AN IMPROVED ENERGY CONSERVING IMPLICIT TIME INTEGRATION ALGORITHM FOR NONLINEAR DYNAMIC STRUCTURAL ANALYSIS

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SUMMARY

Introduction. — The recent literature reflects much concern about choice, stability, accuracy and cost of implicit/explicit direct integration schemes as applied to nonlinear structures. The comparative study by Hartung and Ball (1973) indicates that results to nonlinear problems obtained with existing computer codes may "show virtually no agreement". In linear analysis unconditionally stable schemes have been reported unstable for approximate solution schemes, but stable with fully iterative schemes. The Houbolt scheme has been reported stable despite, or perhaps because of its heavy numerical damping. Some authors modify existing schemes, pushing stability limits of approximate solution schemes, while others introduce families of schemes whose higher order members may be too heavy to built into existing computer codes.

This study proposes a general nonlinear algorithm stability criterion; it introduces a nonlinear algorithm, easily implemented in existing incremental/iterative codes, and it applies the new scheme beneficially to problems of linear elastic dynamic snap buckling.

Errors and stability. — The truncation errors commonly recognized in linear applications are the "period elongation" and the "amplitude decay". Nonlinear analysis recognizes in addition errors due to nonconvergence and an error which can be attributed to the inability of the scheme to correctly calculate the changes in stored elastic potential within each time step. For example, the constant acceleration Newmark algorithm (trapezoidal rule), integrating correctly elastic potentials up to second order in the displacements (linear analysis), introduces cyclic errors in amplitude if applied to nonlinear vibrators, which prove to be periodic amplitude overestimations for softening, and underestimations for stiffening oscillators.

The linear A-stability criteria are in general inapplicable to nonlinear systems. Therefore, a new and stronger nonlinear criterion for unconditional stability is proposed, according to which an algorithm, applied to any nonlinear system is unconditionally stable, if, independent of the time step used, the total sum of energy contained in the system at each time step is less than or equal to the total energy of the previous time step.

Improved algorithm. — Based on the concept of energy conservation, the paper outlines an algorithm which degenerates into the trapezoidal rule, if applied to linear systems. The new algorithm conserves energy in systems having elastic potentials up to the fourth order in the displacements. This is true in the important case of nonlinear total Lagrange formulations where linear elastic material properties are substituted.

The scheme is easily implemented in existing incremental/iterative codes with provisions for stiffness reformation and containing the basic Newmark scheme.

Examples. — Numerical analyses of dynamic stability can be dramatically sensitive to amplitude errors, because damping algorithms may mask, and overestimating schemes may numerically trigger, the physical instability. The newly proposed scheme has been applied with larger time steps and less cost to the dynamic snap buckling of simple one and multi degree-of-freedom structures for various initial conditions.

Conclusions. — A general nonlinear dynamic stability criterion of algorithms has been proposed. An implicit algorithm, conserving energy in the important case of total Lagrange formulation has been introduced and successfully applied to problems of dynamic snap buckling.

1. Introduction

This paper concerns the stability and accuracy of solutions obtained in the nonlinear dynamic analysis of structures with the finite element method. Special emphasis is given to physically unstable structures, e.g. structures which might experience dynamic snap through buckling and related dynamic collapse phenomena.

The paper reflects a study |1| recently carried out in order to clarify the factors contributing to the quality of implicit nonlinear direct integration dynamic solutions, such as precision, stability and cost. It also defines a new implicit algorithm better suited for the numerical analysis of problems of dynamic stability, involving thin shell structures.

Contradictory results have been reported in the field of nonlinear structural dynamics. Among others Hartung and Ball |2|, applying several computer codes to an impulse loaded thin shell truncated cone of elastic material, have summarized their findings noting that the nonlinear results may "show virtually no agreement", while the linear elastic results, using the same codes, differed only very little.

Other authors, applying the familiar implicit direct integration techniques, such as the Newmark, Wilson and Houbolt schemes, to nonlinear problems, have reported numerical instabilities for larger time steps, although the mentioned schemes are unconditionally stable in linear applications |3,4|. Those failures appear to occur if the incremental solution at each time step is approximate as for the "pseudo-force" method or for the "load-correction" method.

