$$\underline{\varepsilon} = \frac{1 - 2\nu s}{3E_s}\,\mathrm{tr}(\underline{\sigma})\,\underline{1} + \frac{1 + \nu s}{E_s}\,\underline{s} + \left(1 - \frac{kn}{m}\right)\underline{\alpha}$$

with $E_s = 3E/(3 + 2(1+\nu)k^2/m)$

$$\nu_s = (3\nu + (1+\nu)\,k^2/m)/(3 + 2(1+\nu)\,k^2/m)$$

We can thus derive from experiment the ratios k^2/m and kn/m. The remaining constant will be obtained through a medium speed loading. Table 1, from [5] gives for example the experimental data for stainless steel :

TABLE 1

Loading	E (MPa)	h(MPa)	ν	σ_y(MPa)
Static ($8\ 10^{-2}$ MPa/s)	170 400	138 700	0.33	101
Dynamic (4800 MPa/s)	188 500	149 500	0.35	142

Identified values : μ = 70 000 MPa ν = 0.35 σ_0 = 80 MPa
$k = 1$ $m = 9$ $n = -1$ $\ell = 0.75$
$\eta = 10^7$ MPa/s (medium speed loading)

For this static loading, we have an upper bound for the inelastic strains $\underline{\varepsilon}^p$ as relaxation effects are maxima.

4. Improved bounds

4.1. Purely viscoelastic cyclic response

To derive closer bounds, we have now to describe more precisely the influence of the period. From shakedown theorems, we know that a cyclic stress response exists and is unique. So, for a regular ***B***, cyclic local and global strain responses will always exist. At first, we calculate the purely viscoelastic cyclic response of the structure ; that is to say the cyclic solution of (3) and (5) when $\underset{\sim}{\alpha}$ is zero and σ_0 infinite. The system is reduced here to five independant scalar equations, the solution of which depends on the radial loading :

$$\eta\dot{\beta}_{ij} + 2\mu m\beta_{ij} = ks_{ij} = ks^{o}_{ij} + k\, g(t)\, \frac{\Delta s_{ij}}{2} \qquad (9)$$

where $\underset{\sim}{s}^{o}$ is the midvalue of $\underset{\sim}{s}$, $\underset{\sim}{\Delta s}$ the amplitude of $\underset{\sim}{s}$ and g(t) a periodic function such as

$$-1 \leqslant g(t) \leqslant 1 \qquad g(t+T) = g(t) \qquad (\forall t) \qquad (10)$$

The viscoelastic response β^{v} and viscoelastic global strain ε^{v} can be written :

$$\beta^{v}_{ij} = \frac{k}{2\mu m}\left(s^{o}_{ij} + \frac{1}{2}\Delta s_{ij}\, G(t)\right)$$
$$\varepsilon^{v}_{ij} = \frac{1-2\nu}{3E}(\sigma)_{ij} + \left(1 + \frac{k^2}{2\mu m}\right)s^{o}_{ij} + \frac{\Delta s_{ij}}{4}\left(g(t) + \frac{k^2}{m}\, G(t)\right) \qquad (11)$$

where $(\eta/2\mu m)\dot{G}(t) + G(t) = g(t)$

So we have to solve one linear differential equation of first order.

You will find in table II different forms of G(t) for various g(t)

Table II

Schema	Type	g(t)	G(t)
⎍⎍	rectangular	ε	$\varepsilon(1 - 2\exp(-2\mu m\tau/\eta)/(1+f)$
/\/	linear	$\varepsilon(1 - 4\tau/T)$	$\varepsilon\left[1 - \frac{4\tau}{T} + \frac{2\eta}{\mu mT}\,\frac{(1 - 2\exp(-2\mu m\tau/\eta))}{1+f}\right]$
∿	sinusoïdal	$-\cos(2\pi t/T)$	$-\frac{(\cos(2\pi t/T) + (\pi\eta/mT)\sin(2\pi t/T)}{1 + \pi^2\eta^2/\mu^2 m^2 T^2}$

$f = \exp\left(\frac{-\mu m t}{\eta}\right)$ and $0 \leqslant \tau \leqslant \frac{T}{2}$ $\quad \varepsilon = +1 \quad t = (N + 1/2)T + \tau$ $\quad \varepsilon = -1 \quad t = NT + \tau$

4.2 Improved bounds

Let us study now the difference between this solution and the realone. By putting :

$$\underset{\sim}{\beta} = \underset{\sim}{\beta}^{v} + \underset{\sim}{\beta}^{1} \quad \text{and} \quad \begin{pmatrix} \underset{\sim}{\hat{\alpha}}^{1} \\ \underset{\sim}{\hat{\beta}}^{1} \end{pmatrix} = \boldsymbol{B}\begin{pmatrix} \underset{\sim}{\alpha} \\ \underset{\sim}{\hat{\beta}}^{1} \end{pmatrix} \qquad (12)$$

