A SECOND GENERATION FINITE ELEMENT COMPUTER PROGRAM FOR STRESS ANALYSIS

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SUMMARY

A second generation finite element computer program for stress analysis is under development. Incorporated in the computer program are finite elements which satisfy the completeness and continuity requirements for arbitrary order polynomial approximating functions. The distinguishing feature of the new algorithm is that it permits the user to exercise control over both the number of finite elements and the order of approximation over each element. Consequently, it is not necessary to define more finite elements than needed to specify the geometry of a structure. The order of polynomial approximating functions may be chosen either directly or indirectly; by specifying the required level of precision in terms of the quantities of interest (i.e. stresses, deflections, etc.). An automated iterative process then seeks the degree of approximation which corresponds to the specified level of precision.

An important advantage of the new algorithm is that it substantially increases the computational power of the finite element method. Comparisons with state-of-the-art computer programs indicated significant reductions in the number of finite elements needed and the number of variables employed. The reduction in the number of finite elements was by at least an order of magnitude in all cases. For example, in a benchmark study performed on "Lockheed test problem No. 2", (analysis of a cylindrical shell with rectangular cut-outs, AFFDL-TR-67-194), the number of finite elements employed by others ranged from 476 to 100. The first (research) version of the second generation computer program under development required only 10 elements. The number of variables ranged between 2457 and 637 in solutions presented by others, whereas Version 1 required only 500 variables. It is expected that future improvements will further reduce this number.

The new finite element stress analysis capability is, of course, applicable to all problems which can be solved by current finite element methods. The potential benefits are the greatest in applications where, due to the presence of steep stress gradients, mechanical fatigue controls design and in dynamic and non-linear analyses where the number of variables must be kept to a minimum in order to make numerical analysis feasible.

1. Introduction

Recent advances in the theory of the finite element method made it possible to develop highly user-oriented, very efficient finite element computer programs for stress analysis. Such a computer program is currently under development at Washington University in St. Louis, Missouri. The main features of the program are as follows:

(i) It is based on the displacement method of analysis;

(ii) Each displacement component over each finite element is approximated by a complete polynomial expression of order p, where p is arbitrary and may vary from element to element;

(iii) The polynomial approximating functions satisfy the interelement continuity conditions exactly.

These features permit users to obtain convergence either with respect to progressively reduced element sizes, as in the conventional finite element method, or with respect to increasing orders of polynomial approximation. The latter convergence process is the more attractive one because it does not require mesh refinement when the accuracy of approximation must be increased, and the computational efficiency (measured in terms of the number of variables needed to achieve a given level of precision) is known to increase rapidly as the order of the polynomial approximating functions is increased [2][3]. Also, the second convergence process makes it possible to incorporate an error analysis feature in the computer program which will permit users to specify the desired level of precision in terms of stresses or other computed quantities of interest. An automated process will then seek those orders of polynomial approximation which correspond to the specified level of precision.

A research-oriented version of the computer program has been developed and coding of user-oriented versions is in progress. The objective here is to present the essential elements of the computer program and to discuss briefly the results of computational experience with the new method.

2. The mathematical problem

The general form of the numerical problem associated with finite element approximation of elliptic boundary value problems is the following quadratic programming problem [3]:

$$\min \pi = \sum_{e=1}^{n} \left(\frac{1}{2} \lfloor a^e \rfloor [S^e] \{a^e\} - \lfloor a^e \rfloor \{Z^e\} \right) \qquad (1)$$

subject to the linear equality constraints:

$$[P]\,\{a\} = \{R\} \qquad (2)$$

where:

π	represents the energy functional
n	is the number of finite elements
$[S^e]$	is the unconstrained stiffness matrix of the e^{th} element
$\{a^e\}$	are the unknown coefficients of the polynomial approximating functions defined over the e^{th} finite element
$\{Z^e\}$	is the loading vector for element e
$[P]$	is the constraint matrix. It enforces the conformity requirements and principal boundary conditions
$\{a\}$	contains tha unknown polynomial coefficients for all finite elements
$\{R\}$	represents specified boundary conditions

The constrained minimization problem represented by eqs. (1,2) can be converted to an unconstrained minimization problem by finding linear dependencies among the variables in {a}, expressing the dependent variables in terms of the independent ones and eliminating them from the objective function π. The computational process is simplified considerably when nodal variables are introduced:

$$[T^e]\,\{a^e\} = \{\delta^e\} \qquad (3)$$

The nodal variables are values of the unknown function and/or its derivatives at specified points along the boundaries of element e. They are defined such that the desired interelement continuity conditions can be enforced either entirely or in part by equating the corresponding nodal variables at the common boundary of adjacent elements. Here we are concerned with the problem of enforcing the required (principal) continuity conditions for complete polynomial approximating functions of arbitrary order. We exclude from consideration both non-conforming and hyperconforming schemes for reasons to be discussed later.

