ABSTRACT

BRYAN, TIMOTHEE WILLIAM. Hall-Littlewood Vertex Operators and the Kostka-Foulkes Polynomials. (Under the direction of Naihuan Jing.)

The familiar Hall-Littlewood polynomials, $L_\lambda[X;t]$ form a basis for symmetric functions and are related to the Schur function, $s_\lambda[X]$, basis via

$$L_\lambda[X;t] = \sum_{\lambda \vdash |\mu|} K_{\lambda\mu}(t) s_\lambda[X]$$

where $K_{\lambda\mu}$ is the Kostka-Foulkes Polynomial. Lascoux and Schützenberger proved that for semi-standard Young tableaux

$$L_\lambda[X;t] = \sum_{T \in SST^\lambda} t^{charge(T)} s_{shape(T)}[X]$$

where the charge of a tableau T is a value obtained by weighting the entries of a reading word corresponding to a filling using content μ. We define an explicit algebraic formula for the Kostka-Foulkes polynomials using Hall-Littlewood vertex operators and Jing's Hall-Littlewood inner product which does not utilize Lascoux and Schüteznberger's result. We also introduce new Young-like tableaux to discuss combinatorially significant lattice symmetries which arise during and as a result of the calculations and proof of the Kostka-Foulkes polynomial formula.

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Hall-Littlewood Vertex Operators and the Kostka-Foulkes Polynomials

by
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DEDICATION

In memory of Bonnie Jean Bryan. I miss you, Mom.

BIOGRAPHY

Timothee William Bryan was born on February 17, 1980 in Sioux Falls, South Dakota to Jacky and Bonnie Bryan. He was raised in Sioux Falls and attended Lincoln High School where he met his wife Amanda Leigh. After graduating, he attended the University of Sioux Falls, majoring in mathematics. He then went to North Carolina State University, where he served as a graduate teaching assistant, while working toward master's and doctorate degrees in mathematics.

Timothee's first memory of the joy of mathematics is of playing mathematical computer games while in kindergarten: covering topics of multiplication and division. After working as a mathematics tutor for several years, he developed a love of teaching. After the efforts of extraordinary teachers at the University of Sioux Falls, he discovered his love of research, which pointed him toward his eventual career in mathematics.

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First, I would like to thank my advisors Naihuan Jing and Ernest Stitzinger for all their help and patience and my parents Jacky and Bonnie for giving me the opportunities and support that have led me here.

I would also like to thank all the people who helped me realize how much I love math and led me to this career. Dr. Joy Lind at the University of Sioux Falls taught me to be an encouraging, patient, understanding collaborator and instructor. Dr. Jason Douma at the University of Sioux Falls taught me abstract algebra and geometry, helped to shape my mathematical philosophy, allowed me my first opportunity for research, and taught me to invest myself in a problem and to pursue it beyond what I believed were my capabilities. Dr. Shawn Chiappetta at the University of Sioux Falls taught me linear algebra and analysis amongst other subjects. He believed in and encouraged me at a time when I did not believe in myself. I am blessed to call you my friend and mentor.

To Butch Walker, for always being there and for helping me to say the things I was not ready to say alone.

And to my wife, Amanda. It is through your love, support, and encouragement that I have been able to develop as a student and grow into a mathematician. You are the best part of my life. I am forever grateful. I love you. Simply put, I wouldn't be here without you.

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Introduction

1.1 Partitions and Tableaux

Definition 1.1.1. A *partition* λ consisting of l parts of a positive integer n is an unordered collection of positive integers summing to n.

Example 1.1.2. Consider $n = 3$. The four partitions of 3 are

$$\begin{aligned} 3 &= 3 \\ &= 2+1 \\ &= 1+2 \\ &= 1+1+1. \end{aligned}$$

Because addition is commutative, we could rearrange the entries of third partition of 3 into the second partition. As we will not be concerned with counting the number of partitions of n, we will consider $2+1 = 1+2$ in this paper and subsequently we will consider only those unique partitions of n which are not rearrangements of each other. Hence the partitions of 3 are

$$\begin{aligned} 3 &= 3 \\ &= 2+1 \\ &= 1+1+1. \end{aligned}$$

For the duration of this paper we will represent an l-part partition λ of n by a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ such that $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_l > 0$ and say that λ has weight n or $|\lambda| = n$. We adopt the notational shorthand where an entry j^m in the sequence of λ does not represent the m-th power of j but rather represents m consecutive $\lambda_i = j$ for $1 \leq i \leq l$, i.e. m is the multiplicity of an element j. We say that λ has length l, denoted $l(\lambda) = l$, and will write $\lambda \vdash n$ to denote that λ is a partition of n.

It is possible and often convenient, in this paper and in general, to graphically represent a partition as a collection of stacked cells or boxes called a Young tableau.

Definition 1.1.3. For a partition λ, the *Young tableau*, represented T_λ or simply T when there is no confusion, for λ is a left aligned collection of rows of cells stacked one on top of another via the rule that row i of T contains λ_i boxes for $1 \leq i \leq l$.

Following the English notation style, we arrange T such that row 1 is the top-most row, row 2 is directly below row 1, and so on until row l is the bottom-most row of the tableau. For this reason, λ is said to be the shape of T.

Remark 1.1.4. *Let λ and $\bar{\lambda}$ be partitions. It is clear from Definition 1.1.3 that if T_λ and $T_{\bar{\lambda}}$ have the same shape, then $\lambda_i = \bar{\lambda}_i$ for $1 \leq i \leq l = l(\lambda) = l(\bar{\lambda})$. Thus, $\lambda = \bar{\lambda}$ and there exists a unique Young tableau for each partition λ.*

Example 1.1.5. For $\lambda \vdash 9$ where $\lambda = (3^2, 2, 1)$ and $\bar{\lambda} \vdash 17$ where $\bar{\lambda} = (5, 4, 3^2, 2)$, the Young tableaux T_λ and $T_{\bar{\lambda}}$ are shown in Figure 1.1.

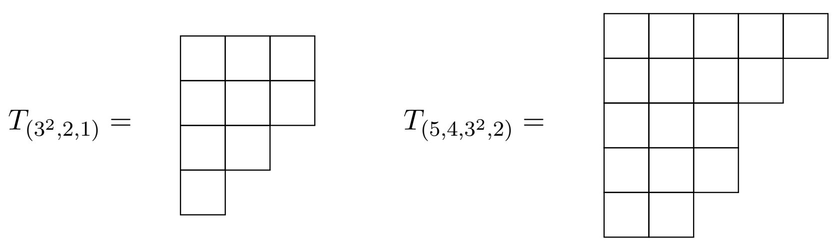


Figure 1.1: Young tableaux of $\lambda = (3^2, 2, 1)$ and $\bar{\lambda} = (5, 4, 3^2, 2)$

Given a partition λ, it is sometimes useful to define the conjugate of a partition.

Definition 1.1.6. For $\lambda \vdash n$ with $l(\lambda) = l$, the *conjugate* of λ, denoted λ', is the partition $\lambda' \vdash n$ whose j-th entry is the number of boxes in the j-th column of λ. Graphically, this can be thought of as the tableau T'_λ obtained by reflecting T_λ about the main diagonal.

Example 1.1.7. For $\lambda \vdash 9$ where $\lambda = (3^2, 2, 1)$ and $\bar{\lambda} \vdash 17$ where $\bar{\lambda} = (5, 4, 3^2, 2)$, the conjugate Young tableaux T'_λ and $T'_{\bar{\lambda}}$ are shown in Figure 1.2.

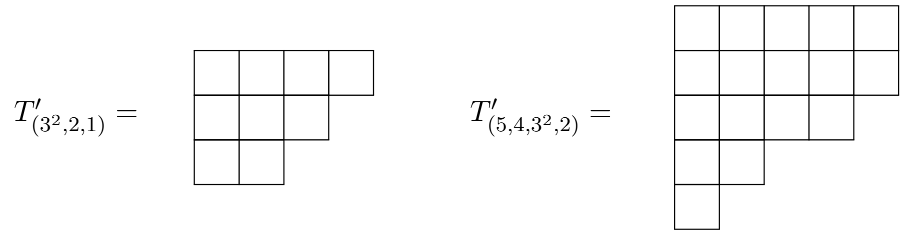


Figure 1.2: Conjugate Young tableau for $\lambda = (3^2, 2, 1)$ and $\bar{\lambda} = (5, 4, 3^2, 2)$

There is another realization of a Young tableau which will be useful later.

Definition 1.1.8. Given a partition λ, locate it's Young tableau such that the upper left corner of the first box in row 1 has (x, y)-coordinate $(0, 0)$ in the plane. Then the entire left edge of column 1 of T_λ is on the negative y-axis and the entire top edge of row 1 is on the positive x-axis. Because λ has a unique tableau it is possible to describe the shape of a partition using a doubly infinite sequence of up and right steps of unit length, denoted U and R respectively, by moving up the negative y-axis until reaching the bottom edge of the tableau and then tracing along the exterior edge of T_λ until reaching the x-axis and moving right. We call the sequence of up and right steps to be the *code*, [5], of the partition λ.

Example 1.1.9. Let $\lambda = (3^2, 2, 1)$ and$\bar{\lambda} = (5, 4, 3^2, 2)$. Then the Young tableau for λ and $\bar{\lambda}$ in quadrant IV of the xy-plane with the bold path formed by moving along the bottom and right edges of the tableau are shown in Figure 1.3.

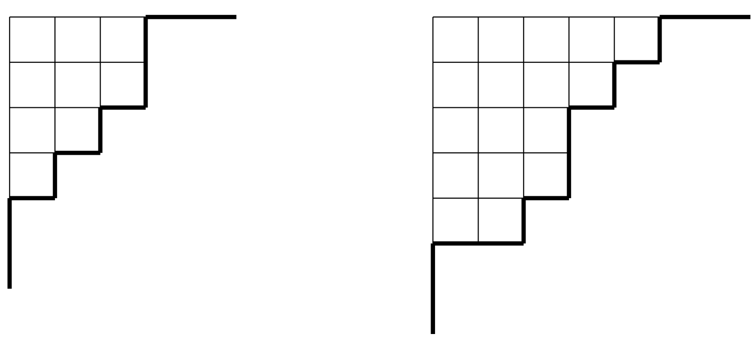


Figure 1.3: Young tableaux and codes for $\lambda = (3^2, 2, 1)$ and $\bar{\lambda} = (5, 4, 3^2, 2)$

	Code
$\lambda = (3^2, 2, 1)$	$\ldots UUURURURUURRR \ldots$
$\bar{\lambda} = (5, 4, 3^2, 2)$	$\ldots UUURRURUURURURRR \ldots$

Regardless of the representation chosen for a partition λ, where $\lambda \neq (1)$, we see there are smaller partitions contained within λ, i.e. by subtracting 1 from λ_l. Because containment will be of central importance in Chapter 3, we define it in the following way.

Definition 1.1.10. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \vdash n$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_k) \vdash m$ with $m \leq n$ and $k \leq l$. If $\mu_j \leq \lambda_i$ for all $1 \leq j \leq i \leq l$, then μ is *contained* within λ and is denoted $\mu \subseteq \lambda$.

We can construct a Young lattice for a partition λ and all tableaux μ contained within λ in the following way. Beginning with the Young tableau of λ we remove one cell at a time from T_λ, such that partition shape is always maintained, to arrive at a subtableau T_μ and write T_μ below T_λ and draw an edge between them.

Observe that, though is it not a partition according to Definition 1.1.1, we can define the empty partition to be the unique "partition" whose Young tableau consists of zero cells and is denoted by $\emptyset$. As $\emptyset$ is contained within each Young tableau, we can extend the Young lattice for any partition λ so that it ends at the Young tableau for the empty partition instead of ending at the Young tableau for the partition $\lambda = (1)$.

Example 1.1.11. Consider the Young tableau corresponding to $\lambda = (3, 2, 1)$, the Young lattice formed by $\mu \subseteq \lambda$ is show in Figure 1.4.

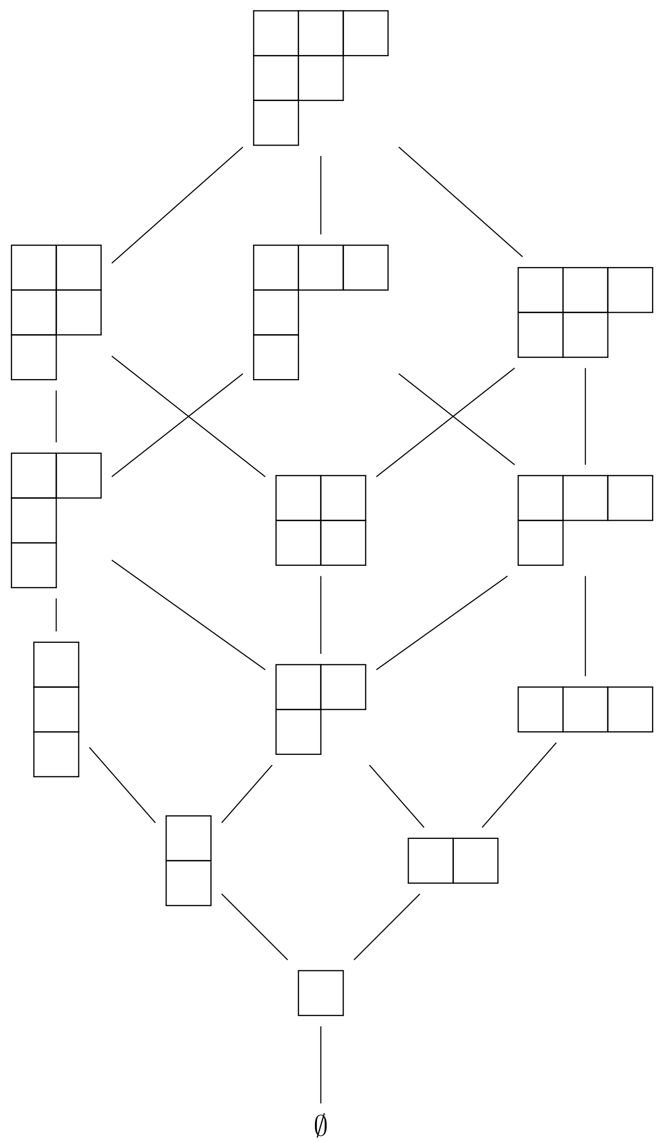


Figure 1.4: Young lattice for $\lambda = (3, 2, 1)$

Having defined containment for any partition λ, we introduce another important type of tableaux created by removing the cells corresponding to a subtableau of λ from T_λ.

Definition 1.1.12. Let λ and μ be partitions with $\mu \subseteq \lambda$. A *skew tableau*, denoted λ/μ, is a tableau formed by removing the upper leftmost cells corresponding to T_μ from the cells of T_λ.

Example 1.1.13. Let $\mu = (3, 2^2)$, $\lambda = (3^2, 2, 1)$ and $\bar{\lambda} = (5, 4, 3^2, 2)$. Then the skew tableaux λ/μ and $\bar{\lambda}/\mu$ are shown, respectively, in Figure 1.5.

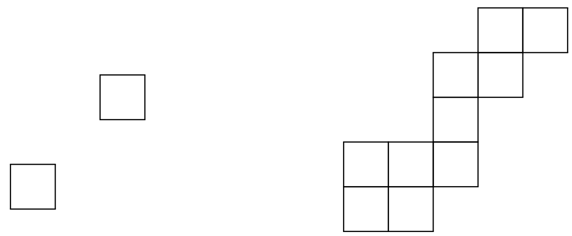


Figure 1.5: Examples of skew tableaux

There exists a special type of connected, skew tableaux, called ribbon tableaux, which we will utilize later in this paper.

