SHELL ANALYSIS THROUGH THE USE OF ASYMPTOTIC THEORIES

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SUMMARY

In the last two decades or so, one of the main activities in the field of shells dealt with the derivation of approximate theories. These theories are in wide use, but because of their approximate character they often lead to failures which could have been predicted only on the basis of more refined theories. However, even the so-called refined theories are themselves derived from the exact three-dimensional elasticity equations on the basis of artificial assumptions whose validity is often open to question. Moreover, in the process of making these assumptions certain contradictions occur which have resulted in shell theories of great variety and complexity.

The reduction of the three-dimensional equations of elasticity to an equivalent set of linear two-dimensional equations is based on the assumption that the shell be thin and that the displacements be small compared to the thickness. The classical derivation of thin shell equations incorporates hypotheses, such as those of Kirchoff, which lead to a priori asssumptions regarding the spatial distribution of displacements and stresses over the thickness of the shell. Another method of deriving shell equations, one which is free from a priori assumptions, is that of the asymptotic integration of the elasticity equations. The method incorporates the use of the boundary layer technique to furnish, depending on the choice of characteristic length scales, different sequences of systems of differential equations. Subsequent integration over the shell thickness and application of the surface boundary conditions yields the desired two-dimensional shell equations. The lowest order system so obtained represents the simplest appropriate shell equations. The higher order systems systematically incorporate thickness corrections associated with the effects of transverse shear and normal stress.

In the present paper, the application of asymptotically derived shell theories will be demonstrated. The particular problem considered is that of a fixed-ended cylinder subjected to a constant line load along a generator. The starting point of the analysis is a set of very general shell equations developed by Goldenveizer. The loading and solution state is represented in the form of a Fourier series in the circumferential direction. Depending on the number of circumferential waves considered, various simplifications of the general shell equations can be carried out on the basis of the results for the asymptotic theories. The complete solution is obtained by a superposition of the solutions of the simplified systems of equations.

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1. Introduction

The method of the asymptotic integration of the elasticity equations has previously been used by many authors [1-8] to derive various shell theories. Relatively little [9-12] has been written, though, about the application of these theories to the solution of actual shell problems. It is the aim of this paper to demonstrate how various asymptotic shell theories can be appropriately superimposed to yield the complete solution. The particular problem considered for the demonstration is a fixed-ended cylindrical shell subjected to a constant line load along a generator.

2. Formulation

In the analysis to follow, a cylindrical shell of middle surface radius a, constant thickness 2h and length L is assumed. A point on the middle surface will be specified by the coordinates ξ and θ (see Fig. 1). Here, ξ is the dimensionless arc-length of the generator and θ the central angle measured from the initial generator. According to Goldenveizer [9], the most general equations governing the unsymmetric deformation of a thin cylindrical shell are given by

$$\left[\frac{\partial^2}{\partial\xi^2} + \left(\frac{1-\nu}{2}\right)\frac{\partial^2}{\partial\theta^2}\right] u + \left(\frac{1+\nu}{2}\right)\frac{\partial^2 v}{\partial\xi\partial\theta} - \nu\frac{\partial w}{\partial\xi} + a^2\,\bar{p}_x = 0 \tag{1}$$

$$\begin{aligned}\left(\frac{1+\nu}{2}\right)\frac{\partial^2 u}{\partial\xi\partial\theta} &+ \left\{\left(\frac{1-\nu}{2}\right)\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\theta^2} + \lambda^2\left[2(1-\nu)\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\theta^2}\right]\right\} v \\ &+ \left\{-\frac{\partial}{\partial\theta} + \lambda^2\left[(2-\nu)\frac{\partial^3}{\partial\xi^2\partial\theta} + \frac{\partial^3}{\partial\theta^3}\right]\right\} w + a^2\,\bar{p}_\theta = 0\end{aligned} \tag{2}$$

$$-\nu\frac{\partial u}{\partial\xi} + \left\{-\frac{\partial}{\partial\theta} + \lambda^2\left[(2-\nu)\frac{\partial^3}{\partial\theta\partial\xi^2} + \frac{\partial^3}{\partial\theta^3}\right]\right\} v + \left[1+\lambda^2\Delta^2\Delta^2\right] w - a^2\,\bar{p}_z = 0 \tag{3}$$

