DETERMINATION OF STRESSES DUE TO DISCONTINUITIES IN FINITE PLATES OF ISOTROPIC AND ORTHOTROPIC MATERIALS

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SUMMARY

Stress concentrations are present in all engineering structures due to holes, cracks, sharp corners and discontinuities. They occur in reactor structural parts like core support structures and reactor vessel domes. The determination of stresses in such cases are of primary importance for assessing and ensuring satisfactory performance in respect of strength, fatigue and fracture. Certain investigations of stress concentrations (originally carried out for aerospace structures) are of relevance to reactor technology and are reported here. In these studies, a continuum approach has been followed and effective techniques have been developed for the accurate determination of stresses around circular holes and corner zones in plates made of isotropic and orthotropic materials.

The solutions and method of analysis are first developed for finite isotropic plates. Plates with different hole sizes including large sized circular holes whose dimensions are comparable to the plate width are successfully dealt with. New solutions and techniques to deal with concentrations in orthotropic domains are then developed and successfully applied to circular and sharp corner discontinuities.

In the first part of the paper, plates perforated with regular square and hexagonal arrays of circular holes are considered. Internal pressure and various external edge conditions are applied. Effects of hole size, geometry and loading are investigated. Effect of orthotropy is studied in the next part. Rectangular orthotropic plates with central circular holes and the axes of orthotropy inclined to the edges are analysed. Both internal and external pressures are dealt with. Finally stress concentrations in corner zones of orthotropic plates are investigated for certain classes of edge conditions.

In all the three studies, stress functions are set up in polar trigonometric series. The conditions on the circular hole are satisfied exactly for isotropic plates and approximately by collocation for orthotropic plates. In either case, conditions on the straight edges are satisfied by suitable approximate techniques.

Rapidly converging results are obtained for all the cases analysed and for hole sizes as large as 0.9 time the plate width. The accuracy of solution depends on stage of approximation, hole size, load system, symmetries in the field and the degree and orientation of orthotropy. The stress concentration factors in case of alternating external stress are much higher than those due to applied uniform loading. The physical reason is that in the cyclic stress case, each segment acts like a beam whereas in the uniform case, each segment acts like an arch.

Effects of the degree and orientation of the orthotropy on the stress concentration factors for a few typical hole sizes and for typical angular corners in wedges are investigated. The results show that reductions in stress concentration factors can be achieved by a suitable choice of orthotropy.

1. INTRODUCTION

Stress concentrations are present in all engineering structures due to holes, cracks, sharp corners and other geometrical discontinuities, material discontinuities and applied stress discontinuities. The determination of stresses in such cases is of importance for assessing and ensuring satisfactory performance in respect of strength, fatigue and fracture. We will discuss here some aspects of geometrical discontinuities that occur in reactor structural parts like core support structures and reactor vessel roofs.

1.1 THE PHENOMENON

The geometrical discontinuities generally encountered in structural practice are of two types: (1) holes with smooth profiles and (2) cracks and sharp corners formed by rectilinear boundaries. In the first type the concentrations are always finite, whereas in the second category they can be, in an elastic sense, finite or infinite. A finite concentration is characterized by its magnitude, whereas an infinite concentration (singularity) is represented by two parameters, viz., the strength and the order of singularity (vide notation). Also, from a fracture mechanics view point, an elastic singularity is often represented by a "stress intensity factor". The actual concentration is dependent on one or more of the following parameters: (i) local geometry and material properties, (ii) local applied loading, (iii) far field geometry and (iv) far field applied loading. At a sharp corner, the possibility of an infinite concentration and its order depends only on local conditions of geometry, material properties and edge conditions, whereas its strength is determined by the loading and far field geometry. If at a corner, a singularity is not developed, the finite concentration is dependent only on the local conditions of geometry, edge supports, materials and loading. The magnitude of a finite concentration due to a smooth hole depends on all local as well as far field conditions. These observations are summarized in Table I.

