THERMODYNAMICS OF SANDWICH STRUCTURES*

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SUMMARY

With the advances in aerospace technology, sandwich structures with high strength-to-weight capability have become an extremely desirable design feature, and owing to this, they have been extensively studied in the literature. The classical Kirchoff-Love model of shells is generally adopted in the earlier studies on this type of shells (plates). The model ignores the skin effects and the cross-sectional distortion, and thereof the coupling of adherent layers. Thus, it is evidently unable to describe the mechanical behaviour of sandwich shells. Further, these studies have been based on an uncoupled analysis thereby eliminating the need for thermodynamics. Such an analysis requires that the governing equations of isothermal sandwich shells be supplemented by the temperature distribution and the appropriate constitutive relations. Also, it is noteworthy that a general coupled analysis has been left untreated even for non-isothermal shells, although its equivalent in continuum theory is available.

The purpose of this paper is to present a general formulation for sandwich shells (plates) within the frame of the three-dimensional theory of linear, non-isothermal, anisotropic, coupled viscoelastodynamics.

The sandwich shell treated here is composed of a single-core matrix having two face reinforcement fibers. Each constituent of the shell is of different thickness and is made of dissimilar anisotropic material. In what follows, starting with the Laplace transformed local equations and using an averaging procedure, we consistently establish the macroscopic equations of motion of sandwich shells together with the initial and boundary conditions for the case when the displacement components of each constituent vary linearly across its thickness. The linearized displacement field which is unknown a priori satisfies the continuity conditions at the interfaces of constituents, and also it allows to include the effects of transverse shear and normal strains, and the rotatory inertia in the matrix and fibers.

Next, by applying the averaging procedure, we similarly construct the generalized Gibbs relation and the balance of thermal energy, revealing the explicit form of the thermomechanical coupling terms, and then we interpret for shells. The temperature distribution is then obtained as well as the constitutive relations. Dynamic interactions are taken into account through the usual continuity conditions at the interfaces. Furthermore, we study the special cases of constituents made of either isotropic thermoviscoelastic or anisotropic thermoelastic materials. Lastly, a simple elastic and viscoelastic example is given in order to illustrate the relative importance of the thermomechanical coupling and inertia as compared to the quasistatic uncoupled elastic solution.

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1. Introduction

In investigating the static and dynamic response of a sandwich structure, we distinguish three main types of theoretical models in two-dimension. The first types of the models are essentially based upon the material behavior of constituents of sandwich structure. They are still widely applied in practice for a sandwich structure consisting of two thin facings of high-strength material and a thick core of low-strength and light-weight material between the facings. The derived governing equations for these models can differ greatly in complexity as a result of certain stiffnesses and energy contributions and/or shear deformations are omitted or included in the core and facings [1]. For the new types of sandwich structures with strong cores, the classical Kirchhoff-Love model of shells (plates) or Bernoulli-Euler model of beams are, in general, adopted. Both of the first strong facing-weak core type and second homogeneous-continuum type models ignore the skin-effects and the cross-sectional distortion, and thereof the coupling of adherent layers. Hence, to describe the mechanical behavior of sandwich structures, a third type of the models, which takes into account all the interactions between constituents is introduced. These type models have been recently used in formulating the set of governing equations for polar and non-polar sandwich shells and plates (see, e.g., [4-6], and references therein).

In this paper, by the use of the latter type of the models, a system of governing equations is derived for sandwich shells within the scope of the three-dimensional theory of linear, non-isothermal, anisotropic coupled viscoelastodynamics.

A résumé of the basic equations of viscoelastodynamics is given in the next section. The displacement and temperature fields, and the geometry of the sandwich shell are studied in Section 3. A three-layer sandwich shell with no singularities of any type is treated in the analysis. The fields are linearized with respect to the thickness coordinate, and then the usual continuity conditions at the interfaces are satisfied by them. We deal with, after defining the stress and heat flux resultants, the constitutive relations in Section 4. Starting with the fields and using an averaging procedure [7], we construct the sandwich shell equations in Section 5. Special cases of interest are taken up in Section 6. Some conclusions regarding the theory are drawn in the last section.

Notation

In the sequel, we use standard tensor notation [8] in a Euclidean 3-space $\mathcal{E}$ of normal coordinates. Latin and Greek indices respectively take the values 1,2,3 and 1,2, and they are summed over these ranges when repeated. These indices are respectively used to indicate space and surface tensors. A comma stands for partial differentiation with respect to the space coordinates θ^i, a superscript or subscript zero for prescribed quantities, a subtilde for vectors, and a superposed dot for time differentiation. A semicolon and a colon denote covariant differentiation using space and surface metrics, respectively. A single prime, double prime and an overtilde represent, in this order, the quantities referring to the lower and upper facings, and the core of sandwich shell (plate). Further, the multiplication symbol is used to denote the Cartesian products of two sets, a star the commutative Stieltjes convolution, namely

$$\varphi * d\psi = \psi * d\varphi = \int_{0^-}^{t_1} \varphi(t-\tau)\, d\psi(\tau) \tag{1}$$

in which φ and ψ are functions of position and time t, and an overbar the one-sided Laplace transform of all time variables, that is,

$$\mathcal{L}\,[\varphi(\theta^i,t)] = \int_0^\infty \varphi(\theta^i,t)\, e^{-st}\, dt = \bar{\varphi}(\theta^i,s) \tag{2}$$

which is assumed to be exist.

2. Résumé of Basic Equations

To render the present work self-contained, this section is devoted to a résumé of the linear theory of viscoelasticity. For a comprehensive background of the subject, we refer the reader to [9,10].

In the three-dimensional Euclidean space $\mathcal{E}$, consider a non-polar thermo-viscoelastic body which occupies a regular region [11] of space, $\mathcal{B} \in \mathcal{E}$, with the closure $\bar{\mathcal{B}}$ and the boundary $\partial\mathcal{B}$. The body is referred to a θ^i-system of fixed geodesic normal coordinates in $\mathcal{E}$. Further, $\mathcal{S}_\alpha$ $(\alpha = u, t)$ and $\mathcal{S}_\beta$ $(\beta = q, \theta)$ denote the complementary regular subsurfaces of $\partial\mathcal{B}$ such that

$$\mathcal{S}_u \cup \mathcal{S}_t = \mathcal{S}_q \cup \mathcal{S}_\theta = \partial\mathcal{B}\ , \quad \mathcal{S}_u \cap \mathcal{S}_t = \mathcal{S}_q \cap \mathcal{S}_\theta = 0 \tag{3}$$

and $\underset{\sim}{n}$ the unit outward normal vector to $\partial\mathcal{B}$. All of the surfaces are required to remain constant with time.