Since dynamic snap phenomena are very nonlinear, only fully iterative solutions are considered here, while some authors concentrate on improving the algorithmic stability of basic algorithms if used with approximate solutions |5|. To this end the paper introduces a broader definition of algorithmic stability, based on local energy balance and proposes an improved scheme by modifying the Newmark algorithm so as to conserve energy in structures having elastic potentials of up to fourth order in the displacements.

This case is important, because the widely used large displacement "total Lagrange" formulation has fourth order elastic potentials, when constant elastic properties are substituted*.

The applications run with the improved scheme show an increase in precision and, due to the larger time steps possible, a significant reduction of analysis cost, if the new algorithm is applied to highly nonlinear dynamic problems such as snap through buckling. The advantage of the scheme is also made apparent in formulations involving higher than fourth order elastic potentials.

2. Errors of Direct Integration Schemes

The errors typically associated with the application of direct integration techniques to linear vibrations are : period elongation and amplitude decay |6,7|.

If applied to nonlinear oscillators, the familiar direct integration schemes |1| produce pseudo-period elongations and amplitude decays, just as in linear applications, plus an extra truncation error if the solution has not converged within each time step, and a new type of error, discussed below, termed local energy imbalance.

*NOTE : while linearly elastic materials in infinitesimal displacement theory produce second order elastic potentials, large displacement "updated Lagrange" formulations, however, have infinite order constant elastic property potentials.

- Pseudo-period elongations and amplitude decay can be reduced through a reduction of the time step.

- The extra truncation error due to non-convergence of the solution can be eliminated through iteration for dynamic equilibrium in each time step |4|.

- The new energy error can also be reduced with smaller time steps ; it appears, however, only in nonlinear applications.

Local Energy Imbalance

A closer look reveals the energy error to be linked to a miscalculation of the elastic energy stored as the solution progresses in time, even if the solution has converged in each time step |8|.

Consider the basic Newmark scheme (trapezoïdal rule) applied to an undamped nonlinear oscillator with the equation of motion :

$$m\ddot{x} + f(x) = 0 \qquad (1)$$

where m = mass ; x, $\dot{x}$, $\ddot{x}$ = displacements, velocities and accelerations and $f(x)$ = nonlinear spring force. Newmark's basic scheme can be written as follows :

$$x_1 - x_0 = \Delta t/2\ (\dot{x}_1 + \dot{x}_0) \qquad (2)$$

$$\dot{x}_1 - \dot{x}_0 = \Delta t/2\ (\ddot{x}_1 + \ddot{x}_0) = -m^{-1}\ \Delta t/2\ (f_1 + f_0) \qquad (3)$$

where, throughout this paper, subscripts 0 and 1 denote the beginning and the end of each time step, $\Delta t = t_1 - t_0$. The increment in kinetic energy, ΔC_{01}, within step, Δt, is given by

$$\Delta C_{01} \equiv 1/2m\ (\dot{x}_1^{\,2} - \dot{x}_0^{\,2}) \equiv 1/2\ (\dot{x}_1 - \dot{x}_0)\ m\ (\dot{x}_1 + \dot{x}_0) \qquad (4)$$

This becomes with the help of eqs.(2) and (3)

$$1/2m\ (\dot{x}_1^{\,2} - \dot{x}_0^{\,2}) = -1/2\ (f_1 + f_0)\ (x_1 - x_0) \qquad (5)$$

and the geometric interpretation of the r.h.s. of eq.(5) shows that the increment in kinetic energy is equated to the trapezoïdal approximation of the increment of elastic energy, as shown in fig. 1.

The shaded area in fig. 1 can be interpreted as the loss in kinetic energy as the solution progresses from state 0 to state 1, and the blank area between the curve and the trapezoïd upper edge represents an under-estimate of the loss in kinetic energy, i.e. the system arrives at state 1 with too high a velocity.

Upon reversal of the direction of motion i.e. if the solution progresses from state 1 to state 0, it is apparent that the blank area represents an underestimate in gain of kinetic energy, and that both errors tend to cancel each other if a constant time step is used over several periods of vibration.