TABLE I : The phenomenon of stress concentration at geometric discontinuities

Types of discontinuities	Holes	Sharp corners		
Nature of concentration	Finite	Finite	Infinite	
Characteristic parameters	Magnitude of concentration	Magnitude of concentration	Order of singularity	Strength of singularity
Influencing agency	Local and far field geometry and loading	Local geometry and loading	Local geometry	Local and far field geometry and loading

1.2 SCOPE OF PRESENTATION

In the present work we choose the problems to study two facets of the stress concentration phenomenon, namely, the magnitude and decay of finite concentrations at smooth holes and the order of singularities at sharp corners. In the first case of finite concentration, we extend the available knowledge of stresses around circular holes in finite isotropic fields and indicate methods of analysing finite orthotropic fields with circular holes. In the second case of sharp corners, we extend the available knowledge of isotropic fields into the domain of orthotropic fields.

We follow throughout a continuum method of approach and a direct (boundary) method of analysis, and develop effective techniques for analysing finite fields with concentrations. We may emphasize here that the concepts and stress functions so developed can be utilized in analytical and numerical solutions of finite fields and, even more important in the finite element treatment of complex problems of single and multiple concentrations by continuum-finite element hybrid methods, such as those of Yamamoto, Morley and Rao et al. [1] .

1.3 REVIEW OF EARLIER WORK

A circular hole in an infinite plate is a classical problem with a simple closed form solution. Attempts to analyse the more practical problems of circular holes in finite plates have gone through various phases, but it is only in recent years that effective analytical methods have been established. The bulk of such work has been on central circular holes in regular polygonal plates. A detailed study of filled and unfilled holes in general rectilinear fields was reported by Venkataraman [2] and the major portion of this work remains unpublished. An extensive review of earlier work till 1950 (mostly on infinite domains) has been given by Savin [3] . Venkataraman [2] has reviewed the work on finite domains upto 1966. More recent work can be seen in Applied Mechanics Reviews (AMR).

The earliest attempts to use stress functions which isolate the singular parts of the solution for stress analysis of isotropic material can be traced back to Klein (1927), Westergaard (1933) and Brahtz (1933)[4]. Their procedure of using biharmonic "corner functions" that identically satisfy the local homogeneous boundary conditions was generalised by Williams [5,6] for two-dimensional extension and flexure. More recently Rao [7] proposed a unified approach for the problems of concentrations in multi-material domains with interfaces between different isotropic materials. This has been extended for anisotropic fields by Dattaguru[8].

2. ISOTROPIC PLATES WITH CIRCULAR HOLES

The method of analysis is conveniently and briefly described with reference to regular polygonal plates with central circular holes, Fig.1.

2.1 REGULAR POLYGONS WITH CENTRAL HOLES

Consider an n-sided regular polygon with a central circular hole under internal pressure p_i and external pressure p_o. The governing differential equation for the domain is the well known biharmonic equation

$$\nabla^4 \phi = 0 \qquad ..(1)$$

The boundary conditions on the inner circular edge are

$$\sigma_r = p_i \quad \text{and} \quad \sigma_{r\theta} = 0 \quad \text{on} \quad r = c \; . \qquad ..(2)$$

A stress function (in a truncated infinite series) satisfying the governing differential equation, the conditions on the inner edge and the n-fold symmetry of the problem is provided by.

$$\begin{aligned} \phi = & \frac{p_i c^2}{2(1-c^2)} \left[\ell n \left(\frac{r}{c}\right)^2 + c^2 - r^2 \right] \\ & + \sum_{m=1}^{M} \frac{A_m}{m^4} \left[r^{mn} - mn\, c^{2mn-2}\, r^{-mn+2} + (mn-1)\, c^{2mn}\, r^{-mn} \right] \cos mn\theta \\ & + \sum_{m=1}^{M} \frac{B_m}{m^4} \left[r^{mn+2} - (mn+1)\, c^{2mn}\, r^{-mn+2} + mn\, c^{2mn+2}\, r^{-mn} \right] \cos mn\theta \end{aligned} \qquad ..(3)$$

The arbitrary parameters A_m, B_m are to be determined by approximately satisfying the boundary conditions on one half of an edge on the outer boundary, say

$$\sigma_x = p_o \quad \text{and} \quad \sigma_{xy} = 0 \quad \text{on} \quad x = a \; . \qquad ..(4)$$

This can be done by a suitable method (see Collatz [9] and Rajaiah & Rao [10]). Here we make use of the polynomial expansion and direct collocation procedures.