Definition 1.1.14. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ and $\mu = (\lambda_2 - 1, \lambda_3 - 1, \ldots, \lambda_l - 1)$ be partitions. Consider the skew tableaux formed by λ/μ. Because λ/μ is a connected strip with no 2×2 subtableaux, graphically, a ribbon tableau is the rightmost edge of T_λ which is still connected and of unit width.

Example 1.1.15. Consider the partitions $\lambda = (3^2, 2, 1)$ and $\bar{\lambda} = (5, 4, 3^2, 2)$. Then for $\mu = (2, 1)$ and $\bar{\mu} = (3, 2^2, 1)$, λ/μ and $\bar{\lambda}/\bar{\mu}$ produce the ribbon tableaux shown in Figure 1.6.

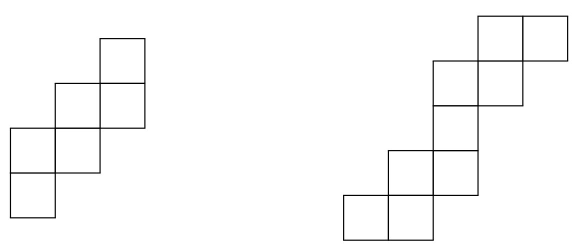


Figure 1.6: Examples of ribbon tableaux

Let λ and $\bar{\lambda}$ be partitions of n such that $\lambda \neq \bar{\lambda}$ for $n \geq 2$. This means that neither partition of n is contained within the other, because there must exist entries i and j where $\lambda_i > \bar{\lambda}_i$ and $\lambda_j < \bar{\lambda}_j$ for $1 \leq i, j \leq \max\{l(\lambda), l(\bar{\lambda})\}$. Hence we need to introduce an ordering for λ and $\bar{\lambda}$ so that we can determine whether λ is less than $\bar{\lambda}$ or vice versa. This ordering is important during the calculation of the main result of Chapter 2 and is motivational to the combinatorial results presented in Chapter 3.

Definition 1.1.16. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ and $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_m)$ be unique partitions of n. We say that $\bar{\lambda}$ is less than λ, denoted $\bar{\lambda} \leq \lambda$, in the *dominance ordering* if $\sum_{i=1}^{k} \bar{\lambda}_i \leq \sum_{i=1}^{k} \lambda_i$ for all $k \geq 1$.

We visualize the dominance ordering for partitions n using a Young lattice where nodes are the tableaux of partitions of n and an edge between nodes denotes that the upper node dominates the lower node.

As Definition 1.1.16 suggests, there is no direct connection between partition containment and the dominance ordering. If we consider the partitions $(5,1)$ and $(4,2)$, it is clear that $(5,1)$ dominates $(4,2)$ even though neither partition of six is contained within the other.

In the following example, we select the first "interesting" value of a partition of weight n for which there exists more than one Young tableau per horizontal level of the Young lattice.

Example 1.1.17. Consider the unique partitions of $n = 6$. Then the dominance ordering for partitions of n are shown in Figure 1.7.

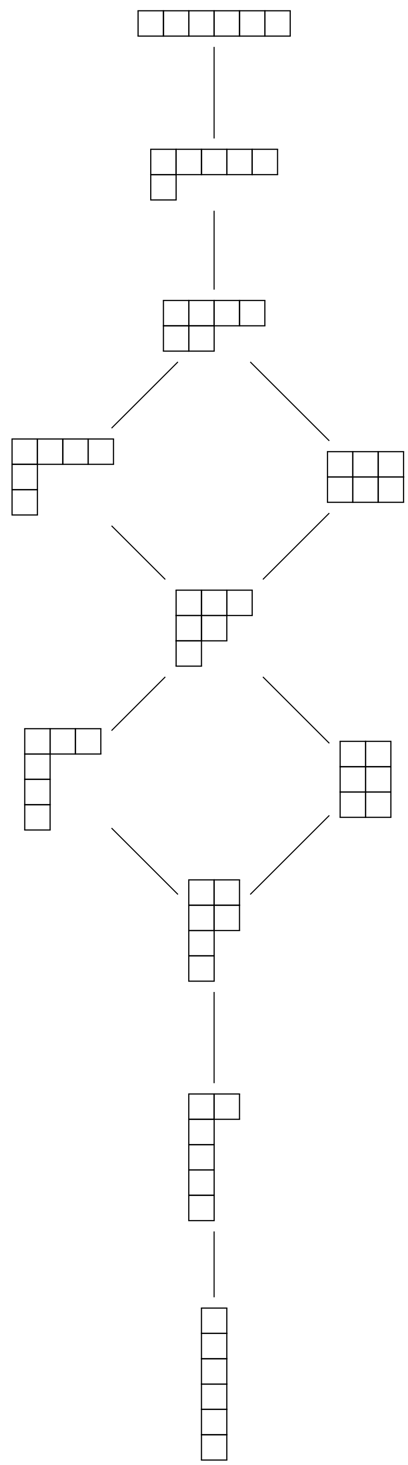


Figure 1.7: Dominance ordering for Young tableau of weight 6

1.2 Content

Definition 1.2.1. For a partition λ of n with Young tableau T and $\mu = (\mu_1, \mu_2, \ldots, \mu_m) \vdash n$ where $\mu \leq \lambda$ in the dominance ordering, we define the *content* of T to be a filling of the boxes of T with μ_i number of entries i for $1 \leq i \leq m$.

Definition 1.2.2. Given a tableau T of shape λ and content μ, we say T is a *standard Young tableau*, or just a *standard tableau*, if the entries in a filling of T are strictly increasing from left to right along the rows and from top to bottom along the columns of T.

Since the rows and the columns of cells must be strictly increasing, a standard filling cannot contain more than more number i in any row or column and, as a result, $\mu_1 = 1$. Since μ is a partition, $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_m > 0$ implies that each $\mu_i = 1$ and hence $\mu = (1^n)$. Thus, a standard tableau T with shape λ corresponding to a partition of n has the entries $\{1, 2, \ldots, n\}$ appearing exactly once in a filling. Observe, a standard tableau T of shape λ may have multiple fillings resulting from permutations of the entries.

Example 1.2.3. Let T be a tableau with shape λ and content μ where $\lambda = (3^2, 2, 1)$ and $\mu = (1^9)$. Two standard fillings of T are shown in Figure 1.8.

1	2	3
4	5	6
7	8	
9		

1	2	4
3	5	6
7	8	
9		

Figure 1.8: Examples of standard tableau of shape $(3^2, 2, 1)$ and content (1^9)

By easing the conditions for a filling of the rows of a tableau, we can define a set of fillings of T where entries can appear with multiplicity greater than one.

Definition 1.2.4. Given a tableau T of shape λ and content μ, we say T is a *semi-standard Young tableau*, or just a *semi-standard tableau*, if the entries in a filling of T are weakly increasing from left to right along the rows and strictly increasing from top to bottom along the columns of T.

Observe that by definition, the set of all standard tableau T of shape λ where $|\lambda| = n$ are precisely those semi-standard tableau with content (1^n).

Example 1.2.5. Let T be a tableau with shape λ and content μ where $\lambda = (3^2, 2, 1)$ and $\mu = (2^4, 1)$. Then the fillings of T which are semi-standard are show in Figure 1.9.

1	1	2
2	3	3
4	4	
5		

1	1	2
2	3	4
3	4	
5		

1	1	2
2	3	4
3	5	
4		

1	1	2
2	3	5
3	4	
4		

1	1	3
2	2	4
3	4	
5		

1	1	3
2	2	4
3	5	
4		

1	1	3
2	2	5
3	4	
4		

1	1	4
2	2	5
3	3	
4		

Figure 1.9: Semi-standard tableau of shape $(3^2, 2, 1)$ and content $(2^4, 1)$

The arrangement of the tableau in Figure 1.9 is meant to be illustrative of a process to assure that one records all possible semi-standard fillings of a tableau. In this case, the first row of tableaux correspond to fixing the 1 entries in the highest rows possible, followed by fixing the 2 entries in the highest remaining rows, and then fixing the 3 entries in the highest remaining rows before recording the possible fillings. The second row of tableaux correspond to fixing the 1 entries in the highest rows possible and then fixing the 2 entries in the highest rows before recording the possible fillings. The third and fourth rows of tableaux proceed in a similar manner to the first and second rows.

1.3 Charge

For semi-standard tableaux T of shape λ and content μ, Lascoux and Schutzenberger, [12], showed that it is possible to define the charge of T, denoted $c(T)$, for all fillings. In order to define the charge for a semi-standard tableau T, it is first necessary to construct the reading word, ω, of T via the following process.

For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, begin by recording each entry of a filling of the tableau from left to right starting at row l and moving upwards until reaching row 1.

Example 1.3.1. Consider $\lambda = (5, 4, 3^2, 2)$ with content $\mu = (3^2, 2^4, 1^3)$ and filling shown in Figure 1.10.

1	1	1	2	4
2	2	3	5	
3	4	7		
5	6	9		
6	8			

Figure 1.10: Example of a tableau of shape $(5, 4, 3^2, 2)$ and content $(3^2, 2^4, 1^3)$

Hence, the reading word ω for the semi-standard filling of T shown in Figure 1.10 is

$$\omega = 68569347223511124.$$

As the content of the previous tableau is not giving by $\mu = (1^{17})$, there are repetitions of cell entries. However, Lascoux and Schutzenberger's charge statistic requires a multiplicity of one for each entry of μ in a reading word. To adjust for multiple entries we need to then form reading subwords, ω_i, of ω for $1 \leq i \leq \mu_1$ by scanning from right to left across ω and recording the position of the first entry 1, followed by the first entry 2, and so on until we either reach the far left end of ω at which point we "wrap" back to the far right end of ω and continue the process or we record m where $l(\mu) = m$. We then construct the first subword, ω_1 of ω, by writing the sequence of numbers $\{1, 2, \ldots, m\}$ in the order they were found in ω from left to right. Next, we delete the entries of ω_1 from ω. We repeat this process until every entry of ω appears in a subword ω_i for $1 \leq i \leq \mu_1$.

For Figure 1.10 the reading subwords of $\omega = 68569347223511124$ are

$$\begin{aligned} \omega_1 &= 869372514 \\ \omega_2 &= 654231 \\ \omega_3 &= 12. \end{aligned}$$

Once we have formed the subwords of ω, we find the charge of ω by weighting the entries j, denoted $w(j)$, of ω_i for $1 \leq i \leq \mu_1$ according to the following rule:

$$\begin{cases} w(j) = 0 & \text{if } j = 1 \\ w(j+1) = w(j) & \text{if } (j+1) \text{ appears to the left of } j \text{ in } \omega_i \\ w(j+1) = w(j) + 1 & \text{if } (j+1) \text{ appears to the right of } j \text{ in } \omega_i \end{cases}$$

and summing of the entry weights of its subwords. The charges of the subwords for the tableau in Figure 1.10 are

$$\begin{aligned} \omega_1 &= 869372514 \\ & \quad\; 213020101 \quad \text{charge} = 10 \\ \\ \omega_2 &= 654231 \\ & \quad\; 111010 \quad \text{charge} = 4 \\ \\ \omega_3 &= 12 \\ & \quad\; 01 \quad \text{charge} = 1. \end{aligned}$$

Hence the charge of the tableau in Figure 1.10 is 15.

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l), \mu = (\mu_1, \mu_2, \ldots, \mu_m) \vdash n$ and $\mu \leq \lambda$. Amongst all possible fillings of a tableau with shape λ and content μ there exist two special situations for the charge, namely those fillings corresponding to the maximum and minimum charges of T.

We begin by considering the minimum charge for a semi-standard tableau. Clearly, for any tableau T, $c(T)$ cannot be negative by the weighting algorithm above. Suppose $\lambda = \mu$. Then $l = m$ and $\lambda_i = \mu_i$ for $1 \leq i \leq l$. Because we have a semi-standard tableau, every entry of a row i in filling of the tableau is exactly i. Hence, there exits only one filling of T with shape λ and content μ. Further, because it is formed by recording the entries from row l upwards to row 1, the reading word ω of T must be in weakly decreasing order. Thus, the weight of every entry in ω is 0. Therefore, when $\lambda = \mu$, the minimum charge of a tableau T is $c(T) = 0$.

We now consider the maximum charge of a tableau T. It is apparent from the weighting algorithm above that the multiplicity and ordering of the entries affect the sum which composes the charge, i.e. to produce a maximum charge we want to arrange the entries j and $(j+1)$ in such a way so as to make $w(j)$ and $w(j+1)$ as large as possible. This is accomplished in two ways. First, to produce a maximum charge, μ must have the form (1^n) because any repetition of an entry j in the reading word can have weight at most $(j-1)$. Second, for every pair of successive entries j and $(j+1)$ of ω, $(j+1)$ must appear to the right of j in ω, otherwise, $w(j) = w(j+1)$, for $1 \leq j \leq l-1$. To achieve this, the length of λ must equal one, because otherwise there must exist some pair of entries j and $(j+1)$ where $(j+1)$ appears in a row beneath the row that j appears in, i.e. $(j+1)$ would appear to the left of j in the reading word. Therefore the maximum charge of a tableau is $\sum_{i=0}^{l-1} i = \frac{(l-1)l}{2}$ and occurs when $\lambda = (n)$ and $\mu = (1^n)$.

Example 1.3.2. For T_λ with $\lambda = (3^2, 2, 1)$ and content $\mu = (3^2, 2, 1)$ and $T_{\bar{\lambda}}$ with $\bar{\lambda} = (9)$ and content $\bar{\mu} = (1^9)$, the reading word and charges of T_λ and $T_{\bar{\lambda}}$ are shown in Figure 1.11.

$T_\lambda =$

1	1	1
2	2	2
3	3	
4		

$T_{\bar{\lambda}} =$

1	2	3	4	5	6	7	8	9

	reading word	charge
T_λ	433222111	0
$T_{\bar{\lambda}}$	123456789	36

Figure 1.11: Examples of maximum and minimum tableau charges

Chapter 2

Symmetric Functions and the Hall-Littlewood Inner Product

2.1 Symmetric Functions

Let $\mathbb{Z}[x_1, x_2, \ldots, x_n]$ be the ring of integer polynomials in n independent variables $x_1, x_2, \ldots, x_n$. The symmetric group S_n acts invariantly upon $\mathbb{Z}[x_1, x_2, \ldots, x_n]$ by permuting the elements of the polynomial, i.e. $S(x_1, x_2, \ldots, x_n)$ is a symmetric polynomial if and only if for every function $\phi \in S_n$, where $\phi : S_n \to S_n$,

$$S(x_1, x_2, \ldots, x_n) = S(x_{\phi(1)}, x_{\phi(2)}, \ldots, x_{\phi(n)}).$$

Provided that n is large enough, the number of variables x_i is usually unimportant. However, it is frequently useful to work with symmetric function rings in infinitely many variables, denoted $\mathbb{Z}[x_1, x_2, \ldots, x_n, \ldots]$. In such cases there exists a natural embedding of $\mathbb{Z}[x_1, x_2, \ldots, x_n, \ldots]$ into $\mathbb{Z}[x_1, x_2, \ldots, x_n]$ by sending x_i to itself for $1 \leq x_i \leq n$ and sending x_i to 0 for $i \geq (n+1)$.