Fig. 2 shows the typical responses of a stiffening and of a softening nonlinear oscillator using the trapezoïdal rule with equilibrium iteration in each time step.

Since the trapezoïdal rule does not exhibit numerical damping, the amplitude errors occur periodically with the same magnitude. Using an algorithm with inherent numerical damping, the amplitude errors will be increasing, however, in time.

Note, that for the stiffening spring the amplitude is underestimated, while for the softening spring the amplitudes are overestimated. Also, an extra period elongation occurs in the softening case, which is due to the physical increase in period with increasing amplitudes.

The local energy imbalance is a far less dangerous error as for example the constant energy injection associated with linearly unstable algorithms. Nevertheless, it can give rise to numerically triggered physical instabilities of potentially unstable structures.

3. Improved Dynamic Algorithms

Stability of algorithms in linear analysis is usually assessed as follows. A linear, undamped oscillator is considered and the solution via an algorithm, starting from some initial conditions, is reduced to the recursive application of an operator matrix, A , whose spectral radius, ρ , must be less than or equal to unity, if the discrete solution sequence is to remain bounded in time.

Because of its generality, the concept of energy balance is chosen here as a basic criterion for the stability of dynamic algorithms in nonlinear applications. For clarity, conservative systems are considered in the following developments, that is systems without dissipation of mechanical energy.

Definition : Let $S(t)$ be a system in evolution, with C = kinetic energy, W = total potential energy. Then, for conservative systems, an algorithm is said to be unconditionally stable, if for all initial conditions and time steps chosen, the following inequality holds :

$$C(\dot{x}_n) + W(x_n) \geq C(\dot{x}_{n+1}) + W(x_{n+1})$$

In this expression, $\dot{x}_n$, $\dot{x}_{n+1}$, x_n, x_{n+1}, are velocities and displacements at time t_n and t_{n+1} respectively, and in linear applications, e.g.

$$W = 1/2\, k\, x^2 - f(x)$$

$$C = 1/2\, m\, \dot{x}^2$$

Improved Energy Conserving Algorithms

As a most basic requirement, stable algorithms must not inject mechanical energy globally into the system. More stringently, energy should also be conserved locally from time step to time step, to avoid, for example, physical instabilities to be predicted too soon, or to be masked and missed by the algorithm solution sequence.

In seeking better algorithms, one may start out from known ones, efficient for linear analyses, and try to improve their energy conservation in nonlinear applications. This is done in some detail below for the well known Newmark trapezoïdal rule.

A better representation of the stored elastic energy and thus of the kinetic energy, than given in eq.(5), will be :

$$1/2m(\dot{x}_1^2 - \dot{x}_0^2) = -1/2(f_1+f_0)\,(x_1-x_0) + 1/12(x_1-x_0)\,(f_{1,x}-f_{0,x})\,(x_1-x_0) \qquad (6)$$

where the second term on the r.h.s. has been added for a better representation of the area under the curve, $f(x)$, in fig. 1.

In eq.(6) $f_{,x} = \partial f/\partial x = k(x)$ = nonlinear spring stiffness.

Using the trapezoïdal rule for the displacements, eq.(2), and working backwards from eq.(6) via eq.(4), one obtains the following difference expressions :

$$x_1 - x_0 = \Delta t/2\ (\dot{x}_1 + \dot{x}_0) \qquad (2)$$

$$\dot{x}_1 - \dot{x}_0 = -M^{-1}\left[\Delta t/2\ (f_1 + f_0) - \Delta t/12\ (k_1 - k_0)\ (x_1 - x_0)\right] \qquad (7)$$

Eqs.(2) and (7) constitute a new algorithm, based on the trapezoïdal rule, conserving energy locally for structures having elastic potentials up to fourth order in the displacements.

It should be noted, that higher order expansion than given through the r.h.s. of eq.(6) can be substituted, which will further improve the algorithm, constituting a whole family of algorithms, based on the trapezoïdal rule. This would complicate considerably the implementation of the algorithms into large scale finite element computer programs, involving derivatives of stiffness matrices w.r.t. the displacements and expensive matrix multiplications. Note, that the whole family based on the trapezoïdal rule does degenerate in linear applications to the trapezoïdal rule, from which all members of the family will inherit the typical period elongation properties of the basic scheme. A more sophisticated version of eq.(7) is therefore not desirable.

