Reliability Index Estimation Based on Covariance Matrix and Finite Element Discretization of Stochastic Processes

Shigeru Nakagiri, Keiko Suzuki
University of Tokyo, Tokyo, Japan

1 INTRODUCTION

Fine element division is considered to result in accurate solution of the deterministic finite element method which affords the expectation of structural response. Finer element division is needed to evaluate the variance of structural response by the stochastic finite element method than the division used for the deterministic finite element analysis (Nakagiri et al., 1987). Probabilistic finite element methods deal with the stochastic process discretized in relation to the finite element division. The response variance by the first-order second-moment method (Ang and Tang, 1975) is calculated in terms of the covariance matrix of basic probabilistic variables which represent the stochastic process. This implies that the adequacy of element division can be judged in regard to the covariance matrix which reflects the discretization of the stochastic process.

The reliability index of the Advanced First-Order Second-Moment (AFOSM) method is calculated as the minimal distance between the origin of the standardized probabilistic variables and the design point on the limit state equation (Hasofer and Lind, 1974), (Grigoriu, 1982). The standardized probabilistic variables with zero mean and unit variance are derived from the independent basic variables through a linear transformation. The transformation is formulated on the basis of the covariance matrix which is input in the form of the autocorrelation function of the basic variables. This means that the adequacy of the discretization of the stochastic process can be examined in terms of the autocorrelation function (Der Kiureghian and Ke, 1988).

This paper deals with the examination of the reliability index estimation from the viewpoint of the covariance matrix. The numerical example is concerned with the reliability index for a critical frequency of a bending vibration model of the BWR Mark-II type reactor building, the Young's modulus of which is taken as spatial stochastic process. Two autocorrelation functions are examined.

2 ITERATIVE SEARCH OF RELIABILITY INDEX

The BWR Mark-II type reactor buliding is modeled as a cantilever beam consisting of ten sections as shown in Fig.1. The length of the model beam is L=73 m. The sections are further divided into several beam elements. The vertical distribution of the weight, cross-sectional area and moment of inertia of the sections is taken deterministic, and only the Young's modulus is assumed as probabilistic. The uncertainty of the Young's modulus E_m of the m-th element is expressed by Eq.1, where x_m stands for a probabilistic variable with zero mean and assigned to an element. The upper bar denotes the expectation term.

$$E_m = \bar{E}_m (1 + x_m) \quad (1)$$

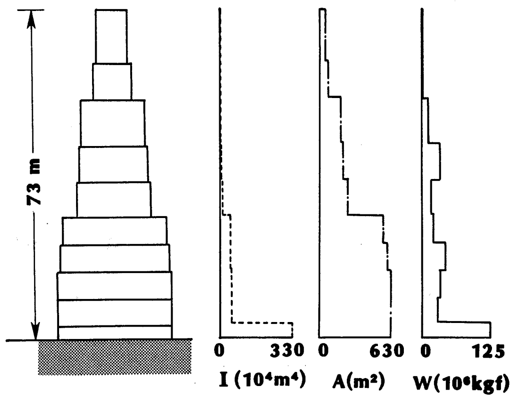

Figure 1 Cantilever beam model of BWR Mark-II reactor building

The uncertain Young's modulus causes the variability of the eigenvalue of undamped bending vibration through the stiffness matrix, while the lumped mass matrix determined by the distributed weight is kept constant. The uncertain eigenvalue can be approximated by the Taylor series expansion with respect to the probabilistic variables in number of M as given by Eq.2 in the truncated form at the first-order.

$$\lambda = \bar{\lambda} + \sum_{m=1}^{M} \lambda_m^I x_m \quad (2)$$

In the above, λ_m^I denotes the first-order rate of change of the eigenvalue with respect to the m-th basic probabilistic variables. The limit state equation is set as Eq.3 under a simple assumption that the model is safe when the eigenvalue is less than or equal to a critical value λ_c. In Eq.3, G_m^I indicates the sensitivity of the limit state equation calculated in terms of the m-th standardized probabilistic variable y_m.

$$\begin{aligned} G(y) &= \lambda_c - \lambda(y) \\ &= \bar{G} + \sum_{m=1}^{M} G_m^I y_m \geqq 0 \qquad \text{safe} \end{aligned} \quad (3)$$

The reliability index is searched by the method of Lagrange multiplier applied to the following functional Π (Shinozuka, 1983). The variability of λ is approximated by using the first-order sensitivities so that the minimization of Π with respect to y_m and Lagrange multiplier μ gives rise to a rough estimation of β^*.

$$\Pi = \sum_{m=1}^{M} y_m^2 + \mu(\lambda_c - \lambda(y)) \quad (4)$$

The iterative search of the exact β^* is devised by introducing the incremental form of y_m given by Eq.5 in order to cope with the deficient first-order approximation. In Eq.5, Δy_m^ℓ means the unknown to be determined, and ℓ denotes the iteration number. The minimization of the functional with respect to Δy_m^ℓ results in the matrix equation 6.