2.2 Symmetric Function Bases

For the symmetric functions there exists five classical bases which can be found in every textbook, [11], [13], [15], on symmetric functions. We will introduce all five bases for the symmetric functions as well as their generating functions, though we will focus exclusively on the last one. In all cases, we let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ be a partition of a positive integer n.

Definition 2.2.1. Define the *monomial symmetric function*, denoted m_λ, as the sum of the product of l variables x_i such that

$$m_\lambda = \sum_{1\le i\le l} x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \dots x_{i_l}^{\lambda_l}$$

where each monomial is distinct having exponents $\lambda_1, \lambda_2, \dots, \lambda_l$.

Definition 2.2.2. Let n be an non-negative integer. The n-th *elementary symmetric function*, denoted e_n, is defined as the sum of the product of n variables x_i, where the x_i's are distinct, such that $e_0 = 1$ and

$$e_n = \sum_{i_1<i_2<\ldots<i_n} x_{i_1} x_{i_2} \dots x_{i_n} = m_{1^n}$$

for $n \ge 1$.

Proposition 2.2.3. *The generating function, denoted $E(t)$, for the elementary symmetric functions is given by*

$$E(t) = \sum_{n\ge 0} e_n(x) t^n = \prod_{i\ge 1} (1 + x_i t)$$

for another variable t.

We observe that if we add all the square-free monomials of a degree n, we obtain the n-th elementary symmetric function e_n.

Definition 2.2.4. Let n be a non-negative integer. The n-th *complete homogeneous*, or simply *complete, symmetric function*, denoted h_n, is defined as the sum of all monomials of degree n of the variables x_i such that $h_0 = 1$ and

$$h_n = \sum_{i_1\le i_2\le\ldots\le i_n} x_{i_1} x_{i_2} \dots x_{i_n} = \sum_{\lambda\vdash n} m_\lambda.$$

We note that, by definition, $e_1 = h_1$ and observe that h_n is the sum of all n-th degree monomials.

Proposition 2.2.5. *The generating function, denoted $H(t)$, for the complete homogeneous symmetric functions is given by*

$$H(t) = \sum_{n\ge 0} h_n(x) t^n = \prod_{i\ge 1} \frac{1}{1 - x_i t}.$$

Definition 2.2.6. Let n be a positive integer. The n-th *power sum symmetric function*, denoted p_n, is defined as the sum of the n variables x_i such that

$$p_n = \sum_{i \geq 1} x_i^n = m_(n).$$

Proposition 2.2.7. *The generating function, denoted $P(t)$, for the power sum symmetric functions is given by*

$$\begin{aligned} P(t) &= \sum_{n \geq 1} \frac{p_n(x)}{t^{n-1}} \\ &= \sum_{i \geq 1} \sum_{n \geq 1} \frac{x_i^n}{t^{n-1}} \\ &= \textstyle\sum_{i \geq 1} \frac{x_i}{1 - x_i t} \\ &= \textstyle\sum_{i \geq 1} \frac{d}{dt} \log\left(\frac{1}{1 - x_i t}\right). \end{aligned}$$

Because they are instructive, we will provide two equivalent definitions for the fifth basis of the symmetric functions, the Schur symmetric functions, first introduced by Issai Schur in his dissertation in 1901 and which bear his name.

Definition 2.2.8. For a partition λ, define the *Schur symmetric functions*, denoted s_λ, to be

$$s_\lambda(x) = \sum_T x^T$$

where the sum is over all semi-standard tableau T of shape λ and $x = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_l^{\lambda_l}$.

This basis for the symmetric functions is often useful to work with as it allows the symmetric functions to be represented by their accompanying Young tableau and is often seen when finding the irreducible characters of symmetric functions.

We note that, when defined this way it is easy to see, the Schur symmetric function basis provides a generalization for the complete homogeneous symmetric function basis when the length of λ is one, i.e. T is a horizontal strip. When $\lambda = (1^n)$, i.e. T is a vertical strip, then the Schur symmetric functions generalize the elementary symmetric functions. Finally, the Schur symmetric functions can be written as the sum of the product of the Kostka numbers, denoted $K_{\lambda,\mu}$, and the monomial symmetric functions, m_μ for a partition $\mu \leq \lambda$. Hence,

$$\begin{aligned} s_{(n)}(x) &= h_n(x), \\ s_{(1^n)}(x) &= e_n(x), \qquad \text{and} \\ s_\lambda &= \sum_{\mu \leq \lambda} K_{\lambda,\mu} m_\mu. \end{aligned}$$

The second definition for the Schur symmetric function basis is expressed as a ratio of matrix determinants and will be motivational in the next section.

Definition 2.2.9. Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, let $\Lambda[x]$ be the ring of symmetric functions in x where x is the set of variables given by $x = \{x_1, x_2, \ldots, x_l\}$. Then by forming the $l \times l$ determinant α_λ given by

$$\alpha_\lambda = \det(x_i^{\lambda_j})$$

for $1 \leq i, j \leq l$, we can define the *Schur symmetric functions*, s_λ, by

$$s_\lambda = \frac{\alpha_{\lambda+\delta}}{\alpha_\delta}$$

where δ is the partition $\delta = (l-1, l-2, \ldots, 0)$, α_δ is the Vandermonde determinant $\prod_{i<j}(x_i - x_j)$, and the addition of λ and δ is done component-wise.

By properties of the determinant of a matrix, we note that the choice of α in the above definition is meant to be evocative of the fact that under any transposition of the variables, α_λ will alternate signs. Then since the Vandermonde determinant is also alternating, we have that the Schur functions are indeed symmetric.

2.3 Hall-Littlewood Functions and Kostka-Foulkes Coefficients

Introduced implicitly by Philip Hall in terms of the Hall algebra and later explicitly defined by D.E. Littlewood, the Hall-Littlewood functions provide a generalization for the complete homogeneous symmetric functions as well as the Schur symmetric functions and serve as the foundation for the operators utilized to prove the main result of the next section.

Definition 2.3.1. Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ and set of variables $x = \{x_1, x_2, \ldots, x_l\}$ and t, define the *Hall-Littlewood symmetric functions*, denoted $L_\lambda[x;t]$, to be

$$L_\lambda[x;t] = \left(\prod_{i \geq 0} \prod_{j=1}^{m(i)} \frac{1-t^i}{1-t^j} \right) \sum_{\omega \in S_l} \omega \left(x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_l^{\lambda_l} \prod_{i<j} \frac{x_i - t x_j}{x_i - x_j} \right)$$

where $m(i)$ is the multiplicity of an element i as defined in chapter 1.

We observe the Hall-Littlewood functions generalize the complete homogeneous symmetric functions by letting $t \to 1$ and generalize the Schur symmetric functions by letting $t \to 0$ in the definition above.

Given their relation, it is not surprising that the Hall-Littlewood symmetric functions can be expanded in terms of the Schur symmetric function basis. For partitions λ and μ of n, with $\mu \leq \lambda$ this expansion is given by

$$L_\lambda[x;t] = \sum_\lambda K_{\lambda,\mu}(t) s_\lambda[x]$$

where the coefficients $K_{\lambda,\mu}(t)$ are the Kostka-Foulkes numbers in variable t with non-negative integer coefficients.

In 1978, Lascoux and Schutzenberger, [12], proved that

$$K_{\lambda,\mu} = \sum_T t^{c(T)}$$

where T is a semi-standard tableau of shape λ and content μ and $c(T)$ is the charge statistic for T.

As an immediate consequence of Lascoux and Schutzenberger's theorem, we see that each semi-standard filling of a tableau contributes exactly one monomial in t to the summand. Combinatorially, this means that the coefficient $c_m \in \mathbb{N}$ of a term in the polynomial

$$K_{\lambda,\mu} = \sum_T t^{c(T)}$$

counts the number of tableaux with charge m and for this reason the coefficients are known as the Kostka-Foulkes numbers.

Example 2.3.2. Let T be a semi-standard Young tableau of shape $\lambda = (4, 1^2)$ with content $\mu = (2, 1^4)$. Every semi-standard filling of T with its respective charge is shown in Figure 2.1. By Lascoux's and Schutzenberger's theorem, the Kostka-Foulkes coefficients for a tableau with shape $\lambda = (4, 1^2)$ with content $\mu = (2, 1^4)$ is

$$K_{(4,1^2)(2,1^4)} = t^3 + t^4 + 2t^5 + t^6 + t^7.$$

Example 2.3.3. By definition of the charge statistic and the combinatorial statement made above, we highlight four special cases for the Kostka-Foulkes coefficients:

$$\begin{array}{rcll} K_{\lambda,\mu}(0) & = & 0 \text{ if } \lambda \neq \mu, & \text{Case (2.3.3.1)} \\ K_{\lambda,\mu}(0) & = & 1 \text{ if } \lambda = \mu, & \text{Case (2.3.3.2)} \\ K_{\lambda,\mu}(t) & = & 0 \text{ if } |\lambda| \neq |\mu|. & \text{Case (2.3.3.3)} \\ K_{\lambda,\mu}(t) & = & 1 \text{ if } |\lambda| = |\mu|. & \text{Case (2.3.3.4)} \end{array}$$

$$\begin{array}{|c|c|c|c|}\hline 1&1&2&3\\\hline \multicolumn{1}{|c|}{4}\\\cline{1-1} \multicolumn{1}{|c|}{5}\\\cline{1-1}\end{array}\qquad\begin{array}{|c|c|c|c|}\hline 1&1&2&4\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1} \multicolumn{1}{|c|}{5}\\\cline{1-1}\end{array}\qquad\begin{array}{|c|c|c|c|}\hline 1&1&2&5\\\hline \multicolumn{1}{|c|}{3}\\\cline{1-1} \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array}$$

charge: 7 6 5

$$\begin{array}{|c|c|c|c|}\hline 1&1&3&4\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1} \multicolumn{1}{|c|}{5}\\\cline{1-1}\end{array}\qquad\begin{array}{|c|c|c|c|}\hline 1&1&3&5\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1} \multicolumn{1}{|c|}{4}\\\cline{1-1}\end{array}\qquad\begin{array}{|c|c|c|c|}\hline 1&1&4&5\\\hline \multicolumn{1}{|c|}{2}\\\cline{1-1} \multicolumn{1}{|c|}{3}\\\cline{1-1}\end{array}$$

charge: 5 4 3

Figure 2.1: $K_{(4,1^2)(2,1^4)}$ utilizing charge

Recall that for a tableau T with shape λ and content μ, it is not possible to find a filling if $\lambda < \mu$. Hence the charge of $T_{\lambda,\mu}$ equals 0 in Case (2.3.3.3) (Case (2.3.3.1) is by convention).

Further, as we saw in section 1.3, there is exactly one filling of T when $\lambda = \mu$ and the reading word is strictly increasing from left to right. Thus, the charge of T is exactly equal to 1 in Case (2.3.3.4) (Case (2.3.3.2) is due to the fact that there is exactly one way to represent the "empty" partition as an "empty" tableau, i.e $\emptyset$).

Although the Kostka-Foulkes coefficients can be calculated utilizing Lascoux and Schutzenberger's charge statistic, the process quickly becomes difficult to perform by hand. This can be seen when considering the difference in the number of semi-standard fillings for a partition of weight and content six as in the previous example versus the number of semi-standard fillings for a partition of weight and content fifteen in the following example, i.e. 6 as compared to 35,035.

Example 2.3.4. Let T be a semi-standard tableau of shape $\lambda = (6, 4, 3, 2)$ with content $\mu = (3, 1^{12})$. Then the Kostka-Foulkes coefficients are $K_{(6,4,3,2),(3,1^{12})} =$

$$
\begin{array}{llllll}
 t^{16} & + \ 3t^{17} & + \ 7t^{18} & + \ 15t^{19} & + \ 28t^{20} & + \ 48t^{21} \\
+ \ 79t^{22} & + \ 122t^{23} & + \ 180t^{24} & + \ 256t^{25} & + \ 351t^{26} & + \ 465t^{27} \\
+ \ 600t^{28} & + \ 751t^{29} & + \ 917t^{30} & + \ 1093t^{31} & + \ 1273t^{32} & + \ 1447t^{33} \\
+ \ 1613t^{34} & + \ 1758t^{35} & + \ 1878t^{36} & + \ 1965t^{37} & + \ 2017t^{38} & + \ 2027t^{39} \\
+ \ 2001t^{40} & + \ 1933t^{41} & + \ 1832t^{42} & + \ 1701t^{43} & + \ 1549t^{44} & + \ 1378t^{45} \\
+ \ 1203t^{46} & + \ 1025t^{47} & + \ 855t^{48} & + \ 695t^{49} & + \ 552t^{50} & + \ 425t^{51} \\
+ \ 320t^{52} & + \ 232t^{53} & + \ 163t^{54} & + \ 110t^{55} & + \ 72t^{56} & + \ 44t^{57} \\
+ \ 26t^{58} & + \ 14t^{59} & + \ 7t^{60} & + \ 3t^{61} & + \ t^{62}. &
\end{array}
$$

While there exist computer software packages, including Maple and Sage, capable of calculating them using the charge, it has long been the goal to develop an explicit algebraic formula for the Kostka-Foulkes coefficients. This goal is both intellectually and practically driven as we clearly would prefer to have a more efficient mechanism for calculating the Kostka-Foulkes coefficients and, thus, a more efficient transition between the Schur symmetric functions and the Hall-Littlewood symmetric functions. Further, the Kostka-Foulkes coefficients appear in the calculations of even more general families of polynomials such as the Lascoux, Leclerc, and Thibon (LLT) polynomials and the Macdonald polynomials, [3], [4], [9], [10]. Moreover, the pursuit of an explicit algebraic formula for the Kostka-Foulkes coefficients is motivated by the hope that such a formula would reveal additional relationships between partitions and their corresponding Young tableaux.

2.4 An Explicit Algebraic Formula for the Kostka-Foulkes Polynomials

To determine an explicit algebraic formula for the Kostka-Foulkes coefficients we develop an explicit algebraic formula for the Kostka-Foulkes polynomials utilizing vertex operators. Let $\mathfrak{h}$ be the Heisenberg algebra generated by the central element c and the n nonzero integer elements h_n, with the relation

$$[h_m, h_n] = m\delta_{m,-n}c.$$

We reintroduce the Hall-Littlewood vertex operators defined by Jing in [7] and utiilized subsequently in [8].

Definition 2.4.1. The *vertex operators* $B(t)$ and $H(t)$ on the space V with parameter t are defined as

$$B(t) = \exp\left\{ \sum_{n\geq 1} \frac{1}{n} h_{-n} t^n \right\} \exp\left\{ -\sum_{n\geq 1} \frac{1-t^n}{n} h_n t^{-n} \right\},$$

$$H(t) = \exp\Big\{\sum_{n\geq 1}\frac{1-t^n}{n}h_{-n}t^n\Big\}\exp\Big\{-\sum_{n\geq 1}\frac{1-t^n}{n}h_n t^{-n}\Big\}.$$

Remark 2.4.2. *The above defintion for the Hall-Littlewood vertex operators differ from those defined by Zabrocki in [18] and utilized in [17] and [19].*

We refer the reader to [7] and [8] for an explicit proofs of the following results.

Corollary 2.4.3. *We have the following relations:*

$$H_n H_{n-1} = tH_{n-1}H_n;$$

$$H_n^* H_{n+1}^* = tH_{n+1}^* H_n^*.$$

Observe for $n \geq 0$, we can consider either the H_n operator or the $H_{-n}*$ operator as an annihilation operator.