Stress functions similar to eq.(3) can be written for various boundary conditions around circular hole, provided the nature of the condition does not change around the periphery. A very important application of this method is in the interference fit problems in joints.

2.2 REGULAR ARRAY OF CIRCULAR HOLES

When the plate has a regular array of holes in an infinite plate, a convenient repeating element can be chosen in the form of a regular polygon of sides 3, 4 or 6. The stress function, eq.(3) is applicable, but the external edge boundary conditions should represent appropriate symmetry conditions. For instance, for a square array with uniform pressure p_o in all directions at infinity, we replace eq.(4) by

$$\phi_x = p_o\, a, \quad \nabla^2 \phi_x = 0 \quad \text{on} \quad x = a. \qquad ..(5)$$

Otherwise the procedure is identical.

2.3 ARBITRARY EXTERNAL SHAPES

The stress function, eq.(3) above can be applied with any number of axes of symmetry down to $n = 1$. When there is no axis of symmetry, we use eq.(3) with $n = 1$ and also introduce two additional series (with additional arbitrary parameters) in which $\cos m\theta$ is replaced by $\sin m\theta$. All the arbitrary parameters are to be determined by approximately satisfying external edge boundary conditions. In fact, in principle, the external edges can be quite arbitrary. However, the shape of external boundary may adversely influence the convergence of solution, in which case a more sophisticated procedure than direct collocation will have to be applied, (see reference [2]). As the differential equation and boundary conditions are identically satisfied around the hole, the order of accuracy is generally well maintained around the hole where the stress concentration occurs.

3. ORTHOTROPIC PLATES WITH CIRCULAR HOLES

The governing differential equation for an orthotropic sheet can be written as

$$k_x \frac{\partial^4 \phi}{\partial x^4} + 2k_{xy} \frac{\partial^4 \phi}{\partial x^2\, \partial y^2} + k_y \frac{\partial^4 \phi}{\partial y^4} = 0 . \qquad ..(6)$$

A simple stress function which simultaneously satisfies this governing differential equation and stipulated boundary conditions on the inner circular edge $(r=c)$ is not available. Thus, we have to set up a general stress function

$$\phi = \sum_{m=1}^{M} r^m \sum_{i=1,2} \left(\frac{\cos\theta}{\cos\theta_i}\right)^m \left[A_{mi} \cos m\theta_i + B_{mi} \sin m\theta_i\right]$$

$$+ \sum_{m=1}^{M} r^{-m} \sum_{i=1,2} \left(\frac{\cos\theta}{\cos\theta_i}\right)^{-m} \left[C_{mi} \cos m\theta_i + D_{mi} \sin m\theta_i\right] , \qquad ..(7)$$

and only approximately satisfy the boundary conditions on both the inner and outer edges. When an axis of symmetry of the stress field coincides with $\theta = 0$, the sine functions drop out. In an orthotropic sheet, irrespective of the order of geometrical or load symmetries, one cannot have a stress field with more than two axes of symmetry. The two symmetry axes occur when the geometrical and the loading axes of symmetry coincide with those of elastic symmetries. If the geometry has any symmetries in it, then the field exhibits skew-symmetry $(F(\theta) = F(\pi + \theta))$ and ϕ is correspondingly simplified by dropping the odd values of m .

4. SHARP CORNERS IN ORTHOTROPIC SHEETS

Consider a corner formed by two rectilinear boundaries in a finite domain. An infinite stress concentration can occur at such a corner. A method of identifying the order of this singularity in isotropic sheets is known, Williams [5,6] , Rao [7] . Herein, we will develop a corresponding analysis for angular corners in orthotropic plates. We proceed to study the effect of local geometry on the order of singularities at crack tips and notches with arbitrary orientation of orthotropy.