C^0 continuity can always be enforced by element level transformations followed by the usual process of assembling the global stiffness matrix. The continuity constraints are satisfied by the assembly process.

C^1 continuity cannot be completely enforced by the same process. The reason is that any set of nodal variables which enforces C^1 continuity (and not more than C^1 continuity) along the boundaries of a finite element results in a singular transformation matrix $[T^e]$. It has been shown that, for complete polynomial approximating functions of order 5 or greater, $[T^e]$ will have at least as many linearly dependent rows as the finite element has vertices [4].

In reference [5] the continuity constraints were separated into two sets. The first and larger set contained the maximum number of linearly independent rows from $[T^e]$. These constraints were identified by the simplex method and were treated much the same way as in the conventional finite element method. The second set of constraints, comprising the linearly dependent rows from $[T^e]$, required rank analysis at the global level. A basis had to be computed and the linearly dependent nodal variables eliminated from the potential energy expression. Identification of the redundant rows and computation of the basis was again performed by means of a modified version of the simplex method.

A great improvement was brought to the computational process recently by clarification of the dimension of the piecewise polynomial space S_p^1. S_p^1 comprises polynomials of degree p, which satisfy C^1 continuity (and not more than C^1 continuity) on a given triangulation. The dimension of this space is simply the number of independent degrees of freedom before the principal boundary conditions are enforced. The available space does not permit detailed discussion of this important subject here; only the result is quoted:

$$\dim S_p^1 = \frac{1}{2}(p+1)(p+2)T - (2p+1)\,E_o + 3V_o + V_S \qquad (4)$$

where:

p = degree of polynomial over each triangular subdomain ($p \geq 5$)

T = number of triangles

E_o = number of interior edges

V_o = number of interior vertices

V_S = number of singular vertices. (A singular vertex is defined as a vertex where four elements meet and their boundaries are given by a pair of intersecting straight lines.)

The dimension of the piecewise polynomial space can be readily computed even when the polynomial orders vary from element to element.

Understanding the linear relationship among the nodal variables in S_p^1 is perhaps the most significant step toward the development of a second-generation finite element computer code. The relationship was discovered independently by Scott [6] [7] and Peano [4] [8]. As a result, it is now possible to identify a set of linearly independent nodal variables for each finite element, perform an elemental level transformation from polynomial coefficients to nodal variables and enforce all continuity conditions, with the exception of the continuity of the mixed second partial derivatives at the vertices, by the usual process of assembling the global stiffness matrix. Continuity of the mixed second partial derivatives is enforced by one of several processes, such as elimination of the dependent variables as in reference [5], application of the Lagrange multiplier technique, application of the penalty method or by the superelement method [4][8]. Significantly, the constraints can be written without incurring redundancies. Details of the computational process are given in references [4],[8],[9]. Only a brief summary of the main features is presented here.

Definition of the nodal variables for S_5^1 is shown on figure 1. At each vertex the nodal variables are: the value of the displacement function w; its first derivatives in the global coordinate directions, $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$; its second derivatives along the edges meeting at the vertex and its mixed second partial derivatives evaluated in the edgewise directions. At each midsize node we define $\frac{\partial^5 w}{\partial s^4 \partial n}$ as nodal variables where s and n are the edgewise and normal variables respectively. For S_6^1 the nodal variables at the vertices are the same but at each of the midsize nodes we add two new nodal variables: $\frac{\partial^6 w}{\partial s^6}$ and $\frac{\partial^6 w}{\partial s^5 \partial n}$, and we define an internal nodal variable. This choice of nodal variables considerably simplifies transformation from polynomial coefficients to nodal variables [9]. In general, for S_p^1 $(p \geq 5)$ we define $6p - 9$ external and $\frac{1}{2}(p^2 - 9p + 20)$ internal nodal variables. It has been remarked already that, in our case, continuity of the mixed second partial derivatives cannot be enforced by the usual process of assembling the global stiffness matrix. The reason for this is that the constraints must be written in terms of the parameters of two elements. Referring to figure 2, we note that continuity of the normal derivatives implies that $\frac{\partial^2 w_1}{\partial s_2 \partial n_2} = \frac{\partial^2 w_2}{\partial s_2 \partial n_2}$ along edge PQ. This condition can be written as:

