Infinite Substructuring for Unbounded Continua

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ABSTRACT

In the design of nuclear power plants it is essential to account for the soil-structure interaction effects. Deformation of the supporting soil media significantly modifies dynamic responses of superstructures. Such observations have generated significant interest in: i) modeling of foundations, ii) identifying dynamic inputs, iii) developing algorithm for interaction computations. In this paper, attention will be confined to a finite element representation of foundations.

It is sufficient to carry out a linear analysis for the purpose of preliminary design. A deep homogeneous soil layer is conventionally idealized as a viscoelastic half space. Frequency dependent dynamic stiffness matrices provide the boundary condition for superstructures. Dynamic analysis is subsequently carried out in the frequency domain according to the substructure method.

An unbounded region is covered with an infinite collection of subregions. These finite element type "cells" are all similar in geometry. Each such cell has an equal number of degrees-of-freedom on the inner and outer faces. It is shown that there exists a family of shape functions related by a scalar multiplier. A quadratic eigenvalue problem can then be constructed in order to numerically evaluate those complex scalar multipliers (as eigenvalues) and the corresponding shape functions (as eigenvectors). The mass and stiffness matrices of a cell furnish the matrix coefficients in the eigenvalue problem. Contraction of outward displacement fields is enforced in order to prevent any incoming wave.

The infinite substructuring operation, which extends over the entire mesh, is shown to be possible. For a cell each "strain energy" term, which is associated with a pair of shape functions, forms a converging geometrical series. This series can be summed up exactly leading to the elements in the dynamic stiffness matrix pertaining to the unbounded medium.

An illustrative numerical example of a three-dimensional half space is presented.

1. INTRODUCTION

Safe designs of nuclear power plants from earthquake hazards have initiated rigorous schemes of analyzing the effects of dynamic interactions of superstructures with supporting soil regions. Conventional finite element modeling [1] for structures has been proven to be numerically accurate as well as computationally economical. The soil domain, on the other hand, is too large to be meshed with reasonable numbers of bounded elements. Hence in almost all computational schemes, e.g. boundary integral method [2], infinite element formulations [3], continuum [4] and associated hybrid [5] models are employed. These methods invariably involve complicated infinite, improper integrals of higher order transcendental functions. Therefore, enormous mathematical difficulties are faced in formulating computational schemes. In this paper an approximate method is presented to evaluate the dynamic stiffness matrices of uniform, linearly elastic unbounded soil regions. The conventional technique of finite element substructuring is utilized. In order to cover the entire unbounded media an infinite number of substructuring operations are implemented algebraically. The latter involves only the summation of the very familiar infinite geometrical series.

The dynamic stiffness matrix $\underline{D}$ for the unbounded medium is illustrated in Fig. 1. The restraining force required at αth degree-of-freedom (d.o.f.) due to a unit harmonic excitation of frequency ω(i.e. $\exp(i\omega t)$, $i = \sqrt{-1}$, t:time) at the βth d.o.f. is $D_{\alpha\beta} \exp(i\omega t)$. The waves, which are generated due to the interaction effects, do not get reflected back to the interface. This accounts for an energy loss (called radiation damping) term. Thus $D_{\alpha\beta}$ contains an imaginary part. An important distinctive characteristic of dymanic stiffness matrices pertaining to elastic unbounded media is that their elements are in general complex (except below the cut-off frequency for two-dimensional layer problems). The purpose of this presentation is to illustrate a method in which the imaginary parts of the dynamic stiffness matrices could be calculated starting from real mass and stiffness matrices of finite element regions.

There were attempts in the past to carry out a large number of individual substructure operations so as to capture the unboundness characteristics of infinite regions [6]. For Laplace's equation (elliptic partial differential equations) such truncation sufficed. However, it is not possible to construct a complex dynamic stiffness matrix via substructuring real mass and stiffness matrices. In the case of elastodynamic solutions (of hyperbolic partial differential equations) the method [6] fails to yield the radiation damping terms. A stand alone finite element scheme, viz. the cloning algorithm, could satisfactorily reproduce the dynamic stiffness matrices, [7] and [8]. An elegant presentation of some important limitations and illustrative examples of that formulation can be found in [9]. In this paper a close alternative is examined via a direct substructuring scheme.