A typical configuration of an angular corner between free-free edges in an orthotropic field is shown in Fig.6. On the edges at the corner, we have four homogeneous boundary conditions,

$$\sigma_{\theta} = 0 \quad , \quad \sigma_{r\theta} = 0 \quad \text{on} \quad \theta = -\beta \quad \text{and} \quad \theta = \alpha - \beta \ . \qquad ..(8)$$

Choose the origin of co-ordinates at the corner and write a stress function in variable separable series

$$\phi = \sum_{m} r^{\lambda_m+1} F(\lambda_m, \theta) \qquad ..(9)$$

where

$$F(\lambda_m, \theta) = \left(\frac{\cos\theta}{\cos\theta_1}\right)^{\lambda_m+1} \left[A_m \cos(\lambda_m+1)\theta_1 + B_m \sin(\lambda_m+1)\theta_1\right]$$

$$+ \left(\frac{\cos\theta}{\cos\theta_2}\right)^{\lambda_m+1} \left[C_m \cos(\lambda_m+1)\theta_2 + D_m \sin(\lambda_m+1)\theta_2\right] ,$$

$\theta_i = \tan^{-1}(\gamma_i \tan\theta)$, $i = 1,2$. $F(\lambda_m, \theta)$ is such that ϕ identically satisfies the governing eq.(6). λ_m's can be real or complex and the requirement of finite displacements at the corner places a restriction that real part of all λ_m's should be positive. Introduction of ϕ from eq.(9) into the boundary conditions eq.(8) yields for any m, four homogeneous linear algebraic equations in terms of the four parameters A_m , B_m , C_m , D_m. For a non-trivial solution of this set, its determinant is set to zero, resulting in a characteristic equation for λ_m. This characteristic equation is conveniently solved by the well-known Regula-Falsi method, Scarborough [11] . An infinite concentration is possible at the corner for each eigenvalue for which $0 < \mathrm{Re}\,\lambda_m < 1$.

The lengthy characteristic equations are given in reference [8] . In practice it is advisable to solve the characteristic equation on a digital computer by writing the programme for the original determinantal form.

A similar procedure is applicable to any combination of homogeneous boundary conditions on the edges at the corner.

The functions obtained from this analysis are not applicable to infinite fields because the stresses grow indefinitely away from the corner. But they can be used for the analysis of finite domains. In an analytical sense, the series of functions does not always provide a complete solution for the problem, but with suitable analytical and numerical techniques it can invariably provide a very satisfactory numerical solution, Dattaguru and Rao [12].

5. RESULTS AND DISCUSSION

Extensive numerical evaluation with careful convergence studies has been carried out for the problems considered. In each case, variations in significant quantities like maximum stress were studied with respect to variations in the parameters of the problem. Some typical information is condensed into Figs. 2 to 6. A brief discussion of these results is also presented below.

5.1 PLATES WITH CIRCULAR HOLES

The square : Isotropic perforated square plates and square repeating elements from infinite plates with a regular array of perforations, both of them under uniform internal pressure around the hole, are analysed for various hole sizes ($c/a = 0.25, 0.5, 0.75, 0.8$ and 0.9). The hoop stress variation around the circular hole is shown in Fig.2. The results show parallel trends for the individual squares and the repeating elements. For small hole sizes (c/a upto about 0.4), the hoop stress σ_θ around the hole is nearly uniform. With increasing hole size the variation of σ_θ on the hole changes its pattern.

For the individual square in the range $c/a = 0.4$ to about 0.75, σ_θ decreases from the diagonal point B' to the central point A'. As c/a increases beyond 0.75, the location of the maximum σ_θ moves from B' towards A'. Further, beyond $c/a = 0.75$, the absolute maximum σ_θ in the plate shifts to A, the center of the external edge (Fig.4).

On the other hand, for square repeating elements with hole sizes c/a less than 0.5 the hoop stress is maximum at A' and decreases towards B'. As c/a increases beyond 0.5, the peak stress moves away from A' towards B'.

The hexagon : The hoop stress variation across the thickness AA' in isotropic hexagonal plates with different hole sizes ($c/a = 0.25, 0.5, 0.704, 0.75$) is given in Fig.3. When the hole size is small (say $c/a \leq 0.5$), the stress distribution is close to that of Lame's solution

$$\sigma_\theta / p_1 = (1 + \frac{a^2}{r^2})/(1 - \frac{a^2}{c^2}) \qquad ..(10)$$

in which the peak hoop stress occurs on the inner edge and is substantially greater than that at the outer edge. However, for larger holes, the maximum hoop stress across the thickness shifts from the inner boundary to the outer boundary.