where u, v, w are the components of displacement in the axial, circumferential and radial directions, respectively; p_x, p_θ, p_z are the components of surface loading; ν is Poisson's ratio; E is the modulus of elasticity; and

$$\Delta^2 = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\theta^2}$$

$$\bar{p}_x = \frac{1-\nu^2}{2Eh}\,p_x, \qquad \bar{p}_\theta = \frac{1-\nu^2}{2Eh}\,p_\theta, \qquad \bar{p}_z = \frac{1-\nu^2}{2Eh}\,p_z \tag{4}$$

$$\lambda^2 = \frac{h^2}{3a^2}$$

In the following, a solution of Eqs. 1-3 is sought for the case of a cylindrical shell rigidly fixed at both ends and loaded by a constant line load along a generator. The line load can be expressed in the form

$$p_z = \frac{P}{2\pi a} + \frac{P}{\pi a}\sum_{m=1}^{\infty}\cos m\theta \tag{5}$$

where P is the load per unit length. As the state of stress is symmetrical, the displacements can be expressed in the form

$$u = u_o(\xi) + \sum_{m=1}^{\infty} u_m(\xi) \cos m\theta$$

$$v = \sum_{m=1}^{\infty} v_m(\xi) \sin m\theta \qquad (6)$$

$$w = w_o(\xi) + \sum_{m=1}^{\infty} w_m(\xi) \cos m\theta$$

The end conditions are given by

$$u = v = w = \frac{\partial w}{\partial \xi} = 0 \qquad (\xi = 0, \ell) \qquad (7)$$

where ℓ is the dimensionless length of the shell,

$$\ell = \frac{L}{a} \qquad (8)$$

On substituting expressions (6) into Eqs. 1-3, the following three systems of differential equations are obtained:

$$\frac{d^2u_o}{d\xi^2} - \nu \frac{dw_o}{d\xi} = 0$$

$$- \nu \frac{du_o}{d\xi} + w_o + \lambda^2 \frac{d^4w_o}{d\xi^4} - \frac{a\ P(1-\nu^2)}{4\pi Eh} = 0 \qquad (9)$$

$$\frac{d^2u_1}{d\xi^2} - (\frac{1-\nu}{2})\ u_1 + (\frac{1+\nu}{2}) \frac{dv_1}{d\xi} - \nu \frac{dw_1}{d\xi} = 0 \qquad (10)$$

$$- (\frac{1+\nu}{2}) \frac{du_1}{d\xi} + (\frac{1-\nu}{2}) \frac{d^2v_1}{d\xi^2} - v_1 + \lambda^2 [2(1-\nu) \frac{d^2v_1}{d\xi^2} - v_1] + w_1 + \lambda^2 [-(2-\nu) \frac{d^2w_1}{d\xi^2} + w_1] = 0$$

$$- \nu \frac{du_1}{d\xi} - v_1 + \lambda^2 [(2-\nu) \frac{d^2v_1}{d\xi^2} - v_1] + w_1 + \lambda^2 [\frac{d^4w_1}{d\xi^4} - 2 \frac{d^2w_1}{d\xi^2} + w_1] - \frac{a\ P(1-\nu^2)}{2\pi Eh} = 0$$

$$\frac{d^2u_m}{d\xi^2} - (\frac{1-\nu}{2})\ m^2\ u_m + (\frac{1+\nu}{2}) \frac{dv_m}{d\xi} - \nu \frac{dw_m}{d\xi} = 0 \qquad (11)$$

$$- (\frac{1+\nu}{2})\ m \frac{du_m}{d\xi} + (\frac{1-\nu}{2}) \frac{d^2v_m}{d\xi^2} - m^2v_m + \lambda^2 [2(1-\nu) \frac{d^2v_m}{d\xi^2} - m^2v_m] + m\ w_m + \lambda^2 [-(2-\nu)\ x$$

$$x\ m \frac{d^2w_m}{d\xi^2} + m^3w_m] = 0$$

$$- \nu \frac{du_m}{d\xi} - m\ v_m + \lambda^2 [(2-\nu)\ m \frac{d^2v_m}{d\xi^2} - m^3v_m] + w_m + \lambda^2 [\frac{d^4w_m}{d\xi^4} - 2m^2 \frac{d^2w_m}{d\xi^2} + m^4w_m] -$$

$$- \frac{a\ P(1-\nu^2)}{2\pi Eh} = 0 \qquad (m \geq 2)$$

3. Solution of the Particular Equations

The following assumptions are now made with regard to obtaining the particular solutions of the three systems of Eqs. 9-11:

1. λ^2 is very small
2. u, v, w are of the same order of magnitude
3. differentiation with respect to ξ of u, v, w does not change the order of magnitude of these quantities.