Lemma 2.4.4. *One has that*

$$\begin{aligned} H_n \cdot 1 = 0; &\quad n > 0; \\ H_{-n}^* \cdot 1 = 0; &\quad n > 0; \quad \textit{and} \\ H_0 \cdot 1 = H_0^* \cdot 1 = 1. & \end{aligned}$$

Proposition 2.4.5. *(1) Given a partition $\lambda = (\lambda_1, \ldots, \lambda_l)$, the element $H_{-\lambda_1}, H_{-\lambda_2}, \cdots H_{-\lambda_l} \cdot 1$ can be expressed as*

$$H_{-\lambda_1}, H_{-\lambda_2}, \cdots H_{-\lambda_l} \cdot 1 = \prod_{i<j}\frac{1-R_{ij}}{1-tR_{ij}}P_{\lambda_1}P_{\lambda_2}\cdots P_{\lambda_l}$$

where R_{ij} is the raising operator given by

$$R_{ij}P_{(\mu_1,\mu_2,\ldots,\mu_l)} = P_{(\mu_1,\mu_2,\ldots,\mu_i+1,\ldots,\mu_j-1,\ldots,\mu_l)}$$

where $P_{(\mu_1,\mu_2,\ldots,\mu_l)} = P_{\mu_1} \cdot P_{\mu_2}, \cdots, P_{\mu_l}$ which ensures that the expression has finitely many terms. Moreover, the product of the vertex operators $H_{-\lambda}.1 = H_{-\lambda_1}H_{-\lambda_2}\cdots H_{-\lambda_l}.1$ are orthogonal such that

$$\langle H_{-\lambda}.1, H_{-\mu}.1\rangle = \delta_{\lambda\mu}b_\lambda(t),$$

where $b_\lambda(t) = \prod_{i\geq 1}\phi_{m_i(\lambda)}(t)$ and $\phi_n(t) = (1-t)(1-t^2)\cdots(1-t^n)$.

(2) Let $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ be any composition. The element of the product of the vertex operators $B_{-\mu_1} B_{-\mu_2} \cdots B_{-\mu_k} \cdot 1$ can be expressed as

$$B_{-\mu_1} B_{-\mu_2} \cdots B_{-\mu_k} \cdot 1 = \prod_{i<j} (1 - R_{ij}) s_{\mu_1} s_{\mu_2} \cdots s_{\mu_k} = s_\mu$$

which is the Schur function associated to μ. In general $s_\mu = 0$ or $\pm s_\lambda$ for a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ such the $\lambda \in S_l(\mu + \delta) - \delta$. Here $\delta = (l-1, l-2, \cdots, 1, 0)$.

We will need to make use of another definition from [7], in the following proposition.

Definition 2.4.6. The *Kostka-Foulkes polynomials* (or the t-analogue of the Kostka-Foulkes numbers) $K_{(\lambda),(\mu)}(t)$ are defined for all $\mu \leq \lambda$ by

$$s_\lambda = \sum_{\mu \leq \lambda} \frac{1}{b_\mu(t)} K_{\lambda\mu}(t) L_\mu(t)$$

where $b_\mu(t) = \prod_{i \geq 1} \Phi_{m_i(\lambda)}(t), \Phi_r(t) = (1-t)(1-t^2) \cdots (1-t^r)$, and $m_i(\lambda)$ is the number of occurrences of i in λ.

Proposition 2.4.7. *The Kostka-Foulkes polynomials are the matrix coefficients of vertex operators. Specifically,*

$$K_{\lambda\mu}(t) = \langle B_{-\lambda} \cdot 1, H_{-\mu} \cdot 1 \rangle.$$

Proposition 2.4.8. *The commutation relations for the Hall-Littlewood inner product are:*

1. $B_m s_n = s_n B_m + s_{n-1} B_{m+1}$ *(It follows for $B(z)s(w) = (1 - \frac{w}{z}) s(w) B(z)$. Here $s(z) = exp\left\{ \sum \frac{h_{-n}}{n} z^n \right\} = \sum s_n z^n$.)*
2. $H_m^* B_n = t^{-1} B_n H_m^* + t^{-1} H_{m-1}^* B_{n-1} + (t^{n-m} - t^{n-m-1}) s_{n-m}$.

Example 2.4.9. Let T be the Young tableau of shape $\lambda = (3, 2, 1)$ with content $(2^2, 1^2)$. By applying second relation from Proposition 2.4.8 for the dual Hall-Littlewood inner product, the Kostka-Foulkes polynomial for T is

$$
\begin{aligned}
K_{(3,2,1),(2^2,1^2)} &= K_{(2^2,1),(2,1^3)} + t^{-1}\langle B_3 H_2^* B_2 B_1, H_2 H_1^2\rangle \\
&\quad -(1-t)[K_{(3,1),(2,1^2)} + K_{(2^2),(2,1^2)} + K_{(2,1^2),(2,1^2)}] \\
&= K_{(2^2,1),(2,1^3)} + t^{-2}\langle B_3 H_1^* B_1^2, H_2 H_1^2\rangle + t^{-2}\langle B_3 B_2 H_2^* B_1, H_2 H_1^2\rangle \\
&\quad -(1-t)[K_{(3,1),(2,1^2)} + K_{(2^2),(2,1^2)} + K_{(2,1^2),(2,1^2)}] - (1-t)t^{-2}K_{(3,1),(2,1^2)} \\
&= K_{(2^2,1),(2,1^3)} - (1-t)K_{(2,1^2),(2,1^2)} - (1-t)K_{(2^2),(2,1^2)} \\
&\quad -(1-t)K_{(3,1),(2,1^2)} - (1-t)t^{-2}K_{(3,1),(2,1^2)} - t^{-2}K_{(3,1),(2,1^2)} \\
&= K_{(2^2,1),(2,1^3)} - (1-t)K_{(2,1^2),(2,1^2)} - (1-t)K_{(2^2),(2,1^2)} \\
&\quad -(-t^{-1}+1-t)K_{(3,1),(2,1^2)} \\
&= (t+t^2) - (1-t)(1) - (1-t)(t) \\
&= -(-t^{-1}+1-t)(t+t^2) \\
&= t + 2t^2 + t^3.
\end{aligned}
$$

The result of the above calculation of the Kostka-Foulkes polynomial for a tableau of shape $(3,2,1)$ with content $(2^2,1^2)$ is easily verified utilizing Lascoux and Schutzenberger's charge statistic or by using mathematics software (our results were verified using the Kostka-Foulkes polynomial function found in Sage).

Given the relative simplicity involved in the calculation of $K_{(3,2,1),(2^2,1^2)}$ through other means we include Example 2.4.9 as a reference to verify two remarks about the action of dual Hall-Littlewood vertex operator found in [1]. First, the action of the dual Hall-Littlewood operator, H_μ^*, on B_λ always leaves the B_{λ_i}s in partition order. Second, the number of dual Hall-Littlewood operator, H_μ^*, actions on B_λ is bounded by $(\mu_1 + l(\lambda) - 2)$.

Remark 2.4.10. *The action of the dual Hall-Littlewood operator, H_μ^*, on B_λ always leaves the B_{λ_i}s in partition order.*

Proof. By part 2) of Proposition 2.4.5, since the Schur function $B_\mu.1$ for any composition μ can always be reduced to that of a partition or it vanishes, the action of the dual Hall-Littlewood vertex operator H_n^* on the $B_\lambda.1$s will terminate in a combination of genuine Schur functions after finitely many steps.

We observe that applying Proposition 2.4.8, one often uses the simple fact

$$H_n^* B_{\lambda_1} B_{\lambda_2} \cdots B_{\lambda_l} \cdot 1 = 0$$

when $n > |\lambda|$.

Assume the inner product is of the form

$$t^{-k}\langle B_{\lambda_{(l-2)}} H_{\mu_\alpha}^* B_{\lambda_{(l-1)}} B_{\lambda_l}, **\rangle$$

and $\mu_\alpha > \lambda_{(l-1)}$. By Proposition 2.4.8 we see that

$$
\begin{aligned}
t^{-k}\langle B_{\lambda_{(l-2)}} H^*_{\mu_\alpha} B_{\lambda_{(l-1)}} B_{\lambda_l}, **\rangle \quad = \quad & t^{-k-1}\langle B_{\lambda_{(l-2)}} H^*_{\mu_\alpha -1} B_{\lambda_{(l-1)}-1} B_{\lambda_l}, **\rangle \\
& + t^{-k-1}\langle B_{\lambda_{(l-2)}} B_{\lambda_{(l-1)}} H^*_{\mu_\alpha} B_{\lambda_l}, **\rangle \\
& -(1-t) t^{\lambda_{(l-1)}-\mu_\alpha-k-1}\langle B_{\lambda_{(l-2)}} s_{(\lambda_{(l-1)}-\mu_\alpha)} B_{\lambda_l}, **\rangle
\end{aligned}
$$

where $s_{(\lambda_{(l-1)}-\mu_\alpha)} B_{\lambda_l}$ is the Littlewood-Richardson multiplication of $s_{(\lambda_{(l-1)}-\mu_\alpha)}$ and B_{λ_l}.

Therefore, the action of the dual Hall-Littlewood operator, H^*_μ, action on B_λ always leaves the B_{λ_i}s in partition order.

□

Remark 2.4.11. *The number of dual Hall-Littlewood operator, H^*_μ, actions on B_λ is bounded by $(\mu_1 + l(\lambda) - 2)$.*

Proof. Combinatorially, we proceed in the manner of a stars and bars proof where $H^*_{\mu_1 - j}$ is inserted into $B_\lambda = B_{\lambda_1} B_{\lambda_2} \cdots B_{\lambda_l} \cdot 1$.

Define $r_{i,j}$ where $i = \#$ of $B_{\lambda^{(i)}}$ in front of $H^*_{\mu_1 - j}$ and j is such that $i+j+1 =$ the exponent k of t^{-k}.

In the second relation of Proposition 2.4.8, $H^*_{\mu_1 - j}$ acts by either removing a box from the B_λ following it and itself or $H^*_{\mu_1 - j}$ acts by trading places with the B_λ following it. This means that minuses from B_λ form a weak composition of j in exactly $i+1$ parts.

We have that i is bounded below by 0 and above by $k-1$ while j is also bounded below by 0 and $k-1$. Hence, there are $\binom{k-1}{i}$ ways to insert $H^*_{\mu_1 - j}$ into B_λ following i entries. Thus there are

$$\sum_{i=0}^{k-1} \binom{k-1}{i}$$

ways to insert $H^*_{\mu_1 - j}$ into B_λ in total.

It is well known that the number of weak compositions of j into $i+1$ parts is counted by

$$\binom{j+i+1-1}{i+1-1} = \binom{j+i}{i}.$$

By construction, $i+j+1 = k$, which gives $i+j = k-1$ and thus there are

$$\sum_{i=0}^{k-1} \binom{k-1}{i}$$

weak compositions of j into $i+1$ parts.

Thus, the number of ways to insert $H^*_{\mu_1-j}$ is equal to the number of weak compositions of j. Hence,

$$|r_{i,j}| = \sum_{i=0}^{k-1} \binom{k-1}{i}$$

for all $k = 1, 2, \ldots$.

Then, assuming a sufficiently large B_{λ_*} the action of H^*_μ is either

- Hold in place (with $\mu_1 - 1) \leq \mu_1 - 1$ times because $H^*_0 B_{\lambda_i} = t^{\lambda_i} B_{\lambda_i}$, or

- Move $\leq l(\lambda) - 1$ times because $H^*_i \cdot 1 = 0$ for all $i > 0$.

Therefore, the action of the dual Hall-Littlewood operator, H^*_μ, on B_λ terminates in $\leq (\mu_1 + l(\lambda) - 2)$ steps. □

We are now ready to prove our main result found in [1], an explicit algebraic formula for the Kostka-Foulkes polynomials utilizing Hall-Littlewood vertex operators found in [7]. In our theorem and subsequent proof, we will make repeated use of removing the j-th entry of a partition λ and adding 1 to each entry λ_i, where $1 \leq i < j \leq l$. Accordingly, we introduce the notational convenience $\lambda^{(i)}$ for this action and define it in the following way.

Definition 2.4.12. Let $\lambda^{(i)} = (\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_{i-1} + 1, \lambda_{i+1}, \ldots, \lambda_l)$.

Theorem 2.4.13. *For partitions $\lambda, \mu \vdash n$ with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$,*

$$H^*_k B_\lambda = \sum_{i=1}^{l} (-1)^{i-1} t^{\lambda_i - k - i + 1} s_{\lambda_i - k - i + 1} B_{\lambda^{(i)}}$$

where s_λ is the multiplication operator by the Schur function s_λ.

Proof. We argue by induction on $|\lambda| + k$, where λ is a partition and k is the degree of the dual vertex operator for the Hall-Littlewood symmetric function. First of all, the initial step is clear.

Thus, the inductive hypothesis implies that

$$\begin{aligned} H^*_{k-1} B_{\lambda_1 - 1} B_{\lambda_2} \cdots B_{\lambda_l} &= t^{\lambda_1 - k} s_{\lambda_1 - k} B_{\lambda_2} \cdots B_{\lambda_l} - t^{\lambda_2 - k} s_{\lambda_2 - k} B_{\lambda_1} B_{\lambda_3} \cdots B_{\lambda_l} - \cdots \\ H^*_k B_{\lambda_2} \cdots B_{\lambda_l} &= t^{\lambda_2 - k} s_{\lambda_2 - k} B_{\lambda_3} \cdots B_{\lambda_l} - t^{\lambda_3 - k - 1} s_{\lambda_3 - k - 1} B_{\lambda_2 + 1} B_{\lambda_4} \cdots B_{\lambda_l} - \cdots . \end{aligned}$$

Now we have

$$\begin{aligned} H_k^* B_\lambda &= t^{-1} B_{\lambda_1} H_k^* B_{\lambda_2} \cdots B_{\lambda_l} \\ &\quad + t^{-1} H_{k-1}^* B_{\lambda_1 - 1} B_{\lambda_2} \cdots B_{\lambda_l} \\ &= t^{-1} B_{\lambda_1} (t^{\lambda_2 - k} s_{\lambda_2 - k} B_{\lambda_3} \cdots B_{\lambda_l} \\ &\quad - t^{\lambda_3 - k - 1} s_{\lambda_3 - k - 1} B_{\lambda_2 + 1} B_{\lambda_4} \cdots B_{\lambda_l} \\ &\quad + t^{\lambda_4 - k - 2} s_{\lambda_4 - k - 2} B_{\lambda_2 + 1} B_{\lambda_3 + 1} B_{\lambda_5} \cdots B_{\lambda_l} - \cdots) \\ &\quad + t^{-1} (t^{\lambda_1 - k} s_{\lambda_1 - k} B_{\lambda_2} \cdots B_{\lambda_l} \\ &\quad - t^{\lambda_2 - k} s_{\lambda_2 - k} B_{\lambda_1} B_{\lambda_3} \cdots B_{\lambda_l} \\ &\quad + t^{\lambda_3 - k - 1} s_{\lambda_3 - k - 1} B_{\lambda_1} B_{\lambda_2 + 1} B_{\lambda_4} \cdots B_{\lambda_l} - \cdots) \\ &\quad + (t^{\lambda_1 - k} - t^{\lambda_1 - k - 1}) s_{\lambda_1 - k} B_{\lambda_2} \cdots B_{\lambda_l}. \end{aligned}$$