Now, let

σ_{ij}	:	symmetric stress tensor,
$\underset{\sim}{u}$	:	displacement vector,
ρ	:	mass density,
$\underset{\sim}{f}$	:	body force vector,
e_{ij}	:	infinitesimal strain tensor,
T_0	:	quiescent reference temperature,
Θ	:	small temperature fluctuations from T_0,
$\underset{\sim}{q}$	:	heat flux vector,
$\underset{\sim}{t}$	:	surface traction vector,
$G_{ijkl}, B_{ijk}, K_{ij}, F_{ij}, M, \varkappa_{ij}$	:	relaxation functions of material,
r	:	heat source function per unit mass,
e	:	specific internal energy,
η, η_0	:	specific and initial entropies

which are defined on $\mathcal{B} \times (-\infty, \infty)$. With this notation, the basic equations of the linear theory are recorded as follows.

Equations of Motion :

$$\sigma^{ij}{}_{;i} + f^{j} - \rho\, \ddot{u}^{j} = 0 \quad , \quad \sigma^{ij} = \sigma^{ji} \qquad \text{on } \mathcal{B} \times [t_o, t_1) \tag{4}$$

where $t_o = 0$, and $t_1 > t_o$ may be infinity.

Strain-Displacement Relations :

$$e_{ij} = \frac{1}{2}(u_{i;j} + u_{j;i}) \qquad \text{on } \mathcal{B} \times [t_o, t_1) \tag{5}$$

Initial Conditions :

$$\Theta = \underset{\sim}{u} = e_{ij} = \sigma^{ij} = 0 \qquad \text{on } \mathcal{B} \times (-\infty, t_o) \tag{6}$$

Boundary Conditions :

$$u_i - u_i^{o} = 0 \quad \text{on } \mathcal{S}_u \times [t_o, t_1) \quad , \quad t^{j} - n_i \sigma^{ij} = 0 \quad \text{on } \mathcal{S}_t \times [t_o, t_1) \tag{7}$$

and

$$\Theta - \Theta_o = 0 \quad \text{on } \mathcal{S}_\theta \times [t_o, t_1) \quad , \quad n_i q^{i} - q_o = 0 \quad \text{on } \mathcal{S}_q \times [t_o, t_1) \tag{8}$$

Here, $\mathcal{S}_u$, $\mathcal{S}_t$, $\mathcal{S}_q$ and $\mathcal{S}_\theta$ stand for the surface portions of $\partial\mathcal{B}$, on which the displacement, traction and heat flux vectors and the temperature field are prescribed, respectively.

Heat Conduction Equations :

Within the scope of infinitesimal theory, the conservation law of energy yields

$$\rho \dot{e} = \sigma^{ij} \dot{e}_{ij} + q^{i}{}_{;i} + \rho r \tag{9}$$

Under quite general constitutive assumptions of viscoelastodynamics, this equation leads to the heat conduction equation :

$$\rho T_o \dot{\eta} = q^{i}{}_{;i} + \rho r \tag{10}$$

where the dissipation functions, non-linear in the strain rates, are omitted, and small temperature fluctuations from a quiescent temperature T_o are assumed.

Constitutive Relations :

$$\left.\begin{aligned} \sigma^{ij} &= G^{ijkl} * de_{kl} + F^{ij} * d\Theta \\ q^{i} &= K^{ij}\Theta_{,j} + B^{ijk} * de_{jk} \\ \eta &= \eta_o + \varkappa^{ij} * de_{ij} + M * d\Theta \end{aligned}\right\} \quad \text{on } \mathcal{B} \times [t_o, t_1) \tag{11 a}$$

with

$$F_{ij} = F_{ji} \quad , \quad B_{ijk} = B_{ikj} \quad , \quad G_{ijkl} = G_{jikl} = G_{klij} = G_{ijlk} \quad , \quad \varkappa_{ij} = \varkappa_{ji} \quad \text{on } \mathcal{B} \times [t_o, t_1)$$

$$M = F_{ij} = B_{ijk} = G_{ijkl} = K_{ij} = \varkappa_{ij} = 0 \qquad \text{on } \mathcal{B} \times (-\infty, 0) \tag{11 b}$$

for anisotropic materials.

3. Geometry. Displacement and Temperature Fields

The sandwich shell treated here is composed of a single core-matrix sandwiched between two thin facings. Each constituent of the shell may possess different thickness $2h$ and different anisotropic material behavior. The shell is referred to the θ^i -system in the space $\mathcal{E}$. The θ^i -system is situated on the middle surface ${}'\mathcal{A}=\mathcal{A}$ of the lower facing. Thus, $\theta^3=0$ defines the middle surface $\mathcal{A}$, and the θ^α-coordinate curves form a set of curvilinear normal coordinates on this surface. Also, we define similar coordinate systems, $\tilde{\theta}^i$ and ${}''\theta^i$, on the middle surface of the core and upper facing ; the following relations hold :

$$ {}'\theta^3=\theta^3 , \quad \tilde{\theta}^3=\theta^3-\tilde{z} , \quad {}''\theta^3=\theta^3-{}''z , \quad {}''\theta^\alpha={}'\theta^\alpha=\tilde{\theta}^\alpha=\theta^\alpha \tag{12a} $$

where

$$ {}'z=0 , \quad \tilde{z}={}'h+\tilde{h} , \quad {}''z=\tilde{z}+\tilde{h}+{}''h \tag{12b} $$

and

$$ -{}'h \leqq \theta^3 \leqq {}''z+{}''h , \quad -\tilde{h} \leqq \tilde{\theta}^3 \leqq \tilde{h} , \quad -{}''h \leqq {}''\theta^3 \leqq {}''h \tag{12c} $$

The edge boundary of the sandwich shell space $\mathcal{V}$ is a right cylindrical surface with the generators perpendicular to $\mathcal{A}$, and it intersects $\mathcal{A}$ along a Jordan curve $\mathcal{C}$. The interface between the lower facing and the core is denoted by ${}'\tilde{\mathcal{A}}$ and that between the core and the upper facing by ${}''\tilde{\mathcal{A}}$ (see Fig. 1).

Let $\mathbf{g}^k, \mathbf{g}_k$ and g^{kl}, g_{kl} respectively designate the base vectors and metric tensors at a generic point of $\mathcal{V}$. They are associated with those of $\mathcal{A}$ by the relations [8, 12] :

$$ \mathbf{g}_\alpha=\mu^\beta_\alpha \mathbf{a}_\beta , \quad \mathbf{g}^\alpha=\lambda^\alpha_\beta \mathbf{a}^\beta , \quad \mathbf{g}_3=\mathbf{a}_3 , \quad \mathbf{g}^3=\mathbf{a}^3 \tag{13a} $$

and

$$ g_{\alpha\beta}=\mu^\nu_\alpha \mu^\sigma_\beta a_{\nu\sigma} , \quad g^{\alpha\beta}=\lambda^\alpha_\nu \lambda^\beta_\sigma a^{\nu\sigma} , \quad g_{\alpha3}=0 , \quad g_{33}=1 \tag{13b} $$

where

$$ \mu^\alpha_\beta=\delta^\alpha_\beta-\theta^3 b^\alpha_\beta , \quad \mu^\alpha_\beta \lambda^\beta_\sigma=\delta^\alpha_\sigma \tag{14} $$