General : Fig.4 is useful to decide whether the peak stress due to internal pressure around a hole in a regular polygon occurs on the hole or on the outer boundary. For instance, the approximate critical hole size (c/a) for the shift is 0.72 for the square, 0.78 for the hexagon and 0.82 for the octagon.

Considering the polygons loaded on the external edges, a uniform stress around the edge causes a modest stress concentration whereas a cyclic loading of tension and compression on alternating edges gives rise to very large stress concentrations. For example, for a square plate with a hole size $c/a = 0.5$, the uniform and the alternating load cases give rise to stress concentrations of 1.72 and 10.4 respectively. This phenomenon is easily explained by the fact that the behaviour of each segment is primarily as an arch for uniform loading and as a beam for cyclic loading (Fig.5).

5.2 SHARP CORNERS IN ORTHOTROPIC PLATES

The variation of the least eigenvalue with corner angle and degree of orthotropy for a specific orientation of the axes of orthotropy is shown in Fig.6. Denoting the longitudinal direction along the bisector of the corner angle, increasing the longitudinal elasticity modulus leads to decrease in the severity of order of the singularity and increasing the transverse modulus leads to an increase in the severity of the singularity. Obviously the relative orientation of the edges and the axes of orthotropy influences the order of singularity. Thus, by choosing suitable orientation and degree of orthotropy, one can control the severity of the singularities. However, for two cases, $\alpha = 180°$ (i.e., loading on a straight edge) and $\alpha = 360°$ (i.e., for a crack tip) the eigenvalues and the singular term are independent of the degree of orthotropy.

6. CONCLUDING REMARKS

In the limited space of this paper, we have indicated certain salient features of the problems under discussion. It is proposed to present a more comprehensive picture in a full length paper in the near future. In particular, it should be emphasized that the sequence of corner functions obtained for orthotropic sheets can be utilised for the analysis of stresses in orthotropic plates with sharp discontinuities.

7. ACKNOWLEDGEMENTS

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NOTATION: **GENERAL**

a	- plate dimensions
A_m, B_m, C_m, D_m	- arbitary parameters, m being integers indicating the sequence
c	- hole radius
M	- order (maximum m) of truncation of infinite series
n	- order of multiple symmetry in the problem
p_i	- internal pressure
x, y	- Cartesian coordinates
r, θ	- polar coordinates
	- angle of the corner
∇^2	$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$
ϕ	- Airy stress function
$\sigma_x, \sigma_y, \sigma_{xy}$	- direct and shear stresses in Cartesian coordinates
$\sigma_r, \sigma_\theta, \sigma_{r\theta}$	- direct and shear stresses in polar coordinates
λ	- sequence of eigenvalues

Subscripts to ϕ

x, y, r, used as a subscript to ϕ denote partial differentiation, (e.g., $\phi_x = \partial\phi/\partial x$, $\phi_y = \partial\phi/\partial y$ etc.)

Stress singularity

singular stresses are represented as $\sigma \sim Ar^{-k}$ and k is the order of singularity and A is the strength of singularity.

ORTHOTROPIC PLATES

$C_{j\ell}$, $j = 1,2,3$; $\ell = 1,2,3$
- elastic constants relating strains ($\epsilon_x, \epsilon_y, \epsilon_{xy}$) and stresses ($\sigma_x, \sigma_y, \sigma_{xy}$), i.e.

$$\begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_{xy} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{12} & C_{22} & 0 \\ 0 & 0 & C_{33} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix}$$

k_x	$= C_{22}$
k_{xy}	$= (2C_{12} + C_{33})/2$
k_y	$= C_{11}$
η_1^2, η_2^2	- roots of the quadratic , $k_x - 2k\,\eta^2 + k_y\,\eta^4 = 0$
θ_i	$= \tan^{-1}(\eta_i \tan\theta)$, $i = 1,2$

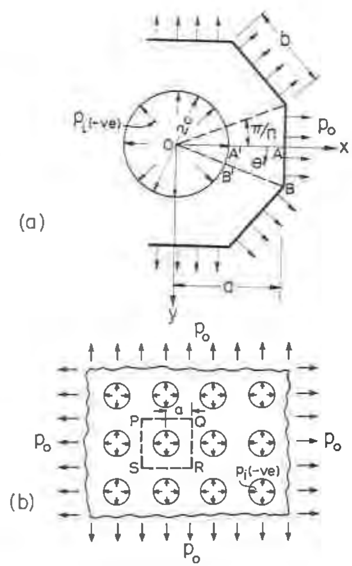

Figure 1 Typical plate configurations with circular holes
(a) Regular polygon with a central circular hole
(b) Regular array of holes with square repeating element