Using the first commutation relation in Proposition 2.4.8, the first parenthesis can be put into

$$\begin{aligned} &(t^{\lambda_2 - k - 1} s_{\lambda_2 - k} B_{\lambda_1} B_{\lambda_3} \cdots B_{\lambda_l} - t^{\lambda_3 - k - 2} s_{\lambda_3 - k - 1} B_{\lambda_1} B_{\lambda_2 + 1} B_{\lambda_4} \cdots B_{\lambda_l} \\ &+ t^{\lambda_4 - k - 3} s_{\lambda_4 - k - 2} B_{\lambda_1} B_{\lambda_2 + 1} B_{\lambda_3 + 1} B_{\lambda_5} \cdots B_{\lambda_l} - \cdots) \\ &+ (t^{\lambda_2 - k - 1} s_{\lambda_2 - k - 1} B_{\lambda_1 + 1} B_{\lambda_3} \cdots B_{\lambda_l} - t^{\lambda_3 - k - 2} s_{\lambda_3 - k - 2} B_{\lambda_1 + 1} B_{\lambda_2 + 1} B_{\lambda_4} \cdots B_{\lambda_l} \\ &+ t^{\lambda_4 - k - 3} s_{\lambda_4 - k - 3} B_{\lambda_1 + 1} B_{\lambda_2 + 1} B_{\lambda_3 + 1} B_{\lambda_5} \cdots B_{\lambda_l} - \cdots). \end{aligned}$$

Combining with the other terms, they are exactly the following sum

$$\sum_{i=1}^{l} (-1)^{i-1} t^{\lambda_i - k - i + 1} s_{\lambda_i - k - i + 1} B_{\lambda^{(i)}}.$$

□

Remark 2.4.14. *We are aware of Zabrocki's result in [18] involving the Kostka-Foulkes polynomials in the action of a modified vertex operator for the Hall-Littlewood operator, $H_{m,\mu}$, on the Schur symmetric function basis and wish to highlight three differences. First, an explicit formula for the calculation of the Kostka-Foulkes polynomials is not provided by the action of the operator $H_{m,\mu}$. Second, the action of the operator in [18] is combinatorial in nature in that it requires fillings for semi-standard tableaux based upon the height their k-snake (or modified k-ribbon) whereas our formula is strictly algebraic. Third, the recursion resulting from Theorem 2.4.13 does not rely upon the iteration defined by Morris in [14].*

In the following remark we highlight several interesting features of the Kostka-Foulkes polynomial formula in Theorem 2.4.13.

Remark 2.4.15. *For partitions $\lambda, \mu \vdash n$ with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$,*

- 2.4.15.1 The formula has at most l nonzero summands, although in many cases there are fewer by the Littlewood-Richardson multiplication rule that $s_{-m}B_{\lambda^{(i)}} = 0$ for a positive integer m.

- 2.4.15.2 The formula depends only upon μ_1 from the content partition.

- 2.4.15.3 The exponent $\lambda_i - k - i + 1$ of t is evocative of the exponents of the determinant defintion of the Schur symmetric functions.

- 2.4.15.4 Despite the alternating signs in the formula, positivity is assured.

Example 2.4.16. Given $\lambda = (4, 1^2)$ and $\mu = (2, 1^4)$, the Kostka-Foulkes plolynomial formula gives

$$\begin{aligned} H_2^* B_\lambda &= \sum_{i=1}^{3} (-1)^{i-1} t^{\lambda_i - \mu_1 - i + 1} s_{\lambda_1 - \mu_1 - i + 1} B_{\lambda^{(i)}} \\ &= t^2 s_2 B_{\lambda^{(1)}} - t^{-2} s_{-2} B_{\lambda^{(2)}} + t^{-3} s_{-3} B_{\lambda^{(3)}} \\ &= t^2 s_2 B_1 B_1 \\ &= t^2 [K_{(3,1),(1^4)} + K_{(2,1^2),(1^4)}] \\ &= t^2 [(t^3 + t^4 + t^5) + (t + t^2 + t^3)] \\ &= t^2 (t + t^2 + 2t^3 + t^4 + t^5) \\ &= t^3 + t^4 + 2t^5 + t^6 + t^7 \\ &= K_{(4,1^2),(2,1^2)}. \end{aligned}$$

We include another example for calculating the Kostka-Foulkes polynomial using our formula to demonstrate its improvement over using Lascoux and Schutzenberger's charge statistic. We consider the semi-standard tableau T shape $\lambda = (6, 4, 3, 2)$ with content $\mu = (3, 1^{12})$. As we saw in Example 2.3.4, there were 35,035 unique fillings of T and, thus, 35,035 summands in the calculation of $K_{(6,4,3,2),(3,1^{12})}$ using the charge.

Example 2.4.17. Given $\lambda = (6, 4, 3, 2)$ and $\mu = (3, 1^{12})$, the dual Hall-Littlewood inner product algorithm gives

$$\begin{aligned} H_3^* B_\lambda &= \sum_{i=1}^{3} (-1)^{i-1} t^{\lambda_i - \mu_1 - i + 1} s_{\lambda_1 - \mu_1 - i + 1} B_{\lambda^{(i)}} \\ &= t^3 s_3 B_{\lambda^{(1)}} - t^0 s_0 B_{\lambda^{(2)}} + t^{-2} s_{-3} B_{\lambda^{(3)}} + t^{-4} s_{-4} B_{\lambda^{(4)}} \\ &= t^3 s_3 B_{\lambda^{(1)}} - B_{\lambda^{(2)}} \end{aligned}$$

Using the Pieri rule, the above formula gives that

$$\begin{aligned}K_{(6,4,3,2),(31^{12})} \quad &= \quad t^3[K_{(7,3,2),(1^{12})} + K_{(6,4,2),(1^{12})} + K_{(6,3^2),(1^{12})} \\ &\quad +K_{(6,3,2,1),(1^{12})} + K_{(5,4,3),(1^{12})} + K_{(5,4,2,1),(1^{12})} \\ &\quad +K_{(5,3^2,1),(1^{12})} + K_{(5,3,2^2),(1^{12})} + K_{(4^2,3,1),(1^{12})} \\ &\quad +K_{(4^2,2^2),(1^{12})} + K_{(4,3^2,2),(1^{12})}] - K_{(7,3,2),(1^{12})}\end{aligned}$$

which is easily verified using Sage or repeated applications of our iterative formula

While example 2.4.17 clearly illustrates the computational superiority of the recursion defined in Theorem 2.4.13, requiring only two summands of a combined 12 Kostka-Foulkes polynomials, we again wish to draw attention to the fact that our result provides an explicit algebraic, instead of combinatorial, formula for the Kostka-Foulkes polynomials.

As consequences of Theorem 2.4.13, in the next section we introduce several combinatorial results about partitions and their corresponding Young tableaux, develop a new lattice structure for tableaux, and define two new classes of symmetric polynomials, the first of which possesses the stronger condition of being monic.

Chapter 3

Littlewood-Richardson Tableaux

3.1 Littlewood-Richardson Cores and Tableaux

In 1981, while extending the Murnaghan-Nakayama formula to generalized characters, James and Kerber, [6], showed any partition λ can be associated with a k-core.

Definition 3.1.1. Define the *k-core* $\lambda_{(k)}$ of a partition λ of n to be the unique partition obtained by removing ribbons of size k, called k-ribbons, from the Young tableau representation of λ such that partition shape is always maintained.

Let $|\lambda_{(k)}| = m$ be the number of k-ribbons removed in acquiring the k-core of λ. Note that, regardless of the method utilized to obtain a k-core for a partition, m is always the same. We then introduce the weight of $\lambda_{(k)}$, denoted μ, to be (1^m). By numbering the removed k-ribbons from 1 to m, where the label 1 represents the last ribbon removed, the number 2 represents the second to last ribbon removed, and so on, we can identify distinct methods of obtaining the k-core.

Example 3.1.2. Let $n = 17$ and $\lambda = (5, 4, 3^2, 2)$. Then the six tableaux of $\lambda/\lambda_{(3)}$ with $\lambda_{(3)} = (1^2)$ and weight $\mu = (1^5)$ are shown in Figure 3.1. Observe that the first and second pairs of tableaux have the exact same k-ribbons to be removed to respectively reach $\lambda_{(3)} = (1^2)$ with differing orderings.

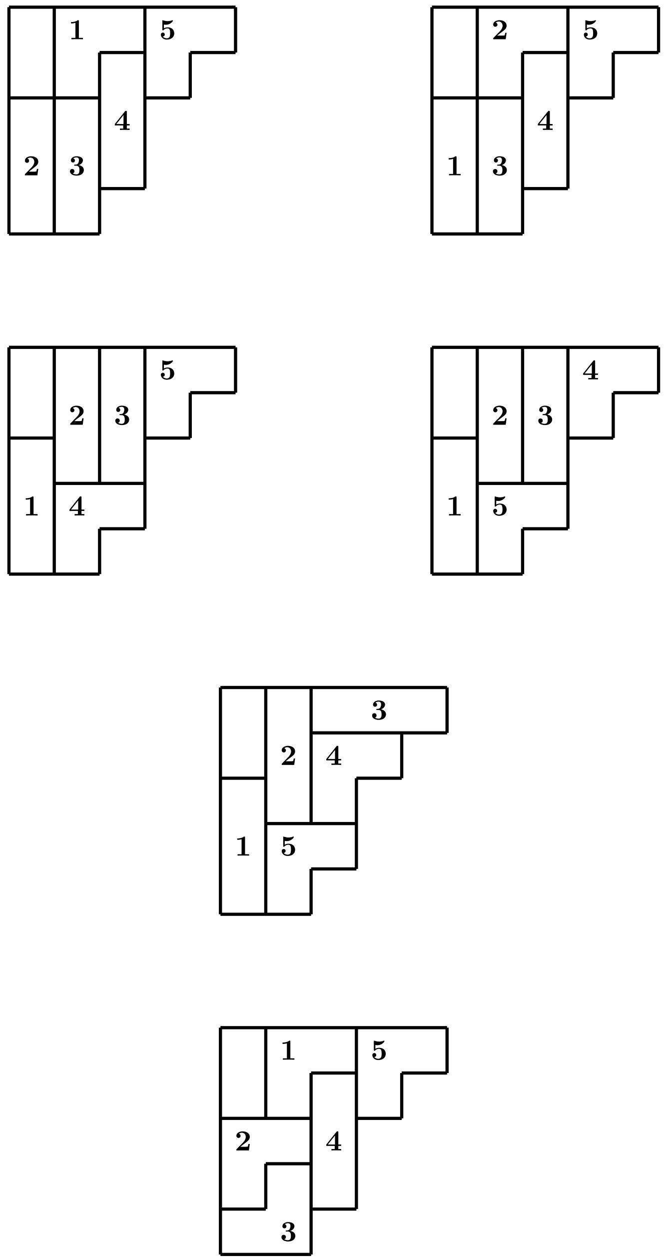


Figure 3.1: $\lambda_{(3)} = (1^2)$ for $\lambda = (5, 4, 3^2, 2)$

As there are often multiple methods for reaching a k-core, we wish to generalize to k-ribbon tableaux of a general shape and weight.

Following Lascoux, Leclerc and Thibon, [10], we denote a k-ribbon by R and consider the upper rightmost cell of R to be its initial cell. Then define a skew shape Ψ to be $\Psi = \gamma/\delta$ and set $\delta_+ = (\gamma_1) \wedge \delta$ such that a horizontal strip consisting of the upper cells of Ψ if formed by δ_+/δ. If Ψ can be tiled by ω k-ribbons with initial cells contained in δ_+/δ, then Ψ is a horizontal k-ribbon strip with weight ω.

Definition 3.1.3. Define the *k-ribbon tableau* T of shape λ/ν and weight $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ to be a chain of partitions

$$\nu = \delta^0 \subset \delta^1 \subset \cdots \subset \delta^r = \lambda$$

such that δ^i/δ^{i-1} is a hortizontal strip k-ribbon strip of weight μ_i.

k-ribbon tableaux have been utilized to calculate the skew Kostka-Foulkes polynomials $K_{\lambda/\nu,\omega}(q)$ where the skew tableaux have shape λ/ν and weight ω as well as playing a role in the creation of LLT polynomials which are single variable symmetric functions in q. Haglund then showed that LLT polynomials are strongly connected to MacDonald polynomials via skew ribbon diagrams.

As we saw in chapter 2, the Littlewood-Richardson rule for the multiplication of Young tableaux was central to the calculation of the algebraic Kostka-Foulkes polynomials $K_{(\lambda)(\mu)}$. Furthermore, we showed that given a partition of shape λ and content μ, every non-zero term in the $K_{(\lambda)(\mu)}$ calculation had a shape containing $(\lambda_2, \lambda_3, \ldots, \lambda_l)$. Similar to the k-cores of Kerber and James, we introduce a a Littlewood-Richardson core (LRC) for a partition λ.

Definition 3.1.4. Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$. Then the *Littlewood-Richardson core* (LRC) of λ is $\lambda_{(k)} = LRC(\lambda) = (\lambda_2, \lambda_3, \ldots, \lambda_l)$. In the case where $l = 1$, we take $LRC(\lambda) = \emptyset$.

In the language of LLT polynomials, this process produces a skew tableau of shape λ/ν where $\nu = (\lambda_2, \lambda_3, \ldots, \lambda_l)$. In the following sections, we will not be producing skew tableaux for a partition λ though we still wish to record the $LRC(\lambda)$. For the remainder of this paper, for a partition λ we will graphically denote the $LRC(\lambda)$ shading the upper leftmost cells of the Young tableau of λ, those having shape $(\lambda_2, \lambda_3, \ldots, \lambda_l)$, in gray. For example, when $\lambda = (3, 2, 1)$, the LRC is shown in Figure 3.2.

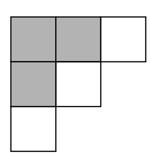


Figure 3.2: Littlewood-Richardson core for $\lambda = (3, 2, 1)$

The next result proves the Littlewood-Richardson core of a partition is always present within the explicit Kostka-Foulkes polynomial algorithm of Theorem 2.4.13.

Theorem 3.1.5. *The Littlewood-Richardson core of a partition λ has shape $\lambda^{(1)}$.*

Proof. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ be partitions of n with $\mu \leq \lambda$ as in Theorem 2.4.13.

Clearly, $(\lambda_2 \lambda_3, \ldots, \lambda_l) = \lambda^{(1)}$.

Consider

$$\begin{aligned} K_{(\lambda),(\mu)} &= t^J K_{(\lambda_1 - 1, \lambda_2, \ldots, \lambda_l),(\mu^*)} + t^{-1} \langle B_{\lambda_1} H^*_{\mu_1} B_{\lambda_2} \cdots B_{\lambda_l}, **\rangle \\ &\quad -(1-t) t^{\lambda_1 - \mu_1 - 1} s_{\lambda_1 - \mu_1} \langle B_{\lambda_2} \cdots B_{\lambda_l}, **\rangle. \end{aligned}$$

We proceed inductively.

Case 1: As $\lambda_1 \geq \mu_1$, $s_{\lambda_1 - \mu_1}$ is non-negative. Thus, each tableau resulting from the Littlewood-Richardson multiplication rule has length either l or $(l-1)$. Hence, $(\lambda_1 - \mu_1)$ cells may be added to $\lambda^{(1)}$. This means a row may have at most $(\lambda_1 - \mu_1)$ added to it. Therefore the intersection of the tableaux resulting from the Littlewood-Richardson multiplication rule is $\lambda^{(1)}$.