Here, $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are the covariant first and second fundamental forms of $\mathcal{A}$, respectively. The third fundamental form $c_{\alpha\beta}$ is defined as

$$ c_{\alpha\beta}=b^\sigma_\alpha b_{\sigma\beta} \tag{15} $$

To develop a non-isothermal theory of sandwich shells, we take the displacements and temperature in the form :

$$ {}'\Theta(\theta^i,t)={}'\varphi(\theta^\alpha,t)+\theta^3\,{}'\psi(\theta^\alpha,t) , \quad \tilde{\Theta}=\tilde{\varphi}+\theta^3\tilde{\psi} , \quad {}''\Theta={}''\varphi+\theta^3\,{}''\psi \tag{16} $$

and

$$ {}'u_i(\boldsymbol{\theta},t)={}'v_i(\theta^\alpha,t)+\theta^3\,{}'w_i(\theta^\alpha,t) , \quad \tilde{u}_i=\tilde{v}_i+\theta^3\tilde{w}_i , \quad {}''u_i={}''v_i+\theta^3\,{}''w_i \tag{17} $$

in which the displacement vector $\mathbf{u}$ is referred to the base vectors $\mathbf{a}_k$ of $\mathcal{A}$, namely

$$ \mathbf{u}=u_\alpha \mathbf{a}^\alpha+u_3\mathbf{a}^3=u^\alpha\mathbf{a}_\alpha+u^3\mathbf{a}_3 \tag{18} $$

Since the displacements and temperatures are continuous at the interfaces ${}'\tilde{\mathcal{A}}$ and ${}''\tilde{\mathcal{A}}$, we can write

$$ {}'u_i=\tilde{u}_i , \quad {}'\Theta=\tilde{\Theta} \text{ on } {}'\tilde{\mathcal{A}} ; \quad {}''u_i=\tilde{u}_i , \quad {}''\Theta=\tilde{\Theta} \text{ on } {}''\tilde{\mathcal{A}} \tag{19} $$

By the use of the continuity conditions (19) in (16) and (17), we obtain :

$$\left.\begin{aligned} \tilde{v}_i &= {}'v_i + {}'h\,{}'w_i - (\tilde{z}-\tilde{h})\tilde{w}_i \quad , \quad {}''v_i = {}'v_i + 2\tilde{h}\tilde{w}_i + {}'h\,{}'w_i - ({}''z - {}''h)\,{}''w_i \\ \tilde{\varphi} &= {}'\varphi + {}'h\,{}'\psi - (\tilde{z}-\tilde{h})\tilde{\psi} \quad , \quad {}''\varphi = {}'\varphi + 2\tilde{h}\tilde{\psi} + {}'h\,{}'\psi - ({}''z - {}''h)\,{}''\psi \end{aligned}\right\} \tag{20}$$

Consequently, we have the independent unknown functions :

$${}'v_i \; , \; {}'w_i \; , \; \tilde{w}_i \; , \; {}''w_i \quad ; \quad {}'\varphi \; , \; {}'\psi \; , \; {}''\psi \; , \; \tilde{\psi} \tag{21}$$

for the three-layer sandwich shell.

4. Stress and Load Resultants. Constitutive Relations.

Before constructing the sandwich shell equations, it is desirable to define the required resultants for various field quantities, and then to give the strain distributions and constitutive relations in this section.

Body Force Resultants :

$$\left\{F^{\alpha}, F^{3}, G^{\alpha}, G^{3}\right\} = \int_{z-h}^{z+h} \mu \left\{\mu^{\alpha}_{\beta}\bar{f}^{\beta}, \bar{f}^{3}, \theta^{3}\mu^{\alpha}_{\beta}\bar{f}^{\beta}, \theta^{3}\bar{f}^{3}\right\} d\theta^{3} \tag{22}$$

Stress Resultants :

$$\left\{N^{\alpha\beta}, M^{\alpha\beta}, P^{\alpha}, T^{\alpha}, N^{33}\right\} = \int_{z-h}^{z+h} \mu \left\{\mu^{\beta}_{\nu}\bar{\sigma}^{\alpha\nu}, \theta^{3}\mu^{\beta}_{\nu}\bar{\sigma}^{\alpha\nu}, \bar{\sigma}^{\alpha 3}, \theta^{3}\bar{\sigma}^{\alpha 3}, \bar{\sigma}^{33}\right\} d\theta^{3} \tag{23}$$

Gross Displacements :

$$\left\{A^{\alpha}, A^{3}, B^{\alpha}, B^{3}\right\} = \int_{z-h}^{z+h} \mu \left\{\mu^{\alpha}_{\beta}\bar{u}^{\beta}, \bar{u}^{3}, \theta^{3}\mu^{\alpha}_{\beta}\bar{u}^{\beta}, \theta^{3}\bar{u}^{3}\right\} d\theta^{3} \tag{24}$$

Entropy Resultants :

$$\left\{S, E, S_{o}, E_{o}\right\} = \int_{z-h}^{z+h} \mu \left\{\bar{\eta}, \theta^{3}\bar{\eta}, \bar{\eta}_{o}, \theta^{3}\bar{\eta}_{o}\right\} d\theta^{3} \tag{25}$$

Gross Temperatures :

$$\left\{T, \tau\right\} = \int_{z-h}^{z+h} \mu \left\{1, \theta^{3}\right\} \bar{\Theta}\, d\theta^{3} \tag{26}$$

Heat Flux and Source Resultants :

$$\left\{Q_i, H_i\right\} = \int_{z-h}^{z+h} \mu \left\{1, \theta^{3}\right\} \bar{q}_i\, d\theta^{3} \quad , \quad \left\{R, H\right\} = \int_{z-h}^{z+h} \mu \left\{1, \theta^{3}\right\} \bar{r}\, d\theta^{3} \tag{27}$$

Load Resultants :

$$p^{\alpha} = \left(\mu\mu^{\alpha}_{\beta}\bar{\sigma}^{3\beta}\right)\Big|_{\theta^{3}=z+h} \; , \quad r^{\alpha} = \left(\mu\mu^{\alpha}_{\beta}\bar{\sigma}^{3\beta}\right)\Big|_{\theta^{3}=z-h} \tag{28}$$

$$\left.\begin{array}{l} p^3 = (\mu\bar{\sigma}^{33})\big|_{\theta^3 = z+h}, \quad r^3 = (\mu\bar{\sigma}^{33})\big|_{\theta^3 = z-h} \\ m^i = p^i - r^i, \quad n^i = (z+h)\,p^i - (z-h)\,r^i \end{array}\right\} \quad (28)$$

Surface Heat Fluxes :

$$\left.\begin{array}{l} a = (\mu q^3)\big|_{\theta^3 = z+h}, \quad b = (\mu q^3)\big|_{\theta^3 = z-h} \\ K = a - b, \quad L = (z+h)\,a - (z-h)\,b \end{array}\right\} \quad (29)$$

Viscoelastic Stiffnesses :

$$\left\{G^{(n)}_{ijkl},\ B^{(n)}_{ijk},\ M^{(n)},\ F^{(n)}_{ij},\ \varkappa^{(n)}_{ij},\ K^{(n)}_{ij}\right\} = \int_{z-h}^{z+h} \mu(\theta^3)^n \left\{G_{ijkl}, B_{ijk}, M, F_{ij}, \varkappa_{ij}, K_{ij}\right\} d\theta^3 \quad (30)$$