These assumptions allow Eqs. 10-11 to be simplified to the following:

$$\begin{aligned}
&\frac{d^2u_1}{d\xi^2} - \left(\frac{1-\nu}{2}\right) u_1 + \left(\frac{1+\nu}{2}\right)\frac{dv_1}{d\xi} - \nu\frac{dw_1}{d\xi} = 0\\
&-\left(\frac{1+\nu}{2}\right)\frac{du_1}{d\xi} + \left(\frac{1-\nu}{2}\right)\frac{d^2v_1}{d\xi^2} - v_1 + w_1 = 0\\
&-\nu\frac{du_1}{d\xi} - v_1 + w_1 = \frac{a\ P(1-\nu^2)}{2\pi Eh}
\end{aligned} \tag{12}$$

$$\begin{aligned}
&\frac{d^2u_m}{d\xi^2} - \left(\frac{1-\nu}{2}\right) m^2u_m + \left(\frac{1+\nu}{2}\right) m\frac{dv_m}{d\xi} - \nu\frac{dw_m}{d\xi} = 0\\
&-\left(\frac{1+\nu}{2}\right) m\frac{du_m}{d\xi} + \left(\frac{1-\nu}{2}\right)\frac{d^2v_m}{d\xi^2} - m^2v_m + \lambda^2\left[2(1-\nu)\frac{d^2v_m}{d\xi^2} - m^2v_m\right] + mw_m +\\
&\qquad + \lambda^2\left[-(2-\nu)m\frac{d^2w_m}{d\xi^2} + m^3w_m\right] = 0\\
&-\nu\frac{du_m}{d\xi} - mv_m + \lambda^2\left[(2-\nu)m\frac{d^2v_m}{d\xi^2} - m^3v_m\right] + w_m + \lambda^2\left[\frac{d^4w_m}{d\xi^4} - 2m^2\frac{d^2w_m}{d\xi^2} + m^4v_m\right] = \frac{a\ P(1-\nu^2)}{2\pi Eh}
\end{aligned} \tag{13}$$

Implicit in the assumptions made here is that the particular solutions not be of the exponential type.

3.1 Solution of Equations (9)

Upon integrating the first equation of (9) and substituting into the second, one obtains

$$\lambda^2\frac{d^4w_o}{d\xi^4} + (1-\nu^2)w_o = \frac{a\ P(1-\nu^2)}{4\pi Eh} + \nu c_1 \tag{14}$$

where c_1 is a constant of integration representing an axial force. A particular solution of Eq. 14 is given by

$$w_o^P = \frac{aP}{4\pi Eh} + \left(\frac{\nu}{1-\nu^2}\right) c_1 \tag{15}$$

while the solution for u_o is

$$u_o^P = \left(\frac{\nu aP}{4\pi Eh} + \frac{c_1}{1-\nu^2}\right)\xi + c_2 \tag{16}$$

Here, c_2 is a constant of integration. Note that (15) represents the solution of the problem of an infinite cylindrical shell under a uniform radial load.

3.2 Solution of Equations (12)

Equation 12 can be rewritten in the following form:

$$L_1(v_1) = -(2+\nu)\frac{d^2w_1}{d\xi^2} + w_1 \tag{17}$$

$$L_1(u_1) = \nu\frac{d^3w_1}{d\xi^3} + \frac{dw_1}{d\xi} \tag{18}$$

$$\frac{d^4w_1}{d\xi^4} = \frac{aP}{2\pi Eh} \tag{19}$$

where L_1 is a differential operator given by

$$L_1 = \frac{d^4}{d\xi^4} - 2\frac{d^2}{d\xi^2} + 1 \tag{20}$$

A solution of (19) is

$$w_1^P = A_o + A_1\xi + A_2\xi^2 + A_3\xi^3 + \frac{aP}{48\pi Eh}\xi^4 \tag{21}$$

where the A's are constants of integration. On substituting Eq. 21 into (17) and (18), an expression for u_1 and v_1 is obtained,

$$u_1^P = A_1 + 6(2+\nu)A_3 + [2A_2 + \frac{(2+\nu)aP}{2\pi Eh}]\xi + 3A_3\xi^2 + \frac{aP}{12\pi Eh}\xi^3 \tag{22}$$

$$v_1^P = A_o - 2\nu A_2 + \frac{a(2\nu+1)P}{2\pi Eh} + (A_1 - 6\nu A_3)\xi + (A_2 - \frac{\nu aP}{4\pi Eh})\xi^2 + A_3\xi^3 + \frac{aP}{48\pi Eh}\xi^4 \tag{23}$$

Solution (21) represents the solution for a beam under uniform transverse loading.