Case 2: If $H^*_{\mu_1} > B_{\lambda_2}$, the Hall-Littlewood inner product from Theorem 2.4.13 equals 0. Else,

$$\begin{aligned} \langle B_{\lambda_1} H^*_{\mu_1} B_{\lambda_2} \cdots B_{\lambda_l}, H_{\mu_2} \cdots H_{\mu_k} \rangle &= t^{-1} \langle B_{\lambda_1} H^*_{\mu_1 - 1} B_{\lambda_2 - 1} B_{\lambda_3} \cdots B_{\lambda_l}, **\rangle \\ &\quad + t^{-1} \langle B_{\lambda_1} B_{\lambda_2} H^*_{\mu_1} B_{\lambda_3} \cdots B_{\lambda_l}, **\rangle \\ &\quad -(1-t) t^{\lambda_2 - \mu_1 - 1} \langle B_{\lambda_1} s_{\lambda_2 - \mu_1} \langle B_{\lambda_3} \cdots B_{\lambda_l}, **\rangle \rangle. \end{aligned}$$

If $\lambda_2 \leq \mu_1$, the Hall-Littlewood inner product becomes $B_{\lambda_1} B_{\lambda_3} \cdots B_{\lambda_l}$. Since $B_{\lambda_1} \geq B_{\lambda_2}$, $\lambda^{(1)}$ is contained within the tableaux resulting from the Littlewood-Richardson multiplication rule. If $\lambda_2 > \mu_1$, then $\lambda^{(1)}$ is contained within the tableaux resulting from the Littlewood-Richardson multiplication rule.

Assume the k-th Hall-Littlewood inner product is not zero. Then the $(k+1)$ Littlewood-Richardson multiplication is either

$$\langle B_{\lambda_1} \cdots B_{\lambda_k} s_{\lambda_{k+1} - \mu_1} \langle B_{\lambda_{k+2}} \cdots B_{\lambda_l}, **\rangle \rangle \qquad \text{or}$$
$$\langle B_{\lambda_1} B_{\lambda_2 - i_1} B_{\lambda_3 - i_2} \cdots B_{\lambda_j - i_{j-1}} s_{\lambda_{j+1} - (\mu_1 - k)} \langle B_{\lambda_{j+1}} \cdots B_{\lambda_l}, H_{\mu_2} \cdots H_{\mu_k} **\rangle \rangle.$$

In the first case, since $B_{\lambda_1} \geq B_{\lambda_2} \geq \ldots B_{\lambda_l} > 0$, $\lambda^{(1)}$ is contained within the $(k+1)$-st Littlewood-Richardson tableaux. In the second case, $B_{\lambda_j - i_{j-1}} \geq B_{\lambda_{j+1}}$ for all $j \in 1, k$ by assumption. Thus, $\lambda^{(1)}$ is contained within the $(k+1)$-st Littlewood-Richardson tableaux. This means that

$$\bigcap_{i=1}^{l} i\text{-th Littlewood-Richardson tableaux } = \lambda^{(1)}.$$

Therefore, the Littlewood-Richardson core of a partition λ has shape $\lambda^{(1)}$. ∎

Remark 3.1.6. *It is because of the common tableaux containment resulting from the Littlewood-Richardson tableaux multiplication rule in Theorem 2.4.13 that we introduced the terminology of Littlewood-Richardson cores.*

If we fix the LRC of the partition and begin removing only those boxes not in the $LRC(\lambda)$, we can produce every tableau contained within λ which still contains the $LRC(\lambda)$ using the dominance ordering. For example, every tableau corresponding to a weight five partition which is contained in $(3, 2, 1)$ with $LRC = (2, 1)$ is shown below.

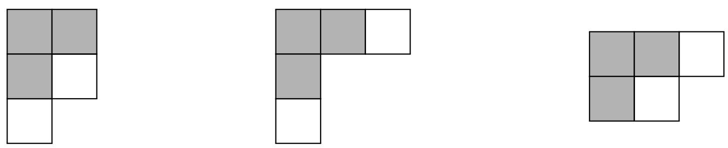


Figure 3.3: Young tableaux contained in $T_{(3,2,1)}$ obtained by removing one cell not contained in the $LRC((3, 2, 1))$

Definition 3.1.7. Given a tableaux T and a partition λ, define the *Littlewood-Richardson tableaux*, LRT, of λ by

$$LRT(\lambda) = \{T | T_{LRC(\lambda)} \subseteq T \subseteq T_\lambda\}.$$

Observe that if $\lambda \vdash n$ has shape (n), then $T = \emptyset$ is an element of $LRT(\lambda)$.

Similar to Young lattices, we can create LRT lattices using the dominance ordering so that a partition shape is always maintained. For example, when $\lambda = (2)$ we get the lattice

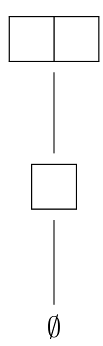


Figure 3.4: LRT lattice for $\lambda = (2)$

Theorem 3.1.8. *Littlewood-Richardson Tableaux are not k-ribbon tableaux for $l(\lambda) > 2$ and $\lambda_2 \nmid \lambda_1$ with $k > 1$.*

Proof. Let $\lambda \vdash n$ with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ and T be the Young tableau of λ. By definition, $|LRC(\lambda)| = (n - \lambda_1)$.

Observe that fixing the upper leftmost cells in T corresponding to $LRC(\lambda)$ means that if $l \geq 3$ there are no cells directly above row l which are not in the Littlewood-Richardson core. As $(n - \lambda_1) \geq 2 \cdot \lambda_l$, the cells in row l cannot be part of a $(n - \lambda_1)$-ribbon. Thus, $LRC(\lambda)$ is not a k-ribbon for $l(\lambda) > 2$.

Assume $l(\lambda) = 2$ and that $\lambda_1 = p$ and $\lambda_2 = q$ with $q \nmid p$. Then, by the division algorithm, $p = mq + r$ and $0 < r < q$. Since $|LRC(\lambda)| = q$, row 1 of T has the rightmost r cells not in a q-ribbon.

Therefore, Littlewood-Richardson Tableaux are not k-ribbon tableaux for $l(\lambda) > 2$ and $\lambda_2 \nmid \lambda_1$ with $k > 1$. ☐

3.2 Littlewood-Richardson Tableaux Lattices

Having defined the Littlewood-Richardson Tableaux in the previous section, we now wish to introduce a standardizing algorithm for recording LRT lattices.

Definition 3.2.1. We define the *Littlewood-Richardson Tableaux lattice algorithm* as follows:

Given a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ of n with Young tableau T

1. Shade the upper leftmost cells in T corresponding to the $LRC(\lambda)$
2. Below T, draw every tableau which has had exactly one cell removed from T which is on the right end of a row and not in $LRC(\lambda)$. Arrange these diagrams such that for $i < j$, the tableau having had a cell removed from row i appears to the left of the tableau having a cell removed from row j for $1 \leq i \leq l - 1$. Thus, every tableau in this row has exactly $(n - 1)$ cells.
3. Draw a line between T and every new tableau which is contained within it.
4. Repeat steps 2) and 3) for each tableau until there are no more non-$LRC(\lambda)$ cells which can be removed.
5. Draw the $LRC(\lambda)$ (Recall if $l(\lambda) = 1$, the $LRC(\lambda) = \emptyset$) and draw lines to it from each tableau in the row immediately above it.

Example 3.2.2. Let $\lambda = (3, 2, 1)$. By Definition 3.2.1, the LRT lattice is shown in Figure 3.5.

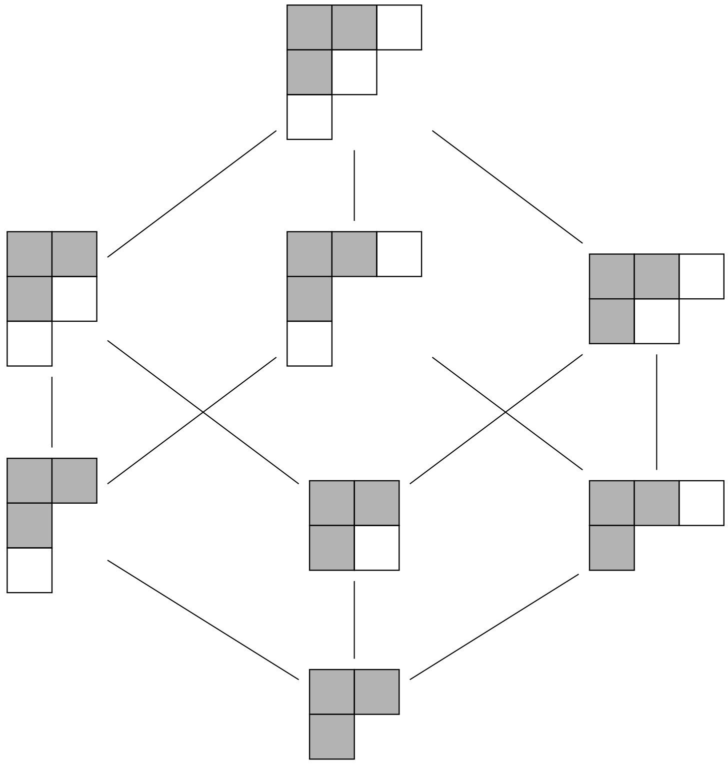


Figure 3.5: LRT lattice for $\lambda = (3, 2, 1)$

Observe that if if we count the number of tableaux appearing in each horizontal level of the LRT lattice and record the results, we see that we have the symmetric and monic sequence of integers $1 - 3 - 3 - 1$. As we will see in the next two examples, fixing non-$LRC(\lambda)$ cells does not, in general, produce a symmetric and monic sequence of integers.

Example 3.2.3. Let $\lambda = (3, 2, 1)$. We know $|LRC(\lambda)| = 3$. Instead of fixing the upper leftmost cells corresponding to the shape of $LRC(\lambda)$, fix λ_1 and shade it as before and proceed with steps 2) through 5) from Definition 3.2.1.

Counting the tableaux appearing in each horizontal level in Figure 3.6 and recording the results produces the integer sequence $1 - 2 - 1 - 1$ which is monic but not symmetric.

Example 3.2.4. Let $\lambda = (3, 2, 1)$. We know $|LRC(\lambda)| = 3$. Instead of fixing the upper leftmost cells corresponding to the shape of $LRC(\lambda)$, fix λ'_1 and shade it as before and proceed with

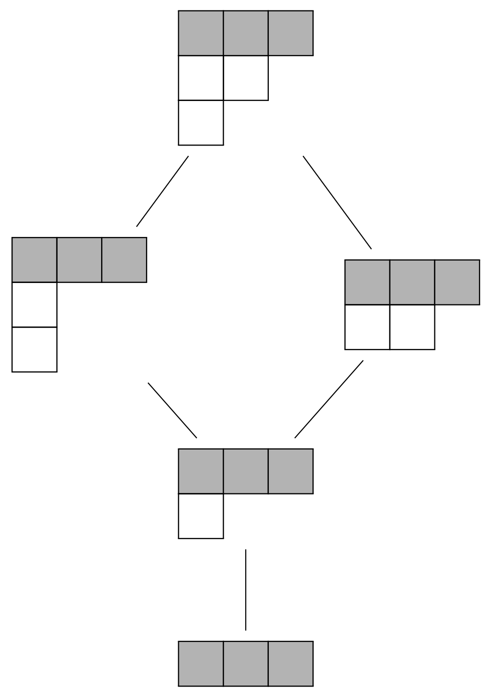


Figure 3.6: Example of fixing λ_1 which does not produce a symmetric lattice

steps 2) through 5) from Definition 3.2.1.

Counting the tableaux appearing in each horizontal level in Figure 3.7 and recording the results produces the integer sequence $1-2-1-1$ which is monic but not symmetric. Observe that we have fixed every set of $|LRC(\lambda)| = 3$ cells in T which have partition shape and only fixing the $LRC(\lambda)$ produced a symmetric and monic integer sequence.

We now define a new polynomial in $\mathbb{Z}[t]$ from the LRT lattice of a partition which is always symmetric and monic and proves that the $LRT(\lambda)$ lattice is also symmetric and monic if you count the number of tableau appearing in each horizontal level.

Definition 3.2.5. Given a partition λ, define the map

$$\phi : LRT(\lambda) \to \mathbb{Z}[t]$$

$$\text{by } T \in LRT(\lambda) \longmapsto t^{|\lambda_T| - |LRC(\lambda)|}$$

where λ_T is the partition corresponding to the tableau T.

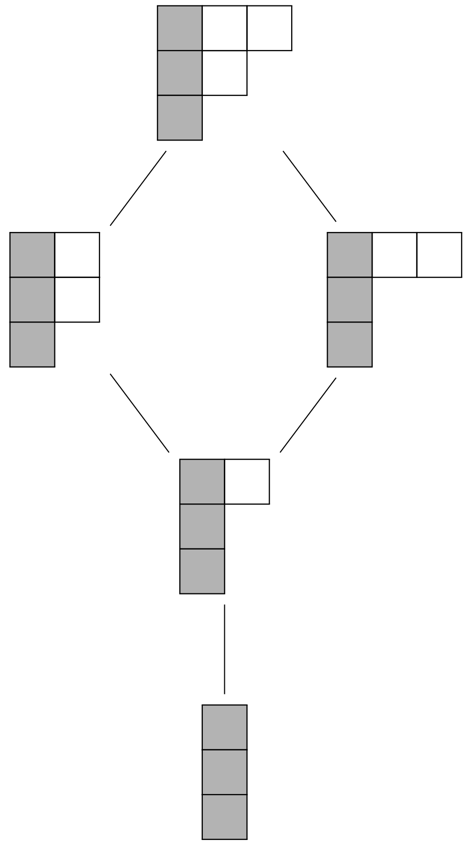


Figure 3.7: Example of fixing λ'_1 which does not produce a symmetric lattice

Define the *T-polynomial*, $T(\lambda)$, of λ by

$$T(\lambda) = \sum_{T \in LRT(\lambda)} t^{|\lambda_T| - |LRC(\lambda)|}.$$

Equivalently,

$$\begin{aligned} T(\lambda) &= \sum_{T \in LRT(\lambda)} t^{|\lambda_T| - |LRC(\lambda)|} \\ &= \sum_{m=0}^{|\lambda| - |LRC(\lambda)|} C_m t^m \\ &= \sum_{m=0}^{\lambda_1} C_m t^m \end{aligned}$$

where C_m is the number of tableaux of weight $|LRC(\lambda)| + m$ where m is the number of boxes added to the $LRC(\lambda)$.

Example 3.2.6. Let $\lambda = (3, 2, 1)$. The T-polynomial for λ is shown in Figure 3.8.