Effective Stress, Heat Flux and Entropy Resultants :

$$\left.\begin{array}{l} \acute{\mathcal{N}}^{\alpha\beta} = \acute{N}^{\alpha\beta} + \tilde{N}^{\alpha\beta} + \acute{\acute{N}}^{\alpha\beta}, \quad \acute{\mathcal{M}}^{\alpha\beta} = \acute{M}^{\alpha\beta} + \acute{h}(\tilde{N}^{\alpha\beta} + \acute{\acute{N}}^{\alpha\beta}), \quad \tilde{\mathcal{M}}^{\alpha\beta} = \tilde{M}^{\alpha\beta} - \acute{h}\tilde{N}^{\alpha\beta} + 2\tilde{h}\acute{\acute{N}}^{\alpha\beta} \\ \acute{\acute{\mathcal{M}}}^{\alpha\beta} = \acute{\acute{M}}^{\alpha\beta} - (\acute{h}+2\tilde{h})\acute{\acute{N}}^{\alpha\beta}, \quad \acute{\mathcal{P}}^{\alpha} = \acute{P}^{\alpha} + \tilde{P}^{\alpha} + \acute{\acute{P}}^{\alpha}, \quad \acute{\mathcal{T}}^{\alpha} = \acute{T}^{\alpha} + \acute{h}(\tilde{P}^{\alpha} + \acute{\acute{P}}^{\alpha}), \\ \tilde{\mathcal{T}}^{\alpha} = \tilde{T}^{\alpha} - \acute{h}\tilde{P}^{\alpha} + 2\tilde{h}\acute{\acute{P}}^{\alpha}, \quad \acute{\acute{\mathcal{T}}}^{\alpha} = \acute{\acute{T}}^{\alpha} - (\acute{h}+2\tilde{h})\acute{\acute{P}}^{\alpha}, \quad \acute{\mathcal{V}}^{\alpha} = \acute{P}^{\alpha} + \acute{h}b^{\alpha}_{\nu}(\tilde{P}^{\nu} + \acute{\acute{P}}^{\nu}), \\ \tilde{\mathcal{V}}^{\alpha} = \tilde{P}^{\alpha} + b^{\alpha}_{\nu}(-\acute{h}\tilde{P}^{\nu} + 2\tilde{h}\acute{\acute{P}}^{\nu}), \quad \acute{\acute{\mathcal{V}}}^{\alpha} = \acute{\acute{P}}^{\alpha} - b^{\alpha}_{\nu}(\acute{h}+2\tilde{h})\acute{\acute{P}}^{\nu} \end{array}\right\} \quad (31)$$

$$\acute{\Omega} = \acute{S} + \tilde{S} + \acute{\acute{S}}, \quad \acute{\Pi} = \acute{E} + \acute{h}(\acute{S} + \acute{\acute{S}}), \quad \tilde{\Pi} = \tilde{E} - \acute{h}\tilde{S} + 2\tilde{h}\acute{\acute{S}}, \quad \acute{\acute{\Pi}} = \acute{\acute{E}} - (\acute{h}+2\tilde{h})\acute{\acute{S}} \quad (32)$$

$$\left.\begin{array}{l} \acute{K}^{\alpha} = \acute{Q}^{\alpha} + \acute{\acute{Q}}^{\alpha} + \tilde{Q}^{\alpha}, \quad \acute{L}^{\alpha} = \acute{H}^{\alpha} + \acute{h}(\tilde{Q}^{\alpha} + \acute{\acute{Q}}^{\alpha}), \quad \tilde{L}^{\alpha} = \tilde{H}^{\alpha} - \acute{h}\tilde{Q}^{\alpha} + 2\tilde{h}\acute{\acute{Q}}^{\alpha} \\ \acute{\acute{L}}^{\alpha} = \acute{\acute{H}}^{\alpha} - (\acute{h}+2\tilde{h})\acute{\acute{Q}}^{\alpha} \end{array}\right\} \quad (33)$$

Effective Body Force, Surface Heat Flux, Loads and Heat Source Resultants, and Gross Displacements :

$$\left.\begin{array}{l} \acute{\mathcal{F}}^{i} = \acute{F}^{i} + \tilde{F}^{i} + \acute{\acute{F}}^{i}, \quad \acute{\mathcal{G}}^{i} = \acute{G}^{i} + \acute{h}(\tilde{F}^{i} + \acute{\acute{F}}^{i}), \quad \tilde{\mathcal{G}}^{i} = \tilde{G}^{i} - \acute{h}\tilde{F}^{i} + 2\tilde{h}\acute{\acute{F}}^{i}, \\ \acute{\acute{\mathcal{G}}}^{i} = \acute{\acute{G}}^{i} - (\acute{h}+2\tilde{h})\acute{\acute{F}}^{i}, \quad \acute{\mathcal{A}}^{i} = \acute{A}^{i} + \tilde{A}^{i} + \acute{\acute{A}}^{i}, \quad \acute{\mathcal{B}}^{i} = \acute{B}^{i} + \acute{h}(\tilde{A}^{i} + \acute{\acute{A}}^{i}), \\ \tilde{\mathcal{B}}^{i} = \tilde{B}^{i} - \acute{h}\tilde{A}^{i} + 2\tilde{h}\acute{\acute{A}}^{i}, \quad \acute{\acute{\mathcal{B}}}^{i} = \acute{\acute{B}}^{i} - (\acute{h}+2\tilde{h})\acute{\acute{A}}^{i} \end{array}\right\} \quad (34)$$

$$\acute{\alpha} = \acute{R} + \tilde{R} + \acute{\acute{R}}, \quad \acute{\beta} = \acute{H} + \acute{h}(\tilde{R} + \acute{\acute{R}}), \quad \tilde{\beta} = \tilde{H} - \acute{h}\tilde{R} + 2\tilde{h}\acute{\acute{R}}, \quad \acute{\acute{\beta}} = \acute{\acute{H}} - (\acute{h}+2\tilde{h})\acute{\acute{R}} \quad (35)$$

$$\acute{R}^{i} = \acute{\acute{p}}^{i} - \acute{r}^{i}, \quad \acute{S}^{i} = \acute{h}(\acute{\acute{p}}^{i} + \acute{r}^{i}), \quad \tilde{S}^{i} = 2\tilde{h}\acute{\acute{p}}^{i}, \quad \acute{\acute{S}}^{i} = 2\tilde{h}\acute{\acute{p}}^{i} \quad (36)$$

$$\acute{m} = \overset{\prime\prime}{a} - \acute{b} \;,\quad \acute{n} = \acute{h}(\overset{\prime\prime}{a} + \acute{b}) \;,\quad \tilde{n} = 2\tilde{h}\overset{\prime\prime}{a} \;,\quad \overset{\prime\prime}{n} = 2\overset{\prime\prime}{h}\overset{\prime\prime}{a} \tag{37}$$