The evolution laws are turned to a simpler form :

$$\dot{\underset{\sim}{\hat{\beta}}}^{1} = -\underset{\sim}{\hat{\beta}}^{1}/\eta = \sigma^{\star(2)}/\eta$$
$$\underset{\sim}{\alpha}^{(1)} = \underset{\sim}{s} - 2\mu\eta\underset{\sim}{\beta}^{v} - \underset{\sim}{\hat{\alpha}}^{1} = \underset{\sim}{s}^{1}_{WT} - \underset{\sim}{\hat{\alpha}}^{1} \in C^{0} \qquad (13)$$

In $\underline{\alpha}$ and $\underline{\beta}^1$ variables, a new problem has been defined with the same $\boldsymbol{B}$ matrix and new localization tensors varying with the period : the influence of the global loading disappears on the dashpots $\underline{\beta}^1$ but is apparent in the block $\underline{\alpha}$ through $\underline{s}^1_T$ equal to $\underline{s} - 2\mu n \underline{\beta}^v$.

In the local residual stresses space ($\hat{\underline{\alpha}}^1, \hat{\underline{\beta}}^1$ space) we can find "dynamic" and "static" bounds for $\hat{\underline{\alpha}}^1$ and $\hat{\underline{\beta}}^1$ as indicated in § 3.1 and § 3.2 . To obtain the first one we have to solve :

$$\begin{aligned} &\beta^{1d}_T \equiv 0 \qquad \hat{\underline{\beta}}^{1d}_T = 2\mu n \underline{\alpha}^d_T \\ &\underline{s}^1_T - 2\mu \ell \hat{\alpha}^{1d}_T = \underline{s}^1_T - 2\mu \ell \hat{\alpha}^{1d}_T \in C_0 \end{aligned} \tag{14}$$

The second bound relies on the hypothesis $\sigma^{*(2)}$ equal zero and is calculated through :

$$\begin{aligned} &\hat{\underline{\beta}}^{1S}_T = 2\mu n \underline{\alpha}^S_T + 2\mu m \, \beta^{1S}_T \equiv 0 \Leftrightarrow \beta^{1S}_T = -(n/m)\alpha^S_T \\ &\underline{s}^1 - \hat{\underline{\alpha}}^{1S} = \underline{s}^1_T - 2\mu (\ell - n^2/m) \underline{\alpha}^S_T \in C_0 \end{aligned} \tag{15}$$

Once again the solution is given in [1]. We come back to pure plastic problem to determine the bounds for $\hat{\alpha}^1$ and $\hat{\beta}^1$.

As $\hat{\underline{\alpha}}$ and $\hat{\underline{\beta}}$ are obtained from $\hat{\underline{\alpha}}^1$ and $\hat{\underline{\beta}}^1$ from a mere translation, we deduce real bounds for $\hat{\underline{\alpha}}$ and $\hat{\underline{\beta}}$ in their plane.

$$\begin{aligned} &\hat{\underline{\alpha}} \in [\hat{\underline{\alpha}}^d_t, \hat{\underline{\alpha}}^S_T] \quad \text{with} \quad \hat{\alpha}^d_T = \hat{\alpha}^{1d}_T - 2\mu n \underline{\beta}^v \\ &\qquad\qquad\qquad\qquad\qquad\quad \hat{\underline{\alpha}} \quad \hat{\underline{\alpha}}^{1d}_T - 2\mu n \underline{\beta}^v \\ &\hat{\underline{\beta}} \in [\hat{\underline{\beta}}^d_t, \hat{\underline{\beta}}^S_T] \quad \text{with} \quad \hat{\beta}^d_T = \hat{\beta}^{1d}_T - 2\mu m \underline{\beta}^v \\ &\qquad\qquad\qquad\qquad\qquad\quad \hat{\beta}^S_T = \hat{\beta}^{1S}_T - 2\mu m \underline{\beta}^v \end{aligned} \tag{16}$$

For a regular $\boldsymbol{B}$ matrix, we obtain then the corresponding bounds for $\underline{\alpha}$ and $\underline{\beta}$

$$\underline{\alpha} \in [\underline{\alpha}^d_T \, , \, \underline{\alpha}^S_T] \qquad \underline{\beta} \in [\underline{\beta}^v \quad \underline{\beta}^v - (n/m)\underline{\alpha}^S_T] \tag{17}$$

and for a positive value of k we finally can found the global inelastic strain between

$$\varepsilon^p \in [\alpha^d_T + k\underline{\beta}^v \, , \, \underline{\alpha}^S_T (1 - kn/m) + k\underline{\beta}^v]$$

For example, we show the results obtained with this method for a one dimensional model with a rectangular loading. One can see in figure 1 the improvements obtained from the simple loading proposed in § 3.