$$\frac{\partial^2 w}{\partial s_2^2}(\cot \phi_1 + \cot \phi_2) = \frac{1}{\sin \phi_1}\frac{\partial^2 w}{\partial s_1 \partial s_2} + \frac{1}{\sin \phi_2}\frac{\partial^2 w}{\partial s_2 \partial s_2} \qquad (5)$$

A derivation of eq. (5) is given in reference [4]. Several significant observations have been made in connection with this equation [4]. The most significant point is, however, that the number of constraints which require special treatment does not depend on the order of polynomial approximation p.

3. Computational experience

Several problems were solved by means of research-oriented finite element codes which admit complete and conforming polynomial approximating functions of arbitrary order. In all instances it was found that the rate of convergence was substantially greater when the polynomial orders were increased than when the number of finite elements were increased. The basis for comparison was the potential energy plotted against the number of degrees of freedom [2]. In one example, the three lowest natural frequencies of a simply supported square plate were computed using 200 3rd-order finite elements (590 degrees of freedom). It was possible to obtain comparable results with only two 6th-order complete and conforming finite elements (20 degrees of freedom) [8].

The question of whether high-order finite elements would yield accurate results in the neighborhood of a singular point was examined also. Lockheed test problem No. 2, a cylindrical shell with two symmetrically arranged rectangular cut-outs, subjected to axial compression, was solved previously by four different computer codes [1] [10]. The number of finite elements ranged from 476 to 100, the number of degrees of freedom from 2457 to 637. Using complete and conforming finite elements it was possible to obtain similar results with only 10 elements and 500 degrees of freedom [11]. The stress distribution in the vicinity of the reentrant corner was found to be almost identical to the stress distribution reported in reference [10].

4. Closing remarks

The finite element algorithm outlined in this paper offers obvious advantages of computational efficiency and convenience. In applications where fatique life considerations govern design, it is often necessary to accurately estimate stresses in areas of steep stress gradients. The capability to increase the orders of approximating functions without mesh refinement is particularly advantageous in such applications. The fact that C^1 continuity is satisfied exactly is important when stresses in the vicinity of shell connections of multishell structures are to be computed. The accuracy of stresses computed by means of non-conforming elements is very sensitive to geometry and load distribution. On the other hand, hyperconforming elements tend to underestimate stresses in those areas.

5. Acknowledgements

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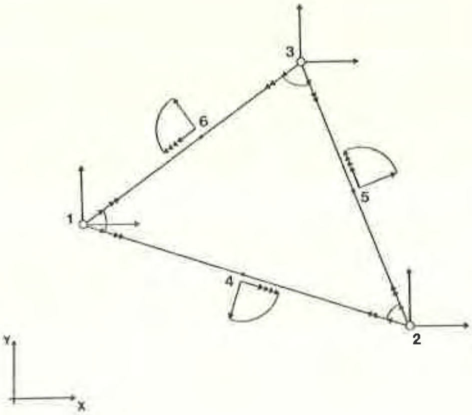

FIGURE 1

NODAL VARIABLES FOR S_5^1

O represents the value of the displacement function; each arrowhead represents partial differentiation in the direction indicated; arrowheads connected by an arc represent mixed partial derivatives.

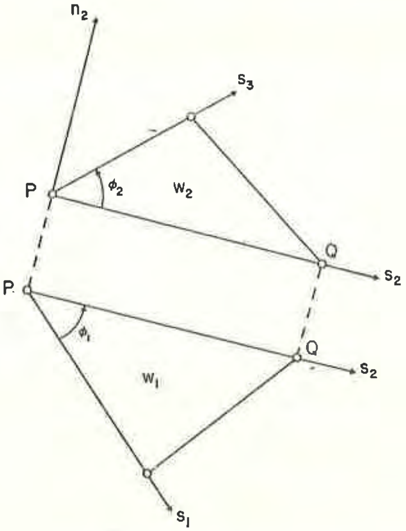

FIGURE 2

NOTATION