Prescribed Traction and Heat Flux Resultants :

$$\left\{N_0^\alpha, M_0^\alpha, P_0^3, T_0^3\right\} = \int_{z-h}^{z+h} \mu \left\{\mu_\beta^\alpha \bar{t}_0^\beta, \theta^3 \mu_\beta^\alpha \bar{t}_0^\beta, \bar{t}_0^3, \theta^3 \bar{t}_0^3\right\} d\theta^3 \tag{38}$$

$$\left\{Q_0, H_0\right\} = \int_{z-h}^{z+h} \mu \left\{1, \theta^3\right\} \bar{q}_0 \, d\theta^3 \tag{39}$$

$$\left.\begin{aligned}
&\acute{\mathcal{N}}_0^\alpha = \acute{N}_0^\alpha + \tilde{N}_0^\alpha + \overset{\prime\prime}{N}_0^\alpha \;,\quad \acute{\mathcal{M}}_0^\alpha = \acute{M}_0^\alpha + \acute{h}(\tilde{N}_0^\alpha + \overset{\prime\prime}{N}_0^\alpha) \;,\quad \tilde{\mathcal{M}}_0^\alpha = \tilde{M}_0^\alpha - \acute{h}\tilde{N}_0^\alpha + 2\tilde{h}\overset{\prime\prime}{N}_0^\alpha \\
&\overset{\prime\prime}{\mathcal{M}}_0^\alpha = \overset{\prime\prime}{M}_0^\alpha - (\acute{h} + 2\tilde{h})\overset{\prime\prime}{N}_0^\alpha \;,\quad \mathcal{P}_0^3 = \acute{P}_0^3 + \tilde{P}_0^3 + \overset{\prime\prime}{P}_0^3 \;,\quad \acute{\mathcal{T}}_0 = \acute{T}_0^3 + \acute{h}(\tilde{P}_0^3 + \overset{\prime\prime}{P}_0^3) \\
&\tilde{\mathcal{T}}_0^3 = \tilde{T}_0^3 - \acute{h}\tilde{P}_0^3 + 2\tilde{h}\overset{\prime\prime}{P}_0^3 \;,\quad \overset{\prime\prime}{\mathcal{T}}_0^3 = \overset{\prime\prime}{T}_0^3 - (\acute{h} + 2\tilde{h})\overset{\prime\prime}{P}_0^3
\end{aligned}\right\} \tag{40}$$

$$\left.\begin{aligned}
&\acute{K}_0 = \acute{Q}_0 + \tilde{Q}_0 + \overset{\prime\prime}{Q}_0 \;,\quad \acute{L}_0 = \acute{H}_0 + \acute{h}(\tilde{Q}_0 + \overset{\prime\prime}{Q}_0) \;,\quad \tilde{L}_0 = \tilde{H}_0 - \acute{h}\tilde{Q}_0 + 2\tilde{h}\overset{\prime\prime}{Q}_0 \\
&\overset{\prime\prime}{L}_0 = \overset{\prime\prime}{H}_0 - (\acute{h} + 2\tilde{h})\overset{\prime\prime}{Q}_0
\end{aligned}\right\} \tag{41}$$

Strain Distributions :

Upon use of the Laplace transformed form of (5) and (17), we obtain the strain distribution to be

$$\bar{e}_{kl}(\theta^i, t) = h_{kl}(\theta^\alpha, t) + \theta^3 m_{kl}(\theta^\alpha, t) + (\theta^3)^2 n_{kl}(\theta^\alpha, t) \tag{42a}$$

where

$$\begin{aligned}
&h_{\alpha\beta} = \tfrac{1}{2}(\bar{v}_{\alpha:\beta} + \bar{v}_{\beta:\alpha} - 2b_{\alpha\beta}\bar{v}_3) \;,\quad h_{\alpha 3} = \tfrac{1}{2}(\bar{w}_\alpha + \bar{v}_{3,\alpha} + b_\alpha^\beta \bar{v}_\beta) \;,\quad h_{33} = \bar{w}_3 \\
&m_{\alpha\beta} = \tfrac{1}{2}(\bar{w}_{\alpha:\beta} + \bar{w}_{\beta:\alpha} - 2b_{\alpha\beta}\bar{w}_3 - b_\alpha^\sigma \bar{v}_{\sigma:\beta} + 2c_{\alpha\beta}\bar{v}_3 - b_\beta^\sigma \bar{v}_{\sigma:\alpha}) \;,\quad m_{\alpha 3} = \tfrac{1}{2}\bar{w}_{3,\alpha} \;,\; m_{33} = 0 \\
&n_{\alpha\beta} = \tfrac{1}{2}(-b_\alpha^\sigma \bar{w}_{\sigma:\beta} - b_\beta^\sigma \bar{w}_{\sigma:\alpha} + 2c_{\alpha\beta}\bar{w}_3) \;,\quad n_{33} = n_{\alpha 3} = 0
\end{aligned} \tag{42b}$$

and for future use we define :

$$h_{kl} = \gamma_{kl}^{(0)} \;,\quad m_{kl} = \gamma_{kl}^{(1)} \;,\quad n_{kl} = \gamma_{kl}^{(2)} \tag{43}$$

Constitutive Relations :

The local constitutive relations (11) may be written :

$$\left.\begin{aligned}
\bar{\sigma}^{ij} &= s\,\bar{G}^{ijkl}\bar{e}_{kl} + s\,\bar{F}^{ij}\bar{\Theta} \\
\bar{q}^i &= K^{ij}\bar{\Theta}_{,j} + s\,\bar{B}^{ijk}\bar{e}_{jk} \\
\bar{\eta} &= \bar{\eta}_0 + s\,\bar{\varkappa}^{ij}\bar{e}_{ij} + s\,\bar{M}\bar{\Theta}
\end{aligned}\right\} \tag{44}$$

in the Laplace transformed form. In conjunction with (23), (25), (27), and the foregoing strain distributions, the macroscopic constitutive relations for an anisotropic thermo-viscoelastic constituent material are found to be