2. DERIVATION

2.1. Geometrical Discretization

An unbounded domain is conceived to be an infinite collection of geometrically similar cells, $\Omega^{(1)}$, $\Omega^{(2)}$,... A cell $\Omega^{(i)}$ is bounded by the inner and outer surfaces, $\partial\Omega^{(i)}$ and $\partial\Omega^{(i+1)}$, respectively, which contain equal numbers of geometrically situated nodes. The characteristic length for $\partial\Omega^{(i)}$ is denoted by $L^{(i)}$, thus:

$$\frac{L^{(2)}}{L^{(1)}} = \frac{L^{(3)}}{L^{(2)}} = \ldots = \frac{L^{(i+1)}}{L^{(i)}} = \gamma \ \text{(say)}. \qquad (1)$$

2.2. Assumption of Static Fields as Shape Functions

In conventional finite element procedures the stiffness, [K], and mass, [M], matrices adequately represent an elastic element. The stiffness matrix characterizes the static behavior hence is calculated from admissible static displacement fields. The consistent mass matrix is also calculated by using the same shape functions. Consequently the frequency dependent dynamic stiffness matrix $[D](\omega)$ assumes the form:

$$[D](\omega) = [K] - \omega^2 [M]. \qquad (2)$$

This frequency independent representation of [K] and [M] is quite satisfactory for the low frequency range which dominates the earthquake excitations. However, for more accurate studies, frequency dependent shape functions have been proposed. The approximate mass and stiffness matrices (which are calculated on the basis of static displacement fields) for an unbounded region will be denoted in this paper by $\underline{X}$ and by $\underline{Z}$, respectively.

2.3. Evaluation of Static Fields

In the case of unbounded media dynamic excitations cause radiation damping losses. It is therefore appropriate to construct a set of complex shape functions for the entire domain. The substructure method [1], furnishes a useful procedure to evaluate such fields. Consider the relation between the static stiffness matrices: $\underline{Z}^{(1)}$, $\underline{Z}^{(2)}$, $\underline{K}^{(1)}$ pertaining to

$$B^{(1)} = \bigcup_{i=1}^{\infty} \Omega^{(i)}, \quad B^{(2)} = \bigcup_{i=2}^{\infty} \Omega^{(i)} \text{ and } \Omega^{(1)}, \quad \text{respectively.}$$

According to the substructure equation:

$$\underline{Z}^{(1)} = \underline{K}_{11}^{(1)} - \underline{K}_{12}^{(1)} [\underline{Z}^{(2)} + \underline{K}_{22}^{(1)}]^{-1} \underline{K}_{21}^{(1)} \qquad (3)$$

The partitioned matrix $\underline{K}_{\alpha\beta}^{(i)}$ refers to the element $\Omega^{(i)}$ and $\alpha,\beta = 1$ for $\partial\Omega^{(i)}$ and $\alpha,\beta = 2$ for $\partial\Omega^{(i+1)}$ degrees-of-freedom.

From dimensional analysis $\underline{Z}^{(2)} = \underline{Z}^{(1)}$ for two-dimensional cases and $\underline{Z}^{(2)} = \gamma \underline{Z}^{(1)}$ for three-dimensional problems. It has been shown in [8] that the eigenvalues $\{\lambda_\alpha\}$ and eigenvectors $[\underline{\chi}_\alpha]$ for the solvent $\underline{Y}$ of

$$\underline{Y}\,\underline{Y} - \underline{Q}\,\underline{Y} + \underline{R} = \underline{0}\ ; \quad \underline{Y}\,\chi_\alpha = \lambda_\alpha\,\chi_\alpha \tag{4}$$

are related to a set of shape functions $[\underset{\sim}{w}_\alpha]$, which attentuates between a pair of consequetive boundaries $\partial\Omega^{(i)}$ and $\partial\Omega^{(i+1)}$ to $[\theta_\alpha \underset{\sim}{w}_\alpha]$. The quadratic matrix equation in eq. (4) is constructed from eq. (3) by setting:

$$\begin{aligned}
\underline{Y} &= [f^{1/2}\,\underline{Z}^{(1)} + f^{1/2}\,K_{22}^{(1)}]^{-21} \\
\underline{Q} &= [f^{1/2}\,\underline{K}_{11}^{(1)} + f^{1/2}\,\underline{K}_{22}^{(1)}]\ [\underline{K}_{21}^{(1)}]^{-1} \\
\underline{R} &= [\underline{K}_{12}^{(1)}]\ [\underline{K}_{21}^{(1)}]^{-1}
\end{aligned} \tag{5}$$