3.3 Solution of Equations (13)

As solutions of the exponential type are not under consideration, the solutions of (13) are taken as

$$u_m,\ v_m,\ w_m = \text{constant} \tag{24}$$

Equations 13 then yield the following particular solutions:

$$w_m^P = \frac{(1-\nu^2)aP}{2\pi Eh}\left[\frac{1}{\lambda^2(m^2-1)^2}\right] \tag{25}$$

$$v_m^P = \frac{(1-\nu^2)aP}{2\pi Eh}\left[\frac{1+\lambda^2m^2}{m\lambda^2(m^2-1)^2}\right] \tag{26}$$

$$u_m^P = 0 \tag{27}$$

where λ^2 when compared with unity has been neglected. Equations 25-27 represent the solution for an infinite cylindrical shell under a self-equilibrating radial load.

4. Solution of the Homogeneous

4.1 Solution of the Homogeneous Part of Equations (9)

Integration of the first equation of (9) with respect to ξ yields

$$\frac{du_o}{d\xi} = \nu w_o \tag{28}$$

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where the constant of integration has been neglected. On substituting this result into the second equation of (9), one obtains

$$\frac{d^4 w_o}{d\xi^4} + 4\beta^4 w_o = 0 \tag{29}$$

where

$$\beta^4 = \frac{1-\nu^2}{4\lambda^2} \tag{30}$$

The solution of (29) and subsequently of (28) is given by

$$w_o^H = \ell^{\beta(\xi-\ell)}[A_1^o \cos\beta(\xi-\ell) + A_2^o \sin\beta(\xi-\ell)] + \ell^{-\beta\xi}(A_3^o \cos\beta\xi + A_4^o \sin\beta\xi) \tag{31}$$

$$u_o^H = \frac{\nu}{2\beta}\{\ell^{\beta(\xi-\ell)}[(A_1^o - A_2^o)\cos\beta(\xi-\ell) + (A_1^o + A_2^o)\sin\beta(\xi-\ell)] + \ell^{-\beta\xi}[-(A_3^o + A_4^o) \times \cos\beta\xi + (A_3^o - A_4^o)\sin\beta\xi]\} \tag{32}$$

4.2 Solution of the Homogeneous Part of Equations (10)

A preliminary order of magnitude analysis discloses that the characteristic length for changes in the axial direction is $0[(ah)^{\frac{1}{2}}]$ and $0(a)$ in the circumferential direction [5-6], and that the correct approximation to Eqs. 10 is given by

$$\frac{d^2 u_1}{d\xi^2} - \nu\frac{dw_1}{d\xi} = 0 \tag{33}$$

$$-\left(\frac{1+\nu}{2}\right)\frac{du_1}{d\xi} + \left(\frac{1-\nu}{2}\right)\frac{d^2 v_1}{d\xi^2} + w_1 = 0 \tag{34}$$

$$-\nu\frac{du_1}{d\xi} + w_1 + \lambda^2\frac{d^4 w_1}{d\xi^4} = 0 \tag{35}$$

Integrating Eq. 33 and substituting the resulting expression into (34) and (35) yields

$$(2+\nu)w_1 + \frac{d^2 v_1}{d\xi^2} = 0 \tag{36}$$

$$\frac{d^4 w_1}{d\xi^4} + 4\beta^4 w_1 = 0 \tag{37}$$

where the constant of integration has been neglected because it is associated with that part of the solution containing an improper length scale. The solution of Eq. 37 and is given by

$$w_1^H = \ell^{\beta(\xi-\ell)}[A_1^1 \cos\beta(\xi-\ell) + A_2^1 \sin\beta(\xi-\ell)] + \ell^{-\beta\xi}[A_3^1 \cos\beta\xi + A_4^1 \sin\beta\xi] \tag{38}$$