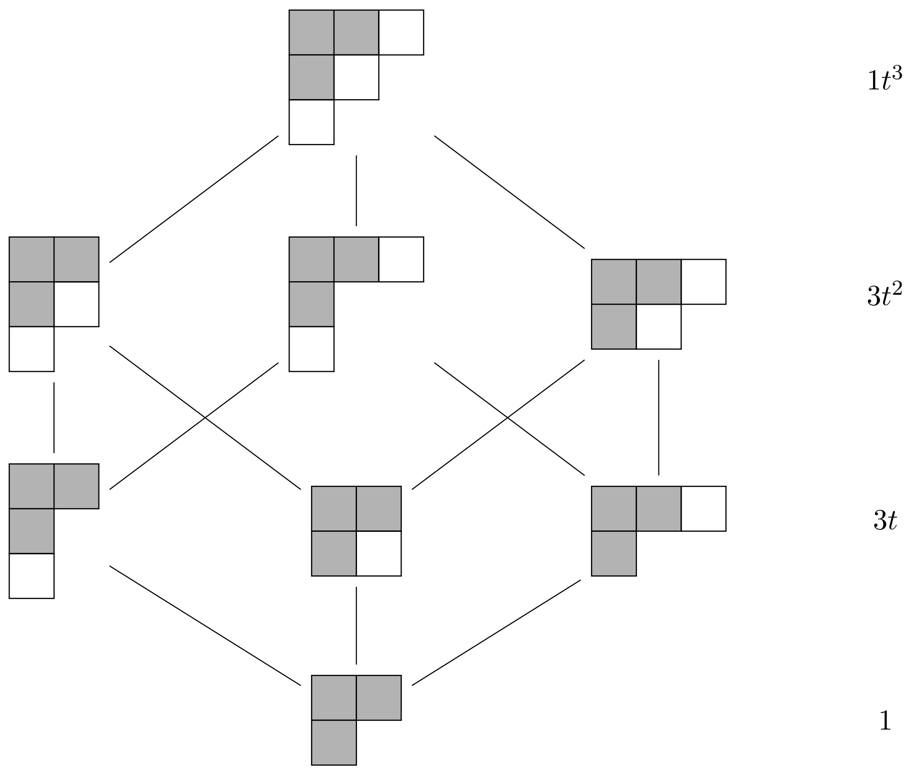


Figure 3.8: T-polynomial for $\lambda = (3, 2, 1)$

Theorem 3.2.7. *For any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ of n, $T(\lambda)$ is symmetric and monic.*

Proof. Since there is exactly one unique Young tableau for each partition λ, there is also exactly one unique Young tableau for $LRC(\lambda)$. This means the first and last terms of $T(\lambda)$ are always 1 and, therefore, $T(\lambda)$ is monic.

We proceed inductively to proof symmetry.

Case 1: Suppose $l(\lambda) = 1$. Then the Young tableau for λ is a horizontal strip and $LRC(\lambda) = \emptyset$. This means that every cell in the Young tableau needs to be removed in order to create the $LRT(\lambda)$ lattice. However, since we must maintain partition shape after each removal, only the rightmost cell may be removed at each step. Hence, there will be exactly $(n + 1)$ levels in the $LRT(\lambda)$ lattice each containing exactly one $LRT(\lambda)$. Thus, $T(\lambda) = t^n + t^{n-1} + \cdots + t^1 + 1$ which is symmetric.

Case 2: Suppose $l(\lambda) = 2$ with $|LRC(\lambda)| = \lambda_2 = m$.

We break this into two cases:

- Suppose that $\lambda = (m, m)$. Then the only cells from the Young tableau of λ which may be removed are in row 2. Thus, we have the same situation as Case 1) involving m removable cells. Hence, $T(\lambda) = t^m + t^{m-1} + \cdots + t^1 + 1$ where $m = \frac{n}{2}$. Therefore, $T((m, m))$ is symmetric.

- Suppose that $\lambda = (m+1, m)$. By definition of the $LRT(\lambda)$, there is only one removable cell in row 1 of the Young tableau of λ, specifically the rightmost cell. Label this cell a.

  If a is the first cell removed from the Young tableau of λ, then beginning one level down and to the left we have the same $LRT(\lambda)$ lattice as when $\lambda = (m, m)$. Moreover, removing a at any step prior to the last step results in a Littlewood-Richardson tableau already contained in $LRT((m, m))$. Thus, we need consider only those $LRT(\lambda)$ where a is fixed.

  However, fixing a leaves the only removable tableau of λ to be precisely those of $\lambda = (m, m)$. Thus fixing a results in the same T-polynomial as when $\lambda = (m, m)$ except it is shifted one degree higher. There are two further cases to consider:

  i. $T((m, m))$ has odd length. Then adding the coefficients of the T-polynomial gives

$$\begin{array}{cccccccl} & & 1, & & \ldots, & 1 & & \text{odd number } m+1 \\ + & 1, & \ldots, & 1 & & & & \text{odd number } m+1 \\ \hline & 1, & 2, & & \ldots, & 2, & 1 & \text{even number } m+2 \end{array}$$

  ii. $T((m, m))$ has even length. Then adding the coefficients of the T-polynomial gives

$$\begin{array}{cccccccl} & & 1, & & \ldots, & 1 & & \text{even number } m+1 \\ + & 1, & \ldots, & 1 & & & & \text{even number } m+1 \\ \hline & 1, & 2, & & \ldots, & 2, & 1 & \text{odd number } m+2 \end{array}$$

  Since the sum of two symmetric polynomials having the same sequence of coefficients and differing by exactly one degree is symmetric, $T((m+1, m))$ is symmetric.

  Assume that $T(\lambda)$ is symmetric for $\lambda = (m+i, m)$ where $i > 1$. Consider $\bar{\lambda} = (m+i+1, m)$. As before, label the last cell in the first row of the Young tableau of $\bar{\lambda}$ with an a. If a is the first cell removed from the Young tableau of $\hat{\lambda}$, then beginning one level down and to the left we have the same $LRT(\bar{\lambda})$ lattice as when $\lambda = (m+1, m)$. Moreover, removing a at any step prior to the last step results in a Littlewood-Richardson tableau already contained in $LRT(\lambda)$. Thus, we need consider only those $LRT(\bar{\lambda})$ where a is fixed.

However, fixing a leaves the only removable cells of $\bar{\lambda}$ to be precisely those of λ. Thus fixing a results in the same T-polynomial as when $\lambda = (m+1, m)$ except it is shifted one degree higher.

Since $T(\lambda)$ is symmetric for $\lambda = (m+i, m)$, by assumption, and $T(\bar{\lambda})$ is obtained by adding the coefficients of $T(\lambda)$ shifted by one degree, $T(\bar{\lambda})$ is also symmetric.

Therefore, for every partition λ of length two, $T(\lambda)$ is symmetric.

Case 3: Assume $l(\lambda) \geq 2$ with $T(\lambda)$ being symmetric. We need only consider two cases for when add a new row to the top of the Young tableau of λ, i.e. when we create a new partition $\bar{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_l)$ from λ.

- Suppose $\lambda_0 = \lambda_1$. Since no additional cells can be removed from the Young tableau of $\bar{\lambda}$ which are not contained in the Young tableau of λ, $T(\bar{\lambda}) = T(\lambda)$. Since $T(\lambda)$ is symmetric, by assumption, $T(\bar{\lambda})$ is also symmetric.

- Suppose $\lambda_0 = (\lambda_1 + 1)$. Label the last cell of the top row of the Young tableau of $\bar{\lambda}$ a as before. Hence, the T-polynomial of $\bar{\lambda}$ is the shifted sum of two symmetric polynomials differing by one degree. Then since $T(\lambda)$ is symmetric, $T(\bar{\lambda})$ is also symmetric by above.

  Assume that $T(\bar{\lambda})$ is symmetric for $\bar{\lambda} = (\lambda_0 + i, \lambda_1, \ldots, \lambda_l)$ where $i > 1$. Consider $\hat{\lambda} = (\lambda_0 + i + 1, \lambda_1, \ldots, \lambda_l)$. As before, label the last cell in the first row of the Young tableau of $\hat{\lambda}$ with an a. If a is the first cell removed from the Young tableau of $\hat{\lambda}$, then beginning one level down and to the left we have the same $LRT(\hat{\lambda})$ lattice as when $\bar{\lambda} = (\lambda_0 + i, \lambda_1, \ldots, \lambda_l)$. Moreover, removing a at any step prior to the last step results in a Littlewood-Richardson tableau already contained in $LRT(\bar{\lambda})$. Thus, we need consider only those $LRT(\hat{\lambda})$ where a is fixed.

  However, fixing a leaves the only removable cells of $\hat{\lambda}$ to be precisely those of $\bar{\lambda}$. Thus fixing a results in the same T-polynomial as when $\bar{\lambda} = (\lambda_0 + i, \lambda_1, \ldots, \lambda_l)$ except it is shifted one degree higher.

  Since $T(\bar{\lambda})$ is symmetric for $\bar{\lambda} = (\lambda_0 + i, \lambda_1, \ldots, \lambda_l)$, by assumption, and $T(\bar{\lambda})$ is obtained by adding the coefficients of $T(\lambda_0, \lambda_1, \ldots, \lambda_l))$ shifted by one degree, $T(\bar{\lambda})$ is also symmetric.

Therefore, for every partition λ of n, $T(\lambda)$ is symmetric. □

Definition 3.2.8. Given a partition λ, define the *i-th shifted T-polynomial of* λ, $T_i(\lambda)$, by adding i to the exponent of $T(\lambda)$ where $1 = t^0$.

Corollary 3.2.9. *Given a partition* λ*,*

$$T(\lambda) \quad = \quad \sum_{i=0}^{\lambda_1-\lambda_2} T_i(LRC(\lambda))$$

where $T_i(LRC(\lambda))$ *is the* i*-th shifted* T*-polynomial of the Littlewood-Richardson core of* λ*.*

Proof. This result follows immediately from Theorem 3.2.7. □

Graphically, we can interpret Corollary 3.2.9 as the $LRT(\lambda)$ lattice is $(\lambda_1-\lambda_2+1)$ diagonally shifted bands of the $LRT(LRC(\lambda))$ lattice. In the next two examples we denote those tableaux contained in a band with a black line between them and the connections between bands of tableaux with red lines.

Remark 3.2.10. *For a partition* λ*, if* $\lambda_1 = \lambda_2$*, then* $T(\lambda) = T(LRC(\lambda))$*.*

Example 3.2.11. Consider the partitions $\lambda = (2,1)$ and $\bar{\lambda} = (3,1)$. In both cases, the $LRC(\lambda) = LRC(\bar{\lambda}) = (1)$. By Theorem 3.2.7, $T(LRC(\lambda)) = T(LRC(\bar{\lambda})) = (1+t)$ and the LRT lattices of $LRC(\lambda)$ and $LRC(\bar{\lambda})$ consist of two levels each containing one tableau.

Thus, for $\lambda = (2,1)$, the Littlewood-Richardson Tableaux lattice consists of two bands each consisting of two levels each containing one tableau, i.e. one band for the $LRC(\lambda)$ and an additional band because $\lambda_1 - \lambda_2 = 1$. The $LRT((2,1))$ lattice is shown in Figure 3.9.

Similarly, for $\bar{\lambda} = (3,1)$, the Littlewood-Richardson Tableaux lattice consists of three bands each consisting of two levels each containing one tableau, i.e. one band for the $LRC(\bar{\lambda})$ and two additional bands because $\bar{\lambda}_1 - \bar{\lambda}_2 = 2$. The $LRT((3,1))$ lattice is shown in Figure 3.10.

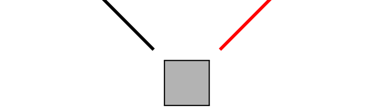


Figure 3.9: LRT lattice for $\lambda = (2,1)$

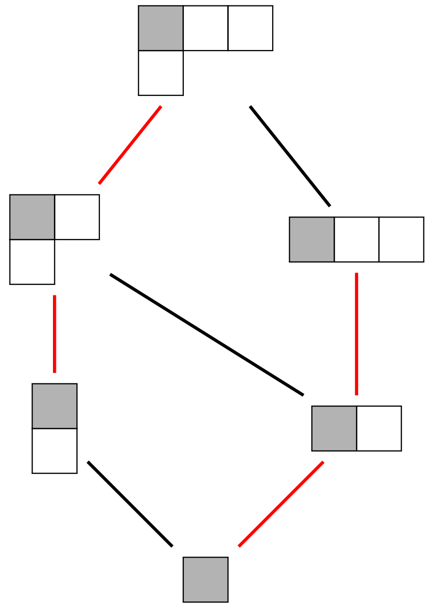


Figure 3.10: LRT lattice for $\lambda = (3, 1)$

Though it is well defined, our next example proves that the function $\phi : LRT(\lambda) \to \mathbb{Z}[t]$, where $T \in LRT(\lambda) \longmapsto t^{|\lambda_T| - |LRC(\lambda)|}$ and λ_T is the partition corresponding to the tableau T, does not have an inverse.

Example 3.2.12. Let λ be a partition and suppose $T(\lambda) = t^4 + 3t^3 + 4t^2 + 3t + 1$. As before, by considering coefficients only we observe below that three shifted sums of $1 - 2 - 1$ and two shifted sums of $1 - 2 - 2 - 1$ equals $1 - 3 - 4 - 3 - 1$.

			1	2	1
+		1	2	1	
+	1	2	1		
	1	3	4	3	1

		1	2	2	1
+	1	2	2	1	
	1	3	4	3	1

Consider the partition $\lambda = (4, 2, 1)$. The $LRT((4, 2, 1))$ lattice is shown in Figure 3.11.

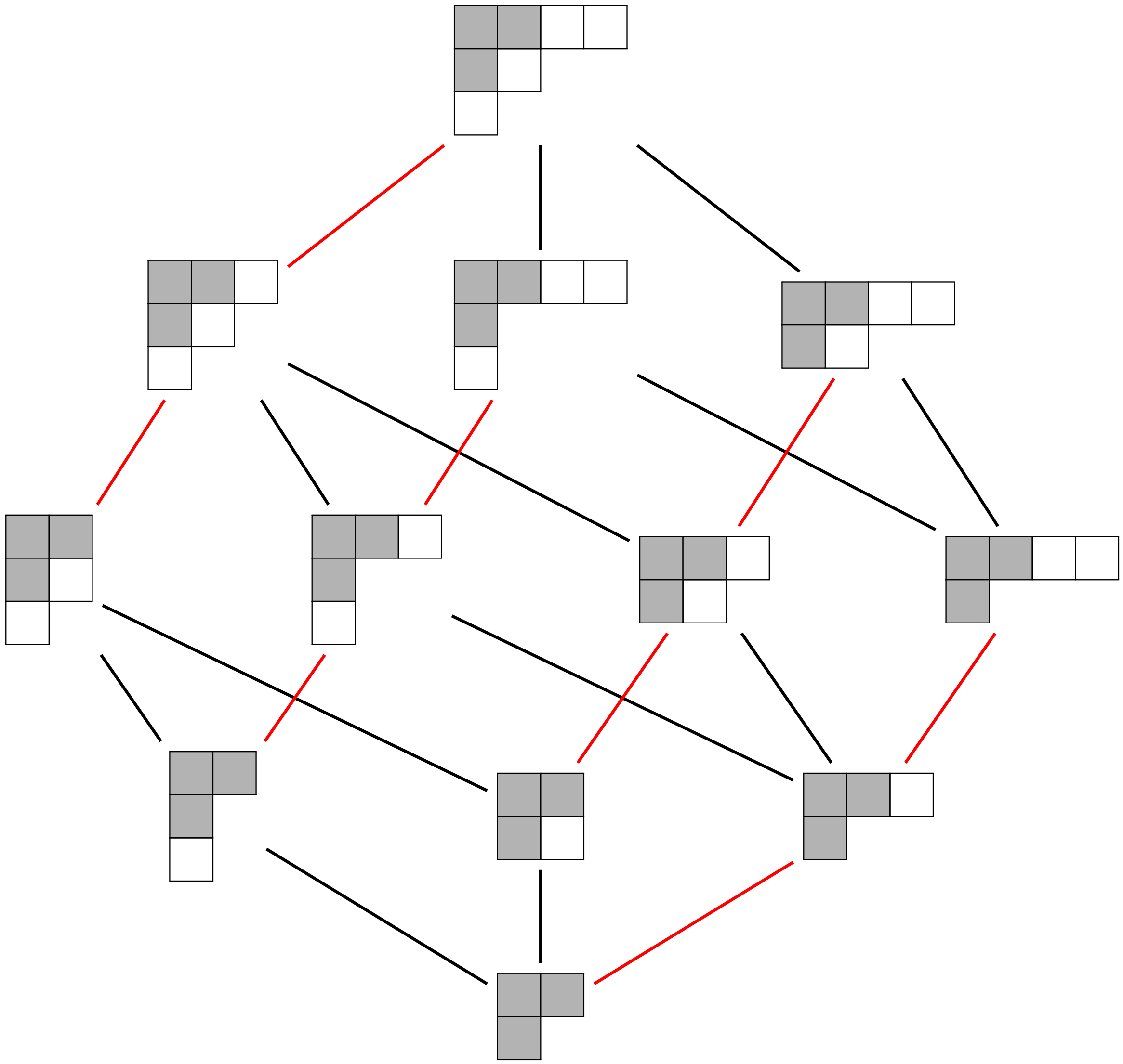


Figure 3.11: LRT lattice for $\lambda = (4, 2, 1)$

Hence, $T((4,2,1)) = t^4 + 3t^3 + 4t^2 + 3t + 1$ and is the result of three shifted sums of $t^2 + t + 1$. Now consider the partition $\bar{\lambda} = (4, 3, 1)$. The $LRT((4,3,1))$ lattice is shown in Figure 3.12.