$$y_m^\ell = y_m^{\ell-1} + \Delta y_m^\ell \quad (5)$$

$$
\left[\begin{array}{ccc|c} 2 & & & G_1^I \\ & \cdot & & \cdot \\ & & \cdot & \cdot \\ & & 2 & G_M^I \\ \hline \text{SYM.} & & & 0 \end{array}\right]
\left\{\begin{array}{c} \Delta y_1^{\ell} \\ \cdot \\ \cdot \\ \Delta y_M^{\ell} \\ \hline \mu \end{array}\right\}
=
\left\{\begin{array}{c} -2y_1^{\ell-1} \\ \cdot \\ \cdot \\ -2y_M^{\ell-1} \\ \hline -\bar{G}^{\ell-1} \end{array}\right\} \tag{6}
$$

When the above procedure of determining Δy_m^{ℓ} converges well to the solutions of $\Delta y_m^{\ell}=0$ by means of renewing the expansion point of y_m^{ℓ}, the reliablity index β^* is calculated by the converged values y^*_m as given below.

$$
\beta^* = \{ \sum_{m=1}^{M} (y^*_m)^2 \}^{1/2} \tag{7}
$$

3 TRANSFORMATION OF BASIC VARIABLES TO STANDARDIZED VARIABLES

The statistical property of the basic probabilistic variables is supposed to be given in the form of the covariance matrix [C]. A linear transformation of the basic variables to the standardized probabilistic variables is defined by a general form of Eq.8, and the inverse transformation by Eq.9.

$$
\{y\} = [A]\ \{x\} \tag{8}
$$

$$
\{x\} = [A]^{-1}\{y\} = [B]\ \{y\} \tag{9}
$$

Transformation Based on Spectral Decomposition

The transformation matrices [A] and [B] are not determined uniquely. A way of the determination is to make use of the spectral decomposition of the covariance matrix (Dias and Nagtegaal, 1985), (Traina et al., 1986). In this way, the inverse transformation matrix, which is employed necessarily in the derivation of G_m^I from λ_m^I, is given by Eq.10, where λ_m and $[\Phi]$ denote the m-th eigenvalue and modal matrix of the covariance matrix, corresponding to the spectral decomposition of the covariance matrix [C] in the form of Eq.11.

$$
[B] = [\Phi]^T \mathrm{diag}[\sqrt{\lambda_m}][\Phi] \tag{10}
$$

$$
[C] = \sum_{m=1}^{M} \lambda_m \{\phi_m\} \{\phi_m\}^T \tag{11}
$$

Considerable effort has to be exerted to compute all the eigenpairs accurately in cases that the condition number and size of the matrix [C] are very large. In such cases, it may be sufficient to make use of the dominant modes only (Liu et al., 1987) to constitute the inverse transformation matrix [B]. The eigenvalue analysis has to be carried out to some extent, however, in order to know which modes are dominant in the spectral decompsition.

Transformation Based on LU Decomposition

It costs shorter CPU time to decompose the matrix [C] into the lower and upper triangular matrices [L] and [U] respectively than to carry out the full eigenvalue analysis. When the LU decomposition is available, the transformation matrix in the lower triangular form [A] can be determined by a simple matrix inversion based on Eq.12, where [I] indicates the unit matrix.

$$
[I] = [A]\ [L]\ [U]\ [A]^T \tag{12}
$$

The availability of the LU decomposition depends on the condition number of the covariance matrix [C]. The computer execution is likely to be interrupted at the first negative argument for square root in the decomposition procedure or to result in false decomposition in the case of large condition number. The modified Cholesky method, which is free from the square root computation, can be employed to cope with the tricky decomposition, but the negative argument for square root takes place again in the procedure to determine the lower triangular transformation matrix [A], giving rise to the halt of computer execution.

4 NUMERICAL EXAMPLE

The standardized probabilistic variables can be obtained easily through the linear transformation based on the spectral decomposition or LU decomposition of the covariance matrix of the basic variables, as stated in the preceding section, in cases that the condition number is small. Consequently, the evaluation of the reliability index can be expected correct. The condition number is governed by the finite element discretization of the stochastic process, and is likely to increase with the number of the element division for a stochastic process. The effect of the element division on the reliability index is examined in the following numerical example, where the covariance matrix is input in terms of the autocorrelation function. The number of the probabilistic variables equals that of the finite element division. The expectation and coefficient of variation c of the Young's modulus are taken equal to 20.6 GPa and 0.05, respectively. The expectation of the first natural frequency is 10.8 Hz. The critical eigenvalue is set as $\lambda_c=1.1\bar{\lambda}$. Equations 13 and 14 indicate the autocorrelation functions examined, where τ, D and 1/a denote the separation of the element centroids and scale of monotonic decay, respectively, and 1/f the wavelength of decay undulation.

$$R(\tau) = c^2 \exp(-\tau^2/D^2) \tag{13}$$

$$R(\tau) = c^2 \exp(-a\tau) \cos(2\pi f\tau) \tag{14}$$

Figure 2 shows the iteration history of the reliability index search in the case of 20 element division corresponding to the condition number equal to 270.0. The autocorrelation function of Eq.13 is input. Both the spectral decomposition and LU decomposition are available to generate the inverse transformation matrix [B. in this case. The LU decomposition is not successful, however, when the element number M is increased to 30. The condition number calculated is 0.8965×10^{16} while six eigenvalues can be judged equal to zero.