$$u_1^H = \frac{\nu}{2\beta}\{\ell^{\beta(\xi-\ell)}[(A_1^1 - A_2^1)\cos\beta(\xi-\ell) + (A_1^1 + A_2^1)\sin\beta(\xi-\ell) + \ell^{-\beta\xi}[-(A_3^1 + A_4^1)\cos\beta\xi + (A_3^1 - A_4^1)\sin\beta\xi]\} \tag{39}$$

$$v_1^H = -\left(\frac{2+\nu}{2\beta^2}\right)\{\ell^{\beta(\xi-\ell)}[-A_2^1 \cos\beta(\xi-\ell) + A_1^1 \sin\beta(\xi-\ell)] + \ell^{-\beta\xi}(A_4^1 \cos\beta\xi - A_3^1 \sin\beta\xi)\} \tag{40}$$

4.3 Solution of the Homogeneous Part of Equations (11)

An order of magnitude analysis of the homogeneous part of Eqs. 11 shows that for $m \geq 2$ but $m << \lambda^{-\frac{1}{2}}$ the solution can be split into two parts: a basic solution associated with stress states which have an axial length scale of $a(a/h)^{\frac{1}{2}}$ and an edge solution which is associated with a boundary layer of width $(ah)^{\frac{1}{2}}$. The equations of the basic solution are

$$\frac{d^2u_m}{d\xi^2} - \left(\frac{1-\nu}{2}\right) m^2u_m + \left(\frac{1+\nu}{2}\right)m \frac{dv_m}{d\xi} - \nu \frac{dw_m}{d\xi} = 0 \tag{41}$$

$$-m\left(\frac{1+\nu}{2}\right) \frac{du_m}{d\xi} + \left(\frac{1-\nu}{2}\right) \frac{d^2v_m}{d\xi^2} - m^2(1+\lambda^2)v_m + m(1+\lambda^2m^2)w_m = 0 \tag{42}$$

$$-\nu \frac{du_m}{d\xi} - m(1+\lambda^2m^2)v_m + (1+\lambda^2m^4)w_m = 0 \tag{43}$$

while those for the edge solution are given by

$$\frac{d^2u_m}{d\xi^2} - \nu \frac{dw_m}{d\xi} = 0 \tag{44}$$

$$-\left(\frac{1+\nu}{2}\right) m \frac{du_m}{d\xi} + \left(\frac{1-\nu}{2}\right) \frac{d^2v_m}{d\xi^2} + mw_m = 0 \tag{45}$$

$$-\nu \frac{du_m}{d\xi} + w_m + \lambda^2 \frac{d^4w_m}{d\xi^4} = 0 \tag{46}$$

Solution of the Basic Equations

The first two basic Eqs. 40-41 can be expressed in the following form:

$$L_2(v_m) = m \frac{d^2w_m}{d\xi^2} \left[-(1+\lambda^2m^2) + \nu\left(\frac{1+\nu}{2}\right)\right] + m^3w_m \left[\left(\frac{1-\nu}{2}\right)(1+\lambda^2m^2)\right] \tag{47}$$

$$L_2(u_m) = \frac{d^3w_m}{d\xi^3} \left[\nu\left(\frac{1-\nu}{2}\right)\right] + m^2 \frac{dw_m}{d\xi}\left[-\nu(1+\lambda^2) + \left(\frac{1+\nu}{2}\right)(1+\lambda^2m^2)\right] \tag{48}$$

where L_2 is a differential operator given by

$$L_2 = \left(\frac{1-\nu}{2}\right) \frac{d^4}{d\xi^4} + m^2 \left[-(1+\lambda^2) + \nu\right] \frac{d^2}{d\xi^2} + m^4(1+\lambda^2)\left(\frac{1-\nu}{2}\right) \tag{49}$$

Operating with L_2 on Eq. 43 and then substituting Eqs. 47-48 into the result thus obtained yields upon requiring that all terms be of the same order of magnitude

$$\frac{d^4w_m}{d\xi^4} + 4\alpha_m^4 w_m = 0 \tag{50}$$

where

$$\alpha_m^4 = \frac{m^4\lambda^2(1-m^2)^2}{4(1-\nu^2)} \tag{51}$$

The solution of Eq. 50 is given by

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$$w_m^b = \ell^{\alpha_m(\xi-\ell)}[A_1^b \cos \alpha_m(\xi-\ell) + A_2^b \sin \alpha_m(\xi-\ell)] + \ell^{-\alpha_m \xi}(A_3^b \cos \alpha_m \xi + A_4^b \sin \alpha_m \xi) \quad (52)$$

The solutions for u_m^b and v_m^b can be obtained from Eqs. 47-48.