$$\left.\begin{aligned}
N^{\alpha\beta} &= s\Big[\sum_{n=0}^{2}(\bar{G}_{(n)}^{\alpha\beta kl}-b_\nu^\beta\bar{G}_{(n+1)}^{\alpha\nu kl})\gamma_{kl}^{(n)}+(\bar{F}_{(0)}^{\alpha\beta}-b_\nu^\beta\bar{F}_{(1)}^{\alpha\nu})\bar{\varphi}+(\bar{F}_{(1)}^{\alpha\beta}-b_\nu^\beta\bar{F}_{(2)}^{\alpha\nu})\bar{\Psi}\Big]\\
M^{\alpha\beta} &= s\Big[\sum_{n=0}^{2}(\bar{G}_{(n+1)}^{\alpha\beta kl}-b_\nu^\beta\bar{G}_{(n+2)}^{\alpha\nu kl})\gamma_{kl}^{(n)}+(\bar{F}_{(1)}^{\alpha\beta}-b_\nu^\beta\bar{F}_{(2)}^{\alpha\nu})\bar{\varphi}+(\bar{F}_{(2)}^{\alpha\beta}-b_\nu^\beta\bar{F}_{(3)}^{\alpha\nu})\bar{\Psi}\Big]\\
P^{\alpha} &= s\Big[\sum_{n=0}^{2}(\bar{G}_{(n)}^{\alpha 3kl}\gamma_{kl}^{(n)})+\bar{F}_{(0)}^{\alpha 3}\bar{\varphi}+\bar{F}_{(1)}^{\alpha 3}\bar{\Psi}\Big]\\
T^{\alpha} &= s\Big[\sum_{n=0}^{2}(\bar{G}_{(n+1)}^{\alpha 3kl}\gamma_{kl}^{(n)})+\bar{F}_{(1)}^{\alpha 3}\bar{\varphi}+\bar{F}_{(2)}^{\alpha 3}\bar{\Psi}\Big]\\
N^{33} &= s\Big[\sum_{n=0}^{2}(\bar{G}_{(n)}^{33kl}\gamma_{kl}^{(n)})+\bar{F}_{(0)}^{33}\bar{\varphi}+\bar{F}_{(1)}^{33}\bar{\Psi}\Big]
\end{aligned}\right\}\qquad(45)$$

$$\left.\begin{aligned}
Q^{i} &= \bar{K}_{(0)}^{i\alpha}\bar{\varphi}_{,\alpha}+\bar{K}_{(1)}^{i\alpha}\bar{\Psi}_{,\alpha}+\bar{K}_{(0)}^{i3}\bar{\Psi}\\
H^{i} &= \bar{K}_{(1)}^{i\alpha}\bar{\varphi}_{,\alpha}+\bar{K}_{(2)}^{i\alpha}\bar{\Psi}_{,\alpha}+\bar{K}_{(1)}^{i3}\bar{\Psi}\\
S &= S_0+s\Big[\sum_{n=0}^{2}(\bar{\varkappa}_{(n)}^{ij}\bar{\gamma}_{ij}^{(n)})+\bar{M}_{(0)}\bar{\varphi}+\bar{M}_{(1)}\bar{\Psi}\Big]\\
E &= E_0+s\Big[\sum_{n=0}^{2}(\bar{\varkappa}_{(n+1)}^{ij}\gamma_{ij}^{(n)})+\bar{M}_{(1)}\bar{\varphi}+\bar{M}_{(2)}\bar{\Psi}\Big]
\end{aligned}\right\}\qquad(46)$$

5. Sandwich Shell Equations

In this section, we construct the remaining governing equations of sandwich shells, and hence complete the non-isothermal theory. To begin with, we rewrite the foregoing local equations (2), (4), (5), (6) and (9) in the Laplace transformed form :

$$\bar{\sigma}^{ij}{}_{;i}+\bar{f}^{j}-\rho s^2\bar{u}^{j}=0 \qquad \text{on } \mathcal{V} \qquad (47)$$

$$\rho T_0(s\bar{\eta}-\bar{\eta}_0)=\bar{q}^{i}{}_{;i}+\rho\bar{r} \qquad \text{on } \mathcal{V} \qquad (48)$$

$$\bar{u}_i-\bar{u}_i^{\circ}=0\ ,\quad \bar{\Theta}-\bar{\Theta}_0=0 \qquad \text{on } \mathcal{S}_f \qquad (49)$$

$$n_i\bar{\sigma}^{ij}-\bar{t}_0^{j}=0\ ,\quad n_i\bar{q}^{i}-\bar{q}_0=0 \qquad \text{on } \mathcal{S}_l \qquad (50)$$

Here, the initial conditions (4) are taken into account, and for simplicity $\mathcal{S}_u=\mathcal{S}_\theta=\mathcal{S}_f$ and $\mathcal{S}_t=\mathcal{S}_q=\mathcal{S}_l$ are assumed, where $\mathcal{S}_f$ and $\mathcal{S}_l$ being the entire face and lateral surface of the sandwich shell, respectively.

Macroscopic Equations of Motion :

Now, we consider the first two equations in (47) by setting $j=\beta$, and multiply by $\mu\mu^{\alpha}_{\beta}$ with the result :

$$\mu\mu^{\alpha}_{\beta}\,(\bar{\sigma}^{i\beta}{}_{;i}+\bar{f}^{\beta}-\varrho s^{2}\bar{u}^{\beta})=0 \tag{51}$$

Using the relations between surface and space metric tensors and their derivatives (see, e.g., [8, 12]), and simplifying, this equation becomes :

$$(\mu\mu^{\alpha}_{\beta}\bar{\sigma}^{\nu\beta})_{:\nu}-\mu b^{\alpha}_{\beta}\bar{\sigma}^{\beta 3}+(\mu\mu^{\alpha}_{\beta}\bar{\sigma}^{\beta 3})_{,3}+\mu\mu^{\alpha}_{\beta}(\bar{f}^{\beta}-\varrho s^{2}\bar{u}^{\beta})=0 \tag{52}$$

Upon integration of (52) with respect to the thickness coordinate θ^{3} between the limits $(z-h)$ to $(z+h)$, we obtain the first two of the macroscopic equations of motion :

$$N^{\beta\alpha}{}_{:\beta}-b^{\alpha}_{\beta}P^{\beta}+F^{\alpha}+m^{\alpha}-\varrho s^{2}A^{\alpha}=0 \tag{53}$$

in which the resultants defined previously are used.

The third equation in (47), that is, when $j=3$, after multiplying by μ and evaluating, may be written :

$$\mu\mu^{\alpha}_{\nu}b_{\alpha\beta}\bar{\sigma}^{\nu\beta}+(\mu\bar{\sigma}^{\alpha 3})_{:\alpha}+(\mu\bar{\sigma}^{33})_{,3}+\mu(\bar{f}^{3}-\varrho s^{2}\bar{u}^{3})=0 \tag{54}$$

which, following integration across the thickness of a constituent, results in

$$P^{\alpha}{}_{:\alpha}+b_{\alpha\beta}N^{\alpha\beta}+F^{3}+m^{3}-\varrho s^{2}A^{3}=0 \tag{55}$$

Here, as before, the resultants of Section 4 and various surface-space tensor relations are used.