However, one may start out from a higher order difference expression than the one given in eq.(2), e.g.

$$x_1 - x_0 = \Delta t/2\ (\dot{x}_1 + \dot{x}_0) - \Delta t^2/12\ (\ddot{x}_1 - \ddot{x}_0) \qquad (8)$$

leading to families of algorithms proposed in |8|. These algorithms, however precise, are difficult to implement in general nonlinear computer programs. Attention is therefore restricted here to the easily programmable modified basic Newmark algorithm, as given through eqs.(2) and (7).

4. Implementation of the new algorithm

Eqs.(2) and (7) can be written as follows :

$$x_1 - x_0 = (\Delta t/2)\ (\dot{x}_1 + \dot{x}_0) \qquad (2')$$

$$M\ (\dot{x}_1 - \dot{x}_0) = (\Delta t/2)\ (W_{1,x} + W_{0,x}) + (\Delta t/12)\ (W_{1,xx} - W_{0,xx})\ (x_1 - x_0) \qquad (7')$$

in which the quantities with subscripts 1 and 0 are evaluated at times t_1, t_0, respectively ; $W_{1,x} = F_1^S - F_1^L$, $W_{1,xx} = K_1^S - K_1^L$ are the first and second derivatives of the potential of the external load vector and internal structural resisting forces, e.g. at time t_1; and M = constant mass matrix. In the following it is further assumed that $K_1^L = 0$ and $K_1 \equiv K_1^S$ = tangent stiffness matrix at t_1.

From above equations one can deduce |1| the following Newton-Raphson iteration sequence for dynamic equilibrium at each time step (i-th iteration)

$$\tilde{K}_1^{(i)}\ \Delta\Delta x_1^{(i+1)} = \Delta\Delta\tilde{P}_1^{(i)} \qquad (9)$$

$\| \Delta\Delta x_1^{(i+1)} \| \equiv \| x_1^{(i+1)} - x_1^{(i)} \| \leqslant \varepsilon$ (where ε is an arbitrarily small quantity) with the dynamic stiffness matrix

$$\tilde{K}_1^{(i)} = (4/\Delta t^2)M + K_1^{(i)} - \underline{(1/6)\ (\Delta K_0^{(i)} + B_1^{(i)})} \qquad (10)$$

and with the equivalent dynamic load term

$$\Delta\Delta\tilde{P}_1^{(i)} = F_1^{L(i)} - F_1^{I(i)} - F_1^{S(i)} + \underline{(1/6)\Delta K_0^{(i)}\ \Delta x_0^{(i)}} \qquad (11)$$

where

$$\Delta K_0^{(i)} = K_1^{(i)} - K_0,\ \Delta x_0^{(i)} = x_1^{(i)} - x_0,\ \text{matrix } B_1^{(i)} = \left[\sum_{k=1}^{NEQ} (\partial K_{1ik}^{(i)}/\partial x_j)\Delta x_{0k}^{(i)}\right],$$

and where the inertia forces, $F_1^{I(i)}$, are obtained with the accelerations of the basic Newmark scheme.

After iterations have converged, the true accelerations at the beginning of a new time increment, leading from state 1 to state 2, follow from

$$\ddot{x}_1 = M^{-1}\ (F_2^{L(0)} - F_1^{S})$$

Eqs.(10) and (11) constitute the new scheme, which, in the form outlined, corresponds in all details to the basic Newmark scheme up to the underlined terms. What makes the scheme different from the trapezoïdal rule is the term $(1/6)\Delta K_0^{(i)}\ \Delta x_0^{(i)}$ in load term, $\Delta\Delta\tilde{P}_1^{(i)}$. For this term, it is required that the tangent stiffness matrix be re-calculated at each iteration. When a full Newton iteration is carried out, this is done anyway, and the new scheme is therefore no more complex and little more time consuming, but more accurate than the basic scheme. Term $B_1^{(i)}$, however, involving derivatives of stiffnesses, would be penalizing, could it not be safely neglected, since any iteration will converge to the right result, however less fast, if only the loads are calculated correctly at each iteration step.