Solution of the Edge Equations

The basic Eqs. 44-46 have a solution similar in form as solutions 38-40 except that in the expressions for w_m^ℓ and u_m^ℓ the constants A_i^1 are replaced by A_i^ℓ while v_m^ℓ is given by

$$v_m^\ell = -m\left(\frac{2+\nu}{2\beta^2}\right)\{\ell^{-\beta(\xi-\ell)}[-A_2^\ell \cos \beta(\xi-\ell) + A_1^\ell \sin \beta(\xi-\ell)] + \ell^{-\beta\xi}(A_4^\ell \cos \beta\xi - A_3 \sin \beta\xi)\} \quad (53)$$

Solution of the Complete Equations

For the case of large $m(m \geq \lambda^{-\frac{1}{2}})$, it is meaningless to divide the solution into a basic and an edge part. One then speaks of a complete solution. It is associated with a length scale of $(ah)^{\frac{1}{2}}$ in both the axial and circumferential directions. An order of magnitude analysis shows the equations to be given by

$$\frac{d^2u_m}{d\xi^2} - \left(\frac{1-\nu}{2}\right) m^2 u_m + \left(\frac{1+\nu}{2}\right) m \frac{dv_m}{d\xi} - \nu \frac{dw_m}{d\xi} = 0$$

$$-\left(\frac{1+\nu}{2}\right) m \frac{du_m}{d\xi} + \left(\frac{1-\nu}{2}\right)\frac{d^2v_m}{d\xi^2} - m^2 v_m + m w_m = 0 \quad (54)$$

$$-\nu \frac{du_m}{d\xi} - m v_m + w_m + \lambda^2\left(\frac{d^4w_m}{d\xi^4} - 2m^2 \frac{d^2w_m}{d\xi^2} + m^4 w_m\right) = 0$$

An eight order equation for w_m can be obtained by following the same procedure as used for the basic equations. This yields

$$\frac{d^8w_m}{d\xi^8} - 4m^2 \frac{d^6w_m}{d\xi^6} + 6m^4 \frac{d^2w_m}{d\xi^4} - 4m^6 \frac{d^2w_m}{d\xi^2} + m^8 w_m + \left(\frac{1-\nu^2}{\lambda^2}\right)\frac{d^4w_m}{d\xi^4} = 0 \quad (55)$$

while the equations for u_m, v_m only are given by

$$L_3 v_m = -m(2+\nu)\frac{d^2w_m}{d\xi^2} + m^3 w_m$$

$$L_3 u_m = \nu \frac{d^3w_m}{d\xi^3} + m^2 \frac{dw_m}{d\xi} \quad (56)$$

Here, L_3 is a differential operator given by

$$L_3 = \frac{d^4}{d\xi^4} - 2m^2 \frac{d^2}{d\xi^2} + m^4 \quad (57)$$

The solution of Eq. 55 can be expressed as

$$
\begin{aligned}
w_m^c &= \ell^{p_1(\xi-\ell)}[A_1^c \sin q_1(\xi-\ell) + A_2^c \cos q_1(\xi-\ell)] \\
&+ \ell^{-p_1(\xi)} (A_3^c \sin q_1 + A_4^c \cos q_1\xi) \\
&+ \ell^{p_2(\xi)} [A_5^c \sin q_2\xi + A_6^c \cos q_2\xi] \\
&+ \ell^{-p_2(\xi-\ell)} [A_7^c \sin q_2(\xi-\ell) + A_8^c \cos q_2(\xi-\ell)]
\end{aligned} \tag{58}
$$

where the p_i and q_i are to be determined from the characteristic equation

$$(k^2 - m^2)^4 + (\frac{1-\nu^2}{\lambda^2})\, k^4 = 0 \tag{59}$$

The expressions for v_m^c and u_m^c can be determined from Eqs. 56.