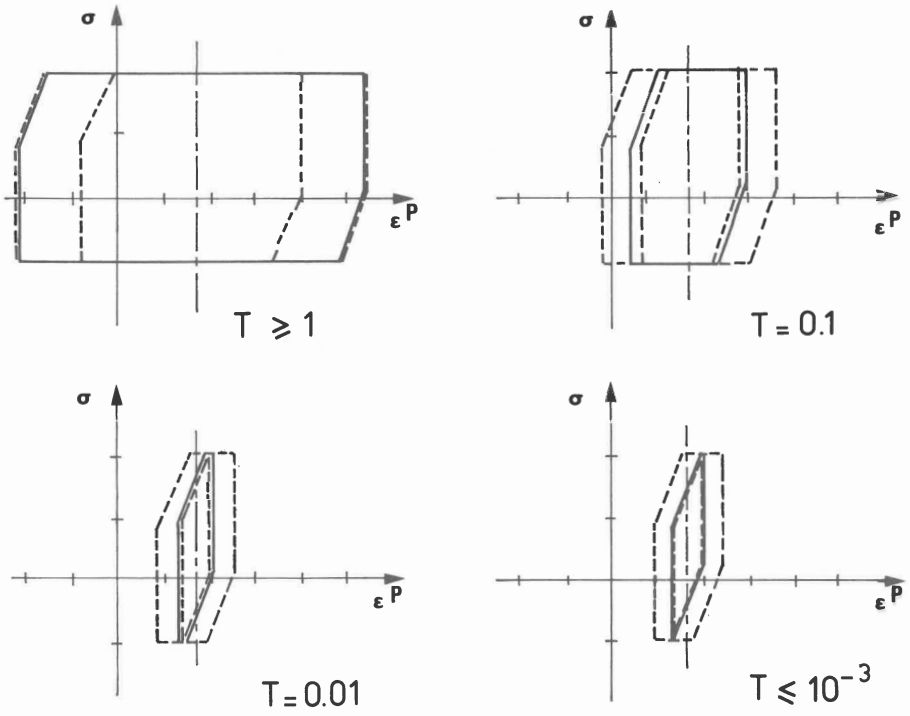

Fig. 1 - Bound's variation against the period T

5. Speed of convergency towards the limiting state during the real loading

Usually, one may notice that viscoplastic shakedown is obtained more quickly than viscoelastic shakedown. Therefore we limit ourselves to a study of the speed of convergence towards the viscoelastic limiting state.

For a given period T, this will occur when all the convex sets $\hat{C}(s_T^1(t))$ deduced from C° by the translations $s_T^1(t)$, for all values of t in a period, have a non-void intersection $\hat{C}_\ell$. As $\hat{\beta}^1$ will always approach zero, we can take $|\hat{\beta}^1(t_o+NT)|/2$ k for a normalized criterion for convergence. N will be the number of cycles and t_o will be chosen later.

The speed of convergence will be determined immediatly when no plastic strains α appears and the representative point $(\hat{\alpha}, \hat{\beta}^1)$ always stay in $\hat{C}_\ell$.

$$\frac{|\hat{\beta}^1|}{2\mu k} \leqslant r \quad \Leftarrow \quad N \geqslant \frac{\eta}{2\mu m T} \cdot \frac{\mathrm{Log}(1/r)}{\mathrm{Log}(|\hat{\beta}^1(o)|/2\mu k)} \tag{19}$$

If we suppose $\hat{\beta}(o)$ to be nul, $\hat{\beta}^1(o)$ will be opposite to $\hat{\beta}^v(o)$.

For a viscoplastic evolution towards a viscoelastic limiting state some additional difficulties appears. The evolution of the various components are linked through the yield criterion :

$$\Sigma (s^1_{Tij} - 2\mu\ell\alpha_{ij} - 2\mu n\beta^1_{ij})^2 \leqslant \gamma_0^2 \tag{20}$$

and the representative point $(\hat{\alpha}_\ell, \hat{\beta}^1_\ell)$ leaves $\hat{C}_\ell$ on each period and comes back on its frontier with a lower value of $|\hat{\beta}^1_\ell|$.

For a one-dimensional system, we proved the speed of convergence N always neighbours

$$\frac{\eta}{2\mu T} \frac{\ell}{\ell^{m-n}} \left[\mathrm{Log}\left(\frac{1}{r}\right) / \mathrm{Log}\left(|\hat{\beta}^1_\ell(o)| / 2\mu k\right) \right]$$

6. Conclusion

A general structure will have exactly the same treatment. Therefore, we are now able to approximate rather precisely the limiting state of a viscoplastic structure following § 3 and 4 for a radial loading. Additional and more detailed results on the speed of convergence will be available in a post-SMIRT Conference.

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