5. Examples

Snap Through Problem

The three hinged bar structure shown in fig.3 is subjected to a step load, P, at the center node.

By increasing P, the critical snap load, P_{cr}, can be found after a few tries. Using the basic Newmark scheme and the new scheme, fig.4 shows the estimates of P_{cr}, with different time steps.

The basic scheme introduces errors in the prediction of the critical step load, starting with a time step 10 times a basic time step chosen at 1/100 the linear period of infinitesimal vibrations about the unstressed configuration. The new scheme gives good results for time steps up to 50 times the basic time step and its error at 100 times the basic time step is 6,5 % and remains still small, compared to a 35 % error using the basic scheme.

Note, that the elastic potential in the simple engineering formulation of the problem has an irrational expression, and is thus of infinite order, whereas the new scheme gives exact energy balance only for polynomial potentials of orders up to four. Nevertheless, the results demonstrate the superiority of the new scheme over the unmodified Newmark scheme.

Two Hinged Circular Arch

The circular arch shown in fig.5a is subjected to an initial velocity field in the form of a sine half wave, the velocity vectors pointing towards the center of the circle.

The arch is modelled with 20 beam elements, and the amplitude, v_0 of the imposed velocity field is varied in order to determine the critical velocity for two dimensional, elastic snap through.

Fig.5b shows the area, A , under the arch and its chord versus time, in a 200 Δt time interval, with $\Delta t_0 = 3.31524 \times 10^{-5}$ sec., for an initial velocity amplitude of $v_0 = 3165.0$ in/sec., using different integration schemes and time steps.

It indicates that the critical snap initial velocity will be predicted differently, depending on the algorithm used, and on the time step used. Curves "B" correspond to the new scheme. Curves B1 and B2, with once and twice, respectively, the basic time step are indistinguishable on the plot of fig.5b and they are accepted as the converged solution.

Increasing the time step to 5 and 10 times the basic step, curves B5 and B10 do not predict snap through. Only curve B20 with 20 times the basic time step, leads to numerical snap through.

The responses obtained using the basic scheme, curves "A", show on the other hand numerical snap through to occur for time steps as small as the basic time step, see curve A1. A halving of the basic time step, represented by curve A05, is required to give results comparable to those obtained with the new scheme. The computer time spent to obtain curve A05 is more than 10 times that needed to obtain curve B10 of comparable accuracy. Using a modified Newton iteration with one stiffness calculation per time step, convergence can also be obtained but the computer time with the basic scheme remains more than 5 times as large as for B10 for comparable accuracy.

6. Conclusion

A more stringent stability criterion for solution sequences via implicit dynamic step-by-step integration of nonlinear structures has been defined, based on a local energy conservation within each time step.

The erroneous energy calculation of familiar techniques in the nonlinear domain has been exposed for the basic Newmark algorithm (trapezoïdal rule).

Based on this, an improvement to the basic Newmark scheme has been introduced, readily implemented in nonlinear structural analysis programs already equipped with the basic scheme and with a provision for Newton-Raphson iteration.

The new scheme conserves energy exactly in the important case of linear elastic total Lagrange formulations. It has been applied successfully, with improved accuracy and savings on computer time, to geometrically highly nonlinear problems of dynamic snap-through.

ACKNOWLEDGMENT

This study was conducted under a DRME/SDR/G.8 Research contract, in collaboration with ONERA, with the participation of MM. MARCOIN and VALID from these organisations.
Dr. I. FARHOOMAND contributed valuable suggestions for the initiation of the project, and C. FARJOT contributed most of the programming effort.

REFERENCES

|1| HAUG, E., NGUYEN, Q.S., de ROUVRAY, A.L., FARJOT, C., "Integration Directe en Dynamique Non Linéaire" - Vols. 1,2 - Engineering System International S.A. 20, rue Saarinen, 94150 Rungis-Silic, France ; June 1976

|2| HARTUNG, R.F., BALL, R.E., "A Comparison of Several Computer Solutions to Three Structural Shell Analysis Problems". Lockheed Palo Alto Research Laboratory - Technical Report AFFDL TR-73-15.