The relations between λ_α, θ_α, χ_α, and $\underset{\sim}{w}_\alpha$ are:

$$\theta_\alpha = -\frac{1}{\sqrt{f}\lambda_\alpha} \quad \text{and} \quad \underset{\sim}{w}_\alpha = [\underline{K}_{21}^{(1)}]^{-1}\,\chi_\alpha \tag{6}$$

in which $f = \gamma$ for three-dimensional problems and $f = 1$ for two-dimensional cases. In order not to admit incoming waves only that set of $\{\lambda_\alpha\}$ and $\{\chi_\alpha\}$ are chosen for which:

$$\left| \sqrt{f}\,\theta_\alpha \right| < 1. \tag{7}$$

2.4 Stiffness and Mass Matrices for Unbounded Domains

Consider a cell $\Omega^{(i)}$ in Fig. 1. If $\partial\Omega^{(1)}$ is subjected to a displacement profile $\underset{\sim}{w}_\alpha$ then the nodal displacements along $\partial\Omega^{(i)}$ and $\partial\Omega^{(i+1)}$ are $\theta_\alpha^{i-1}\underset{\sim}{w}_\alpha$ and $\theta_\alpha^{i}\,\underset{\sim}{w}_\alpha$, respectively. Thus the virtual work (due to strain energy) associate with a pair of such shapes $\underset{\sim}{w}_\alpha$ and $\underset{\sim}{w}_\beta$ is then

$$u_{\alpha\beta}^{(i)} = \begin{matrix} \theta_\beta^{i-1}\underset{\sim}{w}_\beta, & \theta_\beta^{i}\,\underset{\sim}{w}_\beta \end{matrix} \quad \begin{matrix} \underline{K}_{11}^{(i)} & \underline{K}_{12}^{(i)} \\ \underline{K}_{21}^{(i)} & \underline{K}_{22}^{(i)} \end{matrix} \quad \begin{matrix} \theta_\alpha^{i-1} & \underset{\sim}{w}_\alpha \\ \theta_\alpha^{i} & \underset{\sim}{w}_\alpha \end{matrix} \tag{8}$$

$$= (\theta_\beta \theta_\alpha f)^{i-1}\ u_{\alpha\beta}^{(1)} \tag{8}$$

since $\underline{K}_{\alpha\beta}^{(i)} = (f)^{i-1}\ \underline{K}_{\alpha\beta}^{(1)}$. Now the total virtual $U_{\alpha\beta}$ work for the unbounded domain $B^{(1)} \equiv \bigcup_{i=1}^{\infty} \Omega^{(i)}$ is the infinite sum:

$$U_{\alpha\beta} = \sum_{i=1}^{\infty} u_{\alpha\beta}^{(i)} \tag{9}$$

The geometric series eq.(9) can be exactly summed to yield:

$$U_{\alpha\beta} = u_{\alpha\beta}^{(1)} / (1- f\ \theta_\alpha \theta_\beta) \ .$$

By following the same procedure the virtual work due to kinetic energy $V_{\alpha\beta} = v_{\alpha\beta}^{(1)} (1-\gamma^2 f \theta_\alpha \theta_\beta)$ in which v is obtained from eq. (8) by replacing the stiffness submatrices $\underline{K}$'s by the corresponding mass submatrices $\underline{M}$'s. Note that:

$$\underline{M}_{\alpha\beta}^{(i)} = f\gamma^2 \underline{M}_{\alpha\beta}^{(1)}$$

The mass and stiffness matrices $\underline{X}$ and $\underline{Z}$ for $B^{(1)}$ are then

$$\underline{X} = \underline{P}^T\ \underline{V}\ \underline{P}\ ; \quad \underline{Z} = \underline{P}^T\ \underline{U}\ \underline{P}, \quad \underline{P} = [\underset{\sim}{w}_i]^{-1} \tag{10}$$

Hence the dynamic stiffness matrix $\underline{D}(w)$ of eq. (2).

3. NUMERICAL EXAMPLE

Dynamic stiffness matrices of a homogeneous elastic half space with a cubic deletion, as shown in Fig. 2, is evaluated as an illustration of the proposed infinite substructuring scheme. The interaction boundary is discretized with 25 nodes (75 d.o.f.'s). The cell magnification parameter γ in eq. (1) is selected to be 1.2 for the finite element region $\Omega^{(1)}$. Linear displacement field is assumed to evaluate the cell stiffness matrix. Lumped mass model is accepted for that. The steps described in section 2 are implemented leading to the numerical values of $\underline{X}$ and $\underline{Z}$, the (equivalent, approximate) mass and stiffness matrices of the deleted half space.

In order to compare the results with a standard half space problem [4] a cubic block is assembled at the deletion boundary of Fig. 2. The rigid body compliances for horizontal and vertical excitations are compared in Figs. 3a and 3b. The results are quite satisfactory for the low frequency range. As anticipated, the infinite substructuring results, which are obtained from static shape functions, are rather inaccurate for middle and high frequency zones.