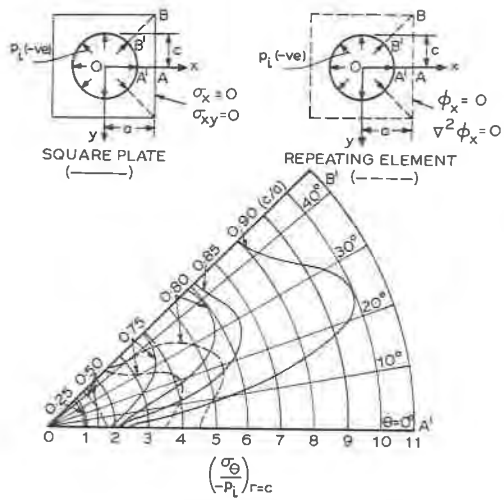

Figure 2 Hoop stress along loaded circular hole: Shift of peak stress position with hole size

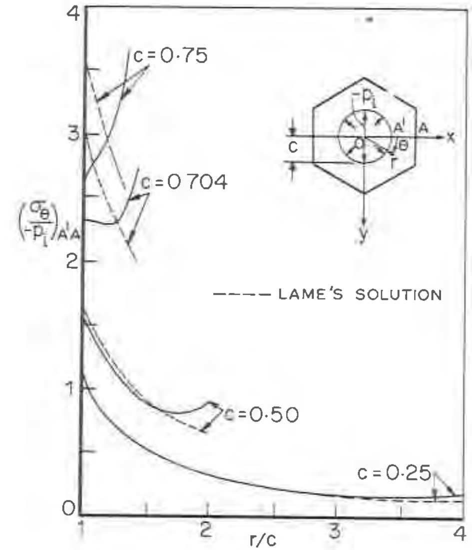

Figure 3 Hoop stress distribution along centre line A'A in hollow hexagon

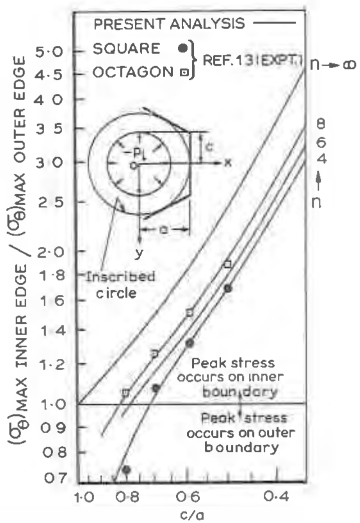

Figure 4 Location of maximum hoop stress due to internal pressure on holes in regular polygons

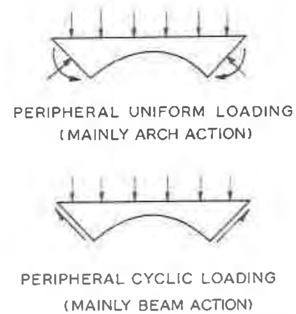

Figure 5 Arch and beam action of segments of hollow polygons under uniform and cyclic external loading

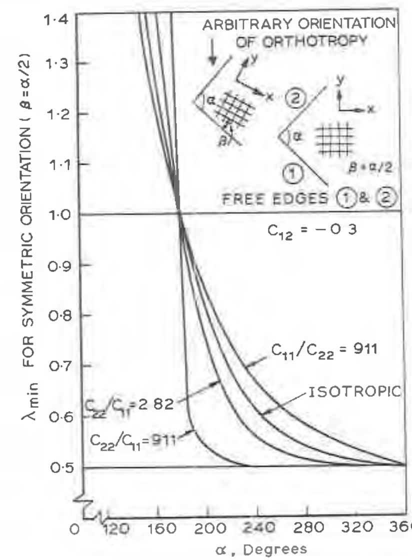

Figure 6 Least eigenvalue for orthotropic plate with free edges and $\beta = \alpha/2$