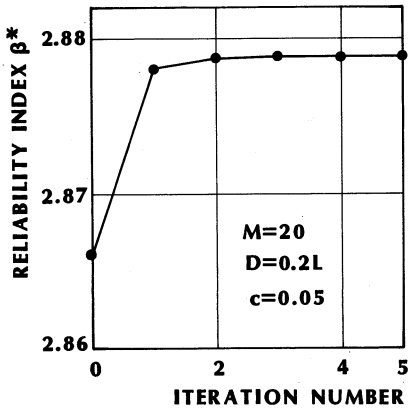

Figure 2 Iteration history of reliability index search

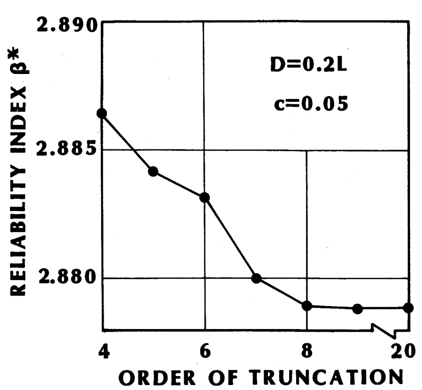

Figure 3 Effect of mode truncation

Figure 3 depicts as for the case of M=20 that only four dominant modes employed in the generation of the matrix [B] results in the false value of β^*, and that more than eight modes used gives rise to the exact value obtained by 20 modes. This implies that the transformation matrix can be approximated efficiently by the dominant modes taken in sufficient number of the truncated spectral decomposition.

Figure 4 illustrates the convergence of the reliability index with respect to the element division in the case that the autocorrelation function of Eq.14 is input. It can be said that more than 50 elements are required to evaluate the reliability index correctly and that the reliability index, converged once, does not vary with the number of the elements. The condition number in the case of M=60 is 491.9, both the spectral decomposition and LU decomposition being available. When the decay parameter a is decreased from a=6/L to a=2/L, that is, the correlation is rather intensified, the condition number is increased. Figure 5 shows the iteration history of the reliability index in the case of M=60. In this case, the condition number is equal to 2645 so that the LU decomposition cannot be exerted. The reliability index can be calculated as shown in the figure by means of the truncated spectral decomposition with 30 dominant modes. The number of the dominant modes is determined by a criterion that the modes, whose eigenvalue is less than 0.001908 of the largest eigenvalue, are omitted.

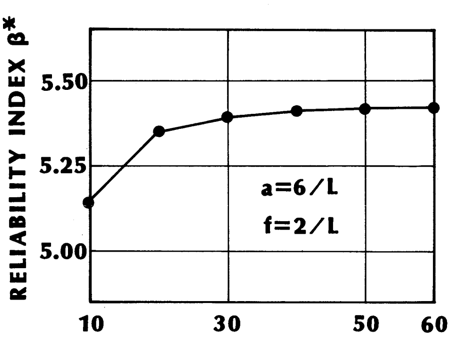

Figure 4 Convergence of reliability index with elements

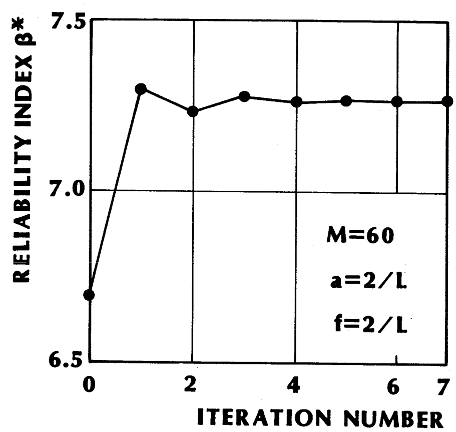

Figure 5 Iteration history obtained by truncated spectral decomposition

5 CONCLUDING REMARKS

The above numerical example indicates that the sufficient number of element division should be examined with respect to the input covariance matrix in order to evaluate the reliability index correctly. Thirty element division, that is, thirty probabilistic variables in this discretization model in a wave are required for the correct evaluation in the case that the autocorrelation function decays by undulation.

The condition number of the covariance matrix is increased with the number of the probabilistic variables for given autocorrelation functions. The LU decomposition of the covariance matrix is not available when the condition number is so large that the transformation matrix from the basic variables to standardized variables cannot be obtained.

The truncated spectral decomposition can be employed when the LU decomposition is not available because of large condition number. In doing so, the modes can be omitted, the eigenvalue of which is less than 0.002 of the largest eigenvalue. About a half of the whole modes can be omitted in the case of small condition number of this case study. The number of the dominant modes should be investigated further to generalize the conclusion for a variety of autocorrelation functions and discretization of stochastic processes.

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