The numerical steps for the example was carried out in a DEC 20 computer. The infinite substructuring scheme required 172.3 seconds to evaluate the mass and stiffness matrices $\underline{X}$ and $\underline{Z}$ for the deleted half space. The hybrid formulation (for which the shape functions are calculated according to the continuum analysis) needed 2 minutes in IBM 370 as reported in [4]. Once the mass and stiffness matrices $\underline{X}$ and $\underline{Z}$ for the infinite region are evaluated the evaluation of the dynamic stiffness matrix, $\underline{D}$, for a given frequency according to $\underline{D}(w) = \underline{Z} - w^2\underline{X}$ hardly takes any computational effort. In the hybrid approach [4], however, for each frequency 2 minutes of time was spent in IBM 370. The relative computational efficiency of the proposed method of the infinite substructuring scheme can be readily ascertained.

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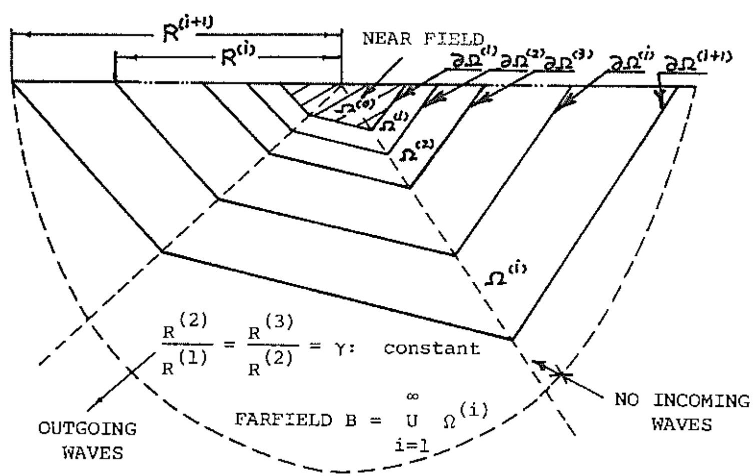

Fig. 1: Discretization

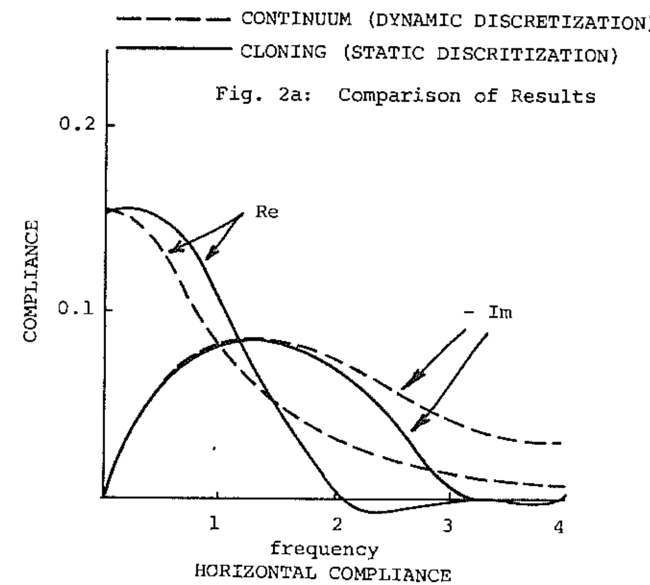

Fig. 2a: Comparison of Results

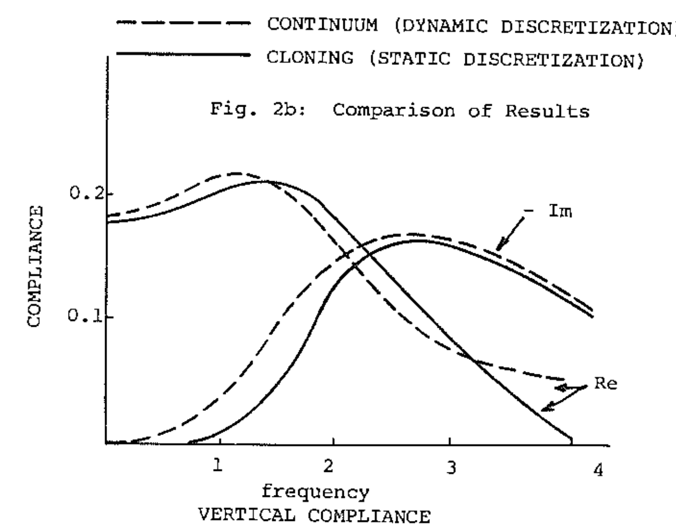

Fig. 2b: Comparison of Results