5. Application of the Boundary Conditions

End conditions (7) are to be satisfied by each term of the expansions (6). In the application of these conditions, two basic simplifications are possible. For small values of $m (m << \lambda^{-\frac{1}{2}})$ one can apply tangential and non-tangential boundary conditions separately. The tangential forces are dominant in the basic solution while in the edge solution the transverse effects are of primary importance. The tangential boundary conditions thus have the form

$$u_m^b + u_m^P = 0 \tag{60}$$

while the non-tangential ones can be written as

$$w_m^b + w_m^\ell + w_m^P = 0 \tag{61}$$

For large values of m and if the length is long the effect of one edge on the solution state can be neglected as being small at the other edge. This is based on the fact that the effect of any self-equilibrating boundary load disappears at some distance from the load.

6. Numerical Results

Numerical calculations were performed for a shell having radius to thickness ratio of 100 and a dimensionless length of 10. Figure 2 shows the variation of the transverse displacement over one-half the length of the shell. The location $\theta = 0^\circ$ is along the load generator. Shown in Fig. 3 is the variation of the transverse displacement at mid-length with angle θ.

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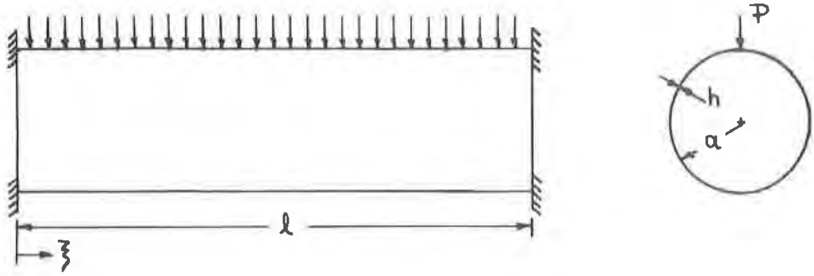

Figure 1: Cylinder with line load along a generator.

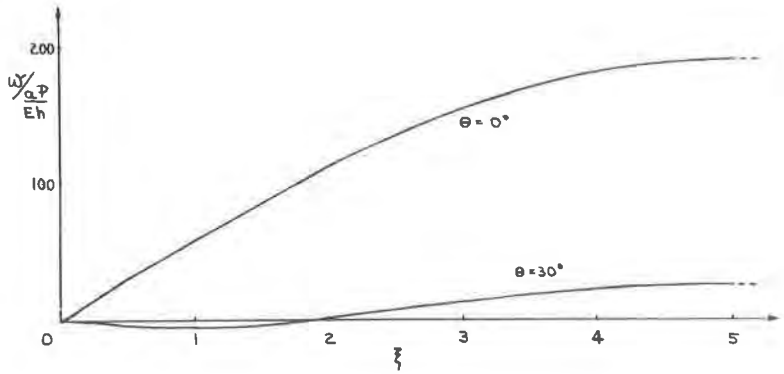

Figure 2: Variation of transverse displacement with length.

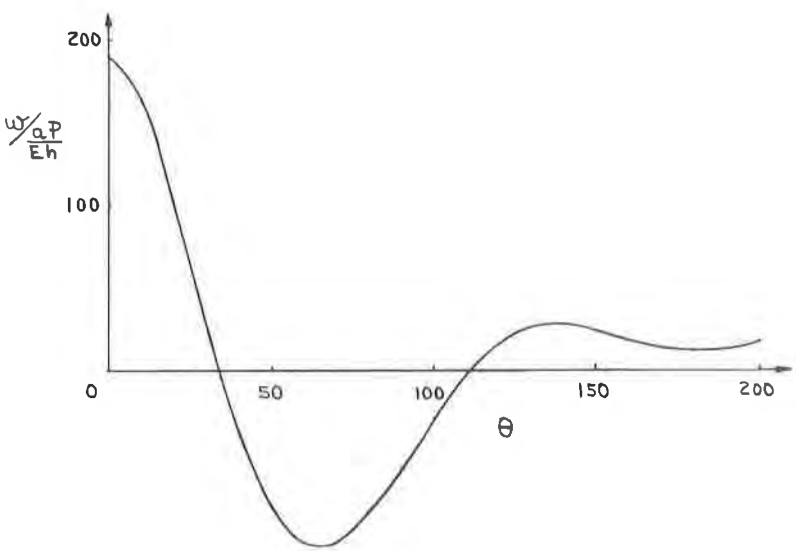

Figure 3: Variation of transverse displacement around circumference (ξ=5)