|3| Mc NAMARA, J.F., "Solution Schemes for Problems of Nonlinear Structural Dynamics", Transactions of the ASME, Journal of Pressure Vessel Technology, ASME Paper NO. 74-PVP-30.

|4| WEEKS, G., "Temporal Operators for Nonlinear Structural Dynamics Problems", Journal of the Engineering Mechanics Division, ASCE, October 1972.

|5| PARK, K.C., "An Improved Stiffly Stable Method for Direct Integration of Nonlinear Structural Dynamic Equations", Transactions of the ASME, Journal of Applied Mechanics, June 1975.

|6| BATHE, K.J., WILSON, E.L., "Stability and Accuracy Analysis of Direct Integration Methods", Earthquake Engineering and Structural Dynamics, Vol. 1, 1973.

|7| HILBER, H.M., HUGHES, T.J.R., TAYLOR, R.L., "Improved Numerical Dissipation for Time Integration Algorithms in Structural Dynamics", Dept. of Civil Engineering and Lawrence Berkeley Laboratory, LBL-4486, University of California, Berkeley, April 1976

|8| ARGYRIS, J.H., DUNNE, P.C., ANGELOPOULOS, T., "Nonlinear Oscillations using the Finite Element Technique", Computer Methods in Applied Mechanics and Engineering, Vol. 2, 1973.

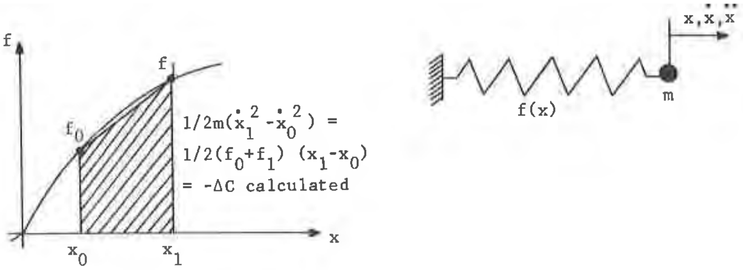


FIGURE 1

NONLINEAR OSCILLATOR

APPROXIMATE KINETIC ENERGY VARIATION ΔC, CALCULATED WITH NEWMARK BASIC SCHEME
(Trapezoïdal rule)

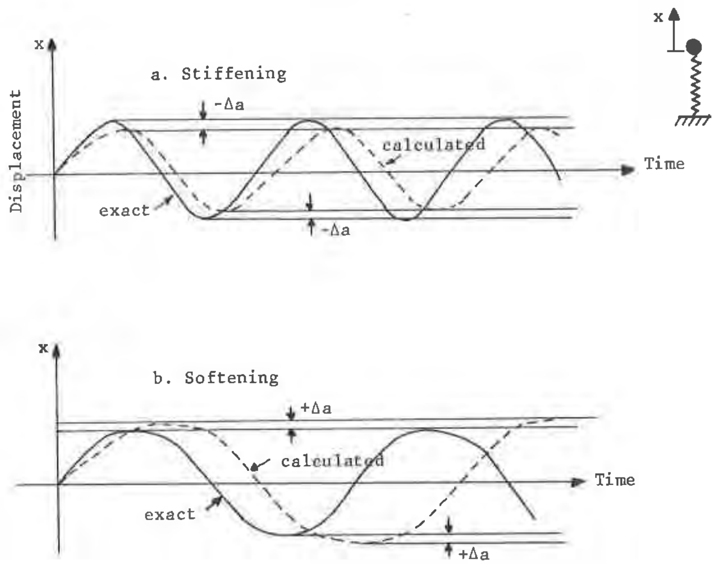


FIGURE 2

TIME RESPONSE OF STIFFENING AND SOFTENING OSCILLATORS WITH NEWMARK BASIC SCHEME,
SHOWING CONSTANT AMPLITUDE ERROR AND PERIOD ELONGATION WITH NON LINEAR SYSTEMS