Likewise, for the balance of angular momentum, we multiply (52) and (54) by θ^{3} and integrate between the limits $(z-h)$ to $(z+h)$ to obtain the remaining equations of motion for a discrete constituent :

$$M^{\beta\alpha}{}_{:\beta}-P^{\alpha}+G^{\alpha}+n^{\alpha}-\varrho s^{2}B^{\alpha}=0 \tag{56}$$

$$T^{\alpha}{}_{:\alpha}-N^{33}+b_{\alpha\beta}M^{\alpha\beta}+G^{3}+n^{3}-\varrho s^{2}B^{3}=0 \tag{57}$$

Next, in view of the independent displacement functions selected in (21), equations (53) and (55) - (57) are combined to construct the macroscopic sandwich shell equations of motion. To do so, we add all of equations (53) and (55), that is, for the facings and the core, to arrive at the equations:

$$\acute{\mathcal{N}}^{\beta\alpha}{}_{:\beta}-b^{\alpha}_{\beta}\acute{\mathcal{P}}^{\beta}+\acute{\mathcal{F}}^{\alpha}+\acute{\mathcal{R}}^{\alpha}-\varrho s^{2}\acute{\mathcal{A}}^{\alpha}=0 \tag{58}$$

$$\acute{\mathcal{P}}^{\alpha}{}_{:\alpha}+b_{\alpha\beta}\acute{\mathcal{N}}^{\alpha\beta}+\acute{\mathcal{F}}^{3}+\acute{\mathcal{R}}^{3}-\varrho s^{2}\acute{\mathcal{A}}^{3}=0 \tag{59}$$

in which the continuity of tractions at the interfaces $\acute{\tilde{\mathcal{A}}}$ and $\acute{\acute{\tilde{\mathcal{A}}}}$ are taken into account, and the effective load and stress resultants of the previous section are used.

Analogously to (53) and (55), in view of (21), the combination of (56) and (57) results in :

$$\acute{\mathcal{M}}^{\beta\alpha}{}_{:\beta}-\acute{\mathcal{N}}^{\alpha}+\acute{\mathcal{G}}^{\alpha}+\acute{S}^{\alpha}-\varrho s^{2}\acute{\mathcal{B}}^{\alpha}=0 \tag{60}$$

$$\acute{\mathcal{T}}^{\alpha}{}_{:\alpha} - \acute{N}^{33} + b_{\alpha\beta}\acute{\mathcal{M}}^{\alpha\beta} + \acute{G}^{3} + \acute{S}^{3} - \varrho s^{2}\acute{\mathcal{B}}^{3} = 0 \qquad (61)$$

$$\tilde{\mathcal{M}}^{\beta\alpha}{}_{:\beta} - \tilde{\mathcal{N}}^{\alpha} + \tilde{G}^{\alpha} + \tilde{S}^{\alpha} - \varrho s^{2}\tilde{\mathcal{B}}^{\alpha} = 0 \qquad (62)$$

$$\tilde{\mathcal{T}}^{\alpha}{}_{:\alpha} - \tilde{N}^{33} + b_{\alpha\beta}\tilde{\mathcal{M}}^{\alpha\beta} + \tilde{G}^{3} + \tilde{S}^{3} - \varrho s^{2}\tilde{\mathcal{B}}^{3} = 0 \qquad (63)$$

$$\overset{\#}{\mathcal{M}}{}^{\beta\alpha}{}_{:\beta} - \overset{\#}{\mathcal{N}}{}^{\alpha} + \overset{\#}{G}{}^{\alpha} + \overset{\#}{S}{}^{\alpha} - \varrho s^{2}\overset{\#}{\mathcal{B}}{}^{\alpha} = 0 \qquad (64)$$

$$\overset{\#}{\mathcal{T}}{}^{\alpha}{}_{:\alpha} - \overset{\#}{N}{}^{33} + b_{\alpha\beta}\overset{\#}{\mathcal{M}}{}^{\alpha\beta} + \overset{\#}{G}{}^{3} + \overset{\#}{S}{}^{3} - \varrho s^{2}\overset{\#}{\mathcal{B}}{}^{3} = 0 \qquad (65)$$

We should note that the foregoing combinations can also be accomplished via a variational procedure (cf., for instance, [6,13]); this method automatically considers the continuity of tractions.

Macroscopic Equations of Heat Conduction :

We now turn to the heat conduction equation (48). After multiplying this equation by μ and $\mu\Theta^{3}$, integrating with respect to Θ^{3} between the limits $(z-h)$ to $(z+h)$ and simplifying, we obtain the macroscopic heat conduction equations :

$$\varrho s T_{0}(S - S_{0}) = \varrho R + Q^{\alpha}{}_{:\alpha} + K \qquad (66)$$

$$\varrho s T_{0}(E - E_{0}) = \varrho H + H^{\alpha}{}_{:\alpha} - Q^{3} + L \qquad (67)$$

for each constituents in terms of the heat flux, source and surface resultants.

In view of (21), as before, we combine the foregoing equations to obtain the macroscopic heat conduction equations of sandwich shells :

$$\varrho s T_{0}(\acute{\Omega} - \acute{\Omega}_{0}) = \varrho\acute{\alpha} + \acute{K}^{\alpha}{}_{:\alpha} + \acute{m} \qquad (68)$$

$$\varrho s T_{0}(\acute{\Pi} - \acute{\Pi}_{0}) = \varrho\acute{\beta} + \acute{L}^{\alpha}{}_{:\alpha} - \acute{Q}_{3} + \acute{n} \qquad (69)$$

$$\varrho s T_{0}(\tilde{\Pi} - \tilde{\Pi}_{0}) = \varrho\tilde{\beta} + \tilde{L}^{\alpha}{}_{:\alpha} - \tilde{Q}_{3} + \tilde{n} \qquad (70)$$

$$\varrho s T_{0}(\overset{\#}{\Pi} - \overset{\#}{\Pi}_{0}) = \varrho\overset{\#}{\beta} + \overset{\#}{L}{}^{\alpha}{}_{:\alpha} - \overset{\#}{Q}_{3} + \overset{\#}{n} \qquad (71)$$

where the continuity of heat fluxes across the interfaces is considered.

Boundary and Initial Conditions :

In a manner similar to the above equations, we integrate (50) to get

$$n_{\alpha}Q^{\alpha} - Q_{0} = 0\,, \quad n_{\alpha}H^{\alpha} - H_{0} = 0 \qquad (72)$$

$$n_{\alpha}N^{\alpha\beta} - N_{0}^{\beta} = 0\,, \quad n_{\alpha}P^{\alpha} - P_{0}^{3} = 0 \qquad (73)$$

$$n_{\alpha}M^{\alpha\beta} - M_{0}^{\beta} = 0\,, \quad n_{\alpha}T^{\alpha} - T_{0}^{3} = 0 \qquad (74)$$

and combine these equations to obtain

$$n_\alpha \acute{K}^\alpha - \acute{K}_o = 0 \;,\quad n_\alpha \acute{L}^\alpha - \acute{L}_o = 0 \;,\quad n_\alpha \tilde{L}^\alpha - \tilde{L}_o = 0 \;,\quad n_\alpha \ddot{L}^\alpha - \ddot{L}_o = 0 \qquad (75)$$

$$n_\alpha \acute{\mathcal{N}}^{\alpha\beta} - \acute{\mathcal{N}}_o^{\beta} = 0 \;,\quad n_\alpha \acute{\mathcal{M}}^{\alpha\beta} - \acute{\mathcal{M}}_o^{\beta} = 0 \;,\quad n_\alpha \tilde{\mathcal{M}}^{\alpha\beta} - \tilde{\mathcal{M}}_o^{\beta} = 0 \;,\quad n_\alpha \ddot{\mathcal{M}}^{\alpha\beta} - \ddot{\mathcal{M}}_o^{\beta} = 0 \qquad (76)$$