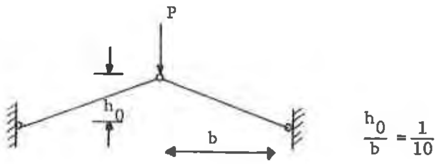


FIGURE 3

THREE HINGED BAR STRUCTURE SUBJECTED TO A STEP LOAD P AT CENTER NODE

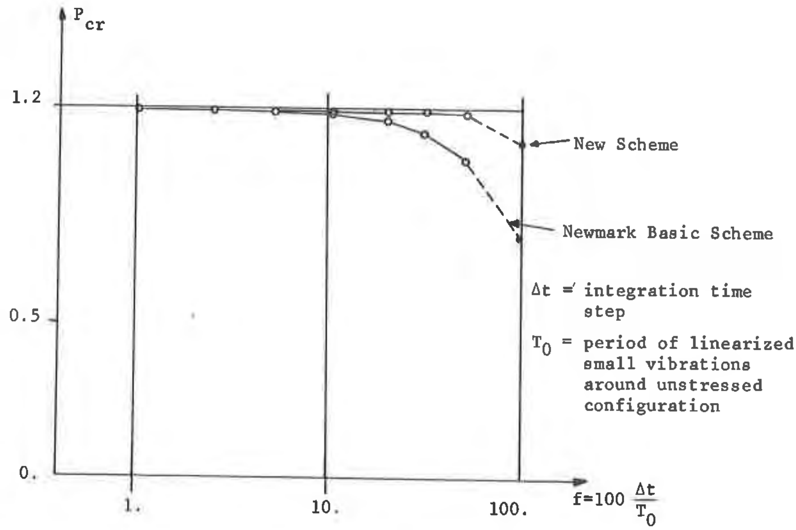
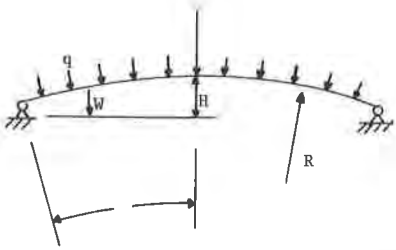


FIGURE 4

CRITICAL SNAP LOAD EVALUATION

FOR DIFFERENT INTEGRATION SCHEMES AND TIME STEPS



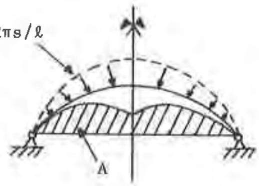
R=67.115 in
h=1.0 in
b=1.0 in
β=15°

$-\gamma=\beta^2\frac{R}{h}=4.6$

$E=10\times10^{10}$ lb/in
ν=0.2
$\rho=2.44\times10^{-4}\ \frac{\text{lb sec}^2}{\text{in}^4}$
$c=\sqrt{\frac{E}{\rho}}=2.024\times10^{5}\ \frac{\text{in}}{\text{sec}}$

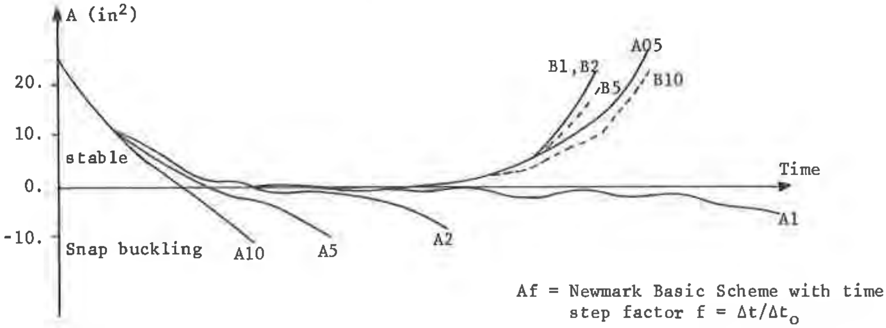
FIGURE 5a

$v=v_o \sin 2\pi s/\ell$
(half sine distribution of initial velocity)



$v_o = -3165.0$ in/sec
$\Delta t_o = 3.31524\times10^{-5}$ sec
$t = N \times \Delta t$

A = area between chord and deformed arch



Af = Newmark Basic Scheme with time step factor $f = \Delta t/\Delta t_o$

Bf = New Scheme with $f = \Delta t/\Delta t_o$

FIGURE 5b

CIRCULAR ARCH

DYNAMIC RESPONSE USING DIFFERENT INTEGRATION SCHEMES AND TIME STEPS