$$n_\alpha \acute{\mathcal{P}}^\alpha - \acute{\mathcal{P}}_o^3 = 0 \;,\quad n_\alpha \acute{\mathcal{C}}^\alpha - \acute{\mathcal{C}}_o^3 = 0 \;,\quad n_\alpha \tilde{\mathcal{C}}^\alpha - \tilde{\mathcal{C}}_o^3 = 0 \;,\quad n_\alpha \ddot{\mathcal{C}}^\alpha - \ddot{\mathcal{C}}_o^3 = 0 \qquad (77)$$

which are the boundary conditions on $\mathcal{C}$

Equations (49) yield the remaining boundary conditions on the faces as

$$\acute{\bar{v}}_i - \acute{\bar{v}}_i^{\,o} = 0 \;,\quad \acute{\bar{w}}_i - \acute{\bar{w}}_i^{\,o} = 0 \;;\quad \ddot{\bar{v}}_i - \ddot{\bar{v}}_i^{\,o} = 0 \;,\quad \ddot{\bar{w}}_i - \ddot{\bar{w}}_i^{\,o} = 0 \qquad (78)$$

$$\acute{\bar{\varphi}} - \acute{\bar{\varphi}}_o = 0 \;,\quad \acute{\bar{\Psi}} - \acute{\bar{\Psi}}_o = 0 \;;\quad \ddot{\bar{\varphi}} - \ddot{\bar{\varphi}}_o = 0 \;,\quad \ddot{\bar{\Psi}} - \ddot{\bar{\Psi}}_o = 0 \qquad (79)$$

Lastly, we record the initial conditions based on (6) :

$$\acute{v}_i = \acute{w}_i = \tilde{w}_i = \ddot{w}_i = 0 \;,\quad \acute{\varphi} = \acute{\Psi} = \tilde{\Psi} = \ddot{\Psi} = 0 \quad \text{on } \mathcal{A} \times (-\infty, t_o) \qquad (80)$$

These equations are already considered in formulating the aforementioned sandwich shell equations.

6. Special Cases

The results of the preceeding sections are now specialized to obtain the following cases of interest.

Viscoelastic Sandwich Plates :

To begin with, we note that in the absence of curvature effects, that is, $b_{\alpha\beta} = 0$, the union of the macroscopic equations of motion (58-65), equations of heat conduction (68-71), constitutive relations (45,46), the strain distributions (42), and the initial and boundary conditions, (80) and (75-79), constitutes a linear, coupled theory of viscoelastic sandwich plates for the case when the displacement and temperature fields are linearized. The theory, as in that of sandwich shells, incorporates the effects of transverse normal strains and shear deformations and the rotatory inertia.

Isotropic Constituents :

When the constituents of sandwich shells are made of isotropic thermo-viscoelastic materials the relaxation functions of (11) take the form :

$$\left.\begin{aligned} &\{F_{ij}\,,\,K_{ij}\,,\,\varkappa_{ij}\} = \{f\,,\,k\,,\,\varkappa\}\, g_{ij} \;,\quad B_{ijk} = 0 \\ &G_{ijkl} = \tfrac{1}{3}(G_2 - G_1)\, g_{ij}\, g_{kl} + \tfrac{1}{2} G_1 (g_{ik}\, g_{jl} + g_{il}\, g_{jk}) \end{aligned}\right\} \qquad (80a)$$

Here, f , k , $\varkappa$, G_1 and G_2 are the independent relaxation functions. The functions k , G_1 and G_2 are respectively defined relative to the states of shear, dilatation and temperature. The functions G_1 and G_2 may be expressed :

$$G_1 = 2\mu \;,\quad G_2 = 3\lambda + 2\mu \qquad (80b)$$

in terms of Lamé's viscoelasticity constants. Consequently, the set of macroscopic constitutive relations simplifies considerably.

Elastic Constituents :

For elastic constituent materials, the relaxation function B_{ijk} vanishes identically, the relaxation function M can be expressed by

$$B_{ijk} = 0 \ , \quad M = (c_0/T_0)\,\delta(t) \qquad (81a)$$

where $\delta(t)$ denoting the Heaviside unit step function and c_0 the specific heat of the material, and the remaining relaxation functions in the Stieltjes convolutions become constant elastic moduli, namely

$$G_{ijkl}(t) = G_{ijkl}\ \delta(t)\ , \quad F_{ij}(t) = -\beta_{ij}\ \delta(t)\ , \quad \varkappa_{ij} = \frac{\beta_{ij}}{\rho}\ \delta(t) \qquad (81b)$$

in which β_{ij} are thermoelastic coefficients.

In order to conserve space, the governing equations of the above special cases are not explicitly written down here.

7. Closure

Thus far, a linear non-isothermal theory of viscoelastic non-polar sandwich shells and plates is constructed, and some special cases of interest are indicated. First, the displacement and temperature fields are linearized with respect to the thickness coordinate. The continuity of the fields at the interfaces between two adjacent constituents is then satisfied automatically by a suitable selection of the displacement and temperature functions. Next, based on the linearized fields, the equations to describe the mechanical behavior of each discrete constituent are derived by the use of an averaging procedure [7] . In the last stage, in view of the selected displacement and temperature functions, the system of equations for discrete layers is combined to yield the macroscopic equations of motion, heat conduction, boundary and initial conditions, and constitutive relations of the three-layer sandwich shell. Moreover, the general results are specialized to get the cases where the constituents of the sandwich shell are made of either isotropic thermo-viscoelastic or anisotropic thermoelastic materials, and the curvature effects disappear, that is, sandwich plates (cf., for instance, [13]).

In the governing equations, if the terms involving, for instance, with the core and upper facing are omitted, we obtain a linear non-isothermal theory of viscoelastic, anisotropic shells (cf., [14, 15] for elastic shells). Also, dropping out the kinematical variables we obtain a set of heat conduction equations for the sandwich shell. Further, for quasi-static problems all the inertia terms can be neglected.

In closing, we emphasize the fact that the consideration of irreversible stresses is left untreated in the present investigation. A general analysis which is based on local equilibrium and reveals irreversibilities associated with the global non-equilibrium [16, 17] is being studied and will be reported elsewhere.

L 2/

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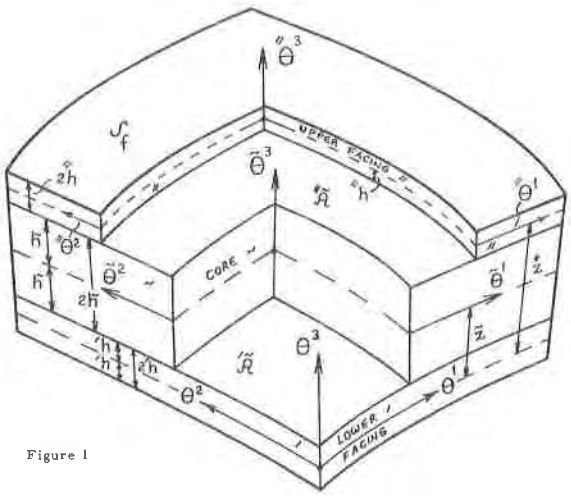

Figure 1