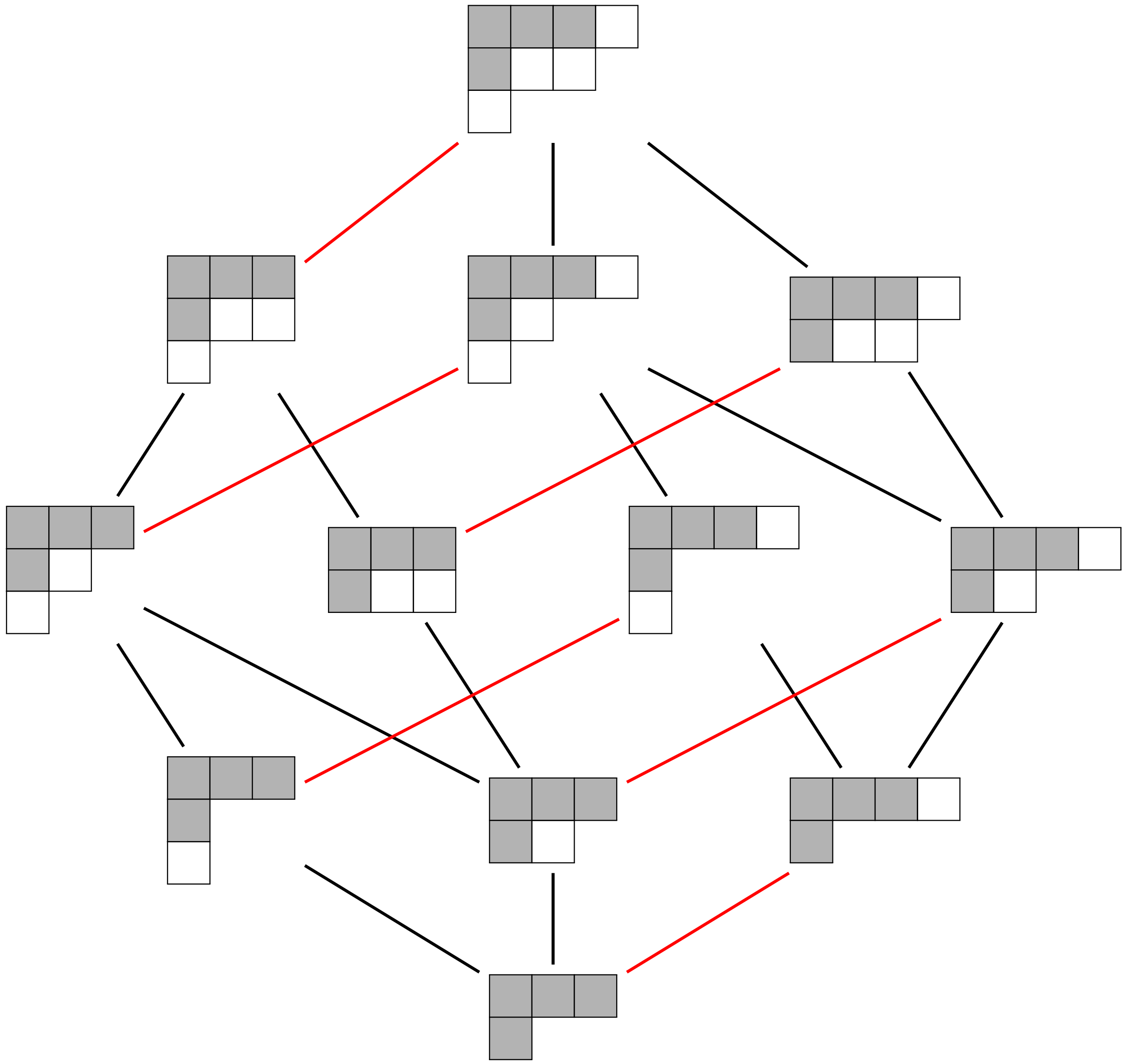


Figure 3.12: LRT lattice for $\lambda = (4, 3, 1)$

Hence, $T((4,3,1)) = t^4+3t^3+4t^2+3t+1$ and is the result of two shifted sums of t^3+2t^2+2t+1. Thus $\phi^{-1}(t^4 + 3t^3 + 4t^2 + 3t + 1)$ is not unique. Therefore, ϕ does not have an inverse.

In addition to creating symmetric lattices, LRT are combinatorially significant in other ways.

Theorem 3.2.13. *There exists a unique k-letter word W consisting of the letters U and D which encodes the T-polynomial of a partition λ where U represents adding a row of $|\lambda_1|$ cells above row one of the previous Young tableau and D represents extending row one of the previous Young tableau by one cell, beginning with $T = \emptyset$, and*

$$k = \Big(\sum_{i=1}^{|\lambda|-1} \lambda_i - \lambda_{i+1}\Big) + \lambda_l.$$

Remark 3.2.14. *The letters U and D were chosen to mirror the arrangement of tableaux within a Young lattice and the diagonal bands of lattices seen in Section 3.4.*

Proof. In each step, we record the letters of W from right to left.

We've already seen that if $\lambda_1 = \lambda_2$, then $T(\lambda) = T(LRC(\lambda))$. In this case, record a U and remove λ_1 from λ.

If $\lambda_1 \neq \lambda_2$, then $T(\lambda) = \bar{T}^{(1)}(\bar{\lambda}^{(1)})$ where $\bar{\lambda}^{(1)} = (\lambda_1 - 1, \lambda_2, \ldots, \lambda_l)$ and $\bar{T}^{(1)}(\bar{\lambda}^{(1)})$ is its T-polynomial. In this case, record a D and remove the rightmost cell from row one of the Young tableau of λ. Repeat until $\lambda_1 = \lambda_2$.

Remove λ_1.

Repeat this process for every entry λ_i of λ for $1 \leq i \leq l - 1$.

Suppose that W_1 and W_2 are words encoding $T(\lambda)$ such that $W_1 \neq W_2$. Let i be the first entry where W_1 differs from W_2. Then through the first $(i-1)$ letters, W_1 and W_2 form the same subpartition, $\bar{\lambda}$, of λ with $|\bar{\lambda}| = m$.

Consider $\bar{\lambda_1}$. When a U is encountered, the next partition formed is $(\bar{\lambda_1}, \bar{\lambda_1}, \bar{\lambda_2}, \ldots, \bar{\lambda_m})$. When a D is encountered, the next partition formed is $(\bar{\lambda_1} + 1, \bar{\lambda_2}, \ldots, \bar{\lambda_m})$.

As this process only adds a new row or extends a row by one cell in the Young tableau at each step, there exists a subpartition formed by W_1 which is different from a subpartition formed by W_2.

This contradicts the fact that every partition has a unique sequence of integers which comprise its entries. Therefore $W_1 = W_2$. □

Reading W then from left to right provides the unique up and diagonal path to λ from $\emptyset$.

Example 3.2.15. Beginning at $T = \emptyset$ and creating its unique Littlewood-Richardson core, the up and diagonal path to $\lambda = (5, 4, 3^2, 2)$ is given by the code $DDUDUUDUD$ and is shown in Figure 3.13.

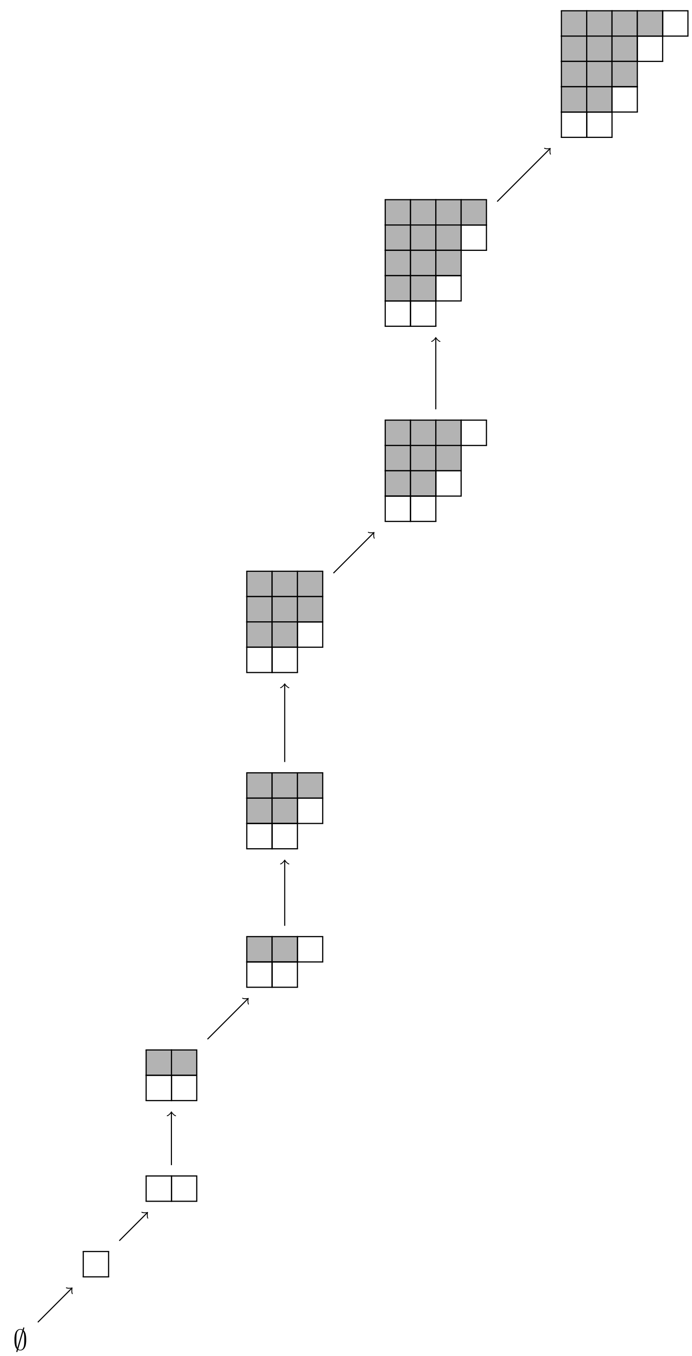


Figure 3.13: Littlewood-Richardson up and diagonal paths from $\emptyset$ to $\lambda = (5, 4, 3^2, 2)$

Corollary 3.2.16. *For a partition $\lambda \neq \emptyset$, there exist exactly $C(k,r)$ unique partitions reachable from λ with a k-length word W with r letter Ds. If $\lambda = \emptyset$, there exist exactly $C(k-1, r-1)$ unique partitions reachable from λ with a k-length word W with r letter Ds.*

Proof. For $\lambda \neq \emptyset$, since the order of the Ds doesn't matter and there are $(k-r)$ Us in W, there exist $\binom{k}{r,k-r} = C(k,r)$ words of length k with r Ds. Therefore, there are $C(k,r)$ unique partitions reachable from $\lambda \neq \emptyset$ with a k-length word W with r letter Ds.

For $\lambda = \emptyset$, the first letter of W must always be a D. This is because a U represents adding a new row above λ of length λ_1. However, if $\lambda = \emptyset$, $\lambda_i = 0$ for $i \geq 1$ and adding a new row does not change λ. Therefore, since the first letter of W is fixed as a D, we have $(r-1)$ remaining Ds to place in the remaining $(k-1)$ spaces of W. Hence, there exist $\binom{k-1}{r-1,(k-1)-(r-1)} = \binom{k-1}{r-1} = C(k-1, r-1)$ words of length k with r Ds. Therefore, there are $C(k-1, r-1)$ unique partitions reachable from $\lambda = \emptyset$ with a k-length word W with r letter Ds. □

3.3 T-polynomials and t-analogues

In 1997, Lascoux, Leclerc, and Thibon [10] introduced a family symmetric functions called LLT polynomials which are the product of q-Analogues of the Schur symmetric functions arising from the rows of a semi-standard Young tableau for a partition. Because Haglund, Haiman, Loehr, [3], proved the Macdonald polynomials, the symmetric functions which generalize the Hall-Littlewood symmetric functions and thus the Schur symmetric functions, can be expanded in terms of them, the LLT polynomials are of particular interest. This connection was further enhanced in 2015, when Grojnowski and Haiman [2] proved the positivity conjecture for the LLT and Macdonald polynomials.

In this section we prove that the T-polynomial of a partition is the product of t-analogues, an introduced statistic closely related to q-analogues and so-named to match the notation of T-polynomials.

Though they appear in many contexts, see [3], [10], [17], q-analogues frequently arise in combinatorics. For this reason, we use a standard combinatorial definition for q-analogues as defined in [16].

Definition 3.3.1. Let $n \in \mathbb{N}$. Define the *q-analogue of $n!$* to be the polynomial $1(1+q)(1+q+q^2)\cdots(1+q+q^2+\cdots+q^{n-1})$ and denote it $(\mathbf{n}!)$. Moreover, we let $(\mathbf{n})$ denote the *q-analogue of n* where

$$(\mathbf{n}) = 1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$$

so that

$$(\mathbf{n})! = (\mathbf{1})(\mathbf{2})\cdots(\mathbf{n}).$$

Using Stanley's definition of a q-analogue is done for two reasons. First, specifying the value of q to be 1 in the definition of a q-analogue returns the original integer n. Second, while q-analogues overwhelmingly require more than just the previous property in their definitions, there is no universal agreement about what "more" is required.

Definition 3.3.2. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ be a partition of n. For each row i of the Young tableau $\lambda/LRC(\lambda)$, for $1 \leq i \leq l$, number the cells $(1, 2, \ldots, \lambda_i - \lambda_{i+1})$ from right to left where $\lambda_{i+1} = 0$ if $i + 1 > l$. This is equivalent to numbering the order in which the cells of each row will be removed in the process of moving between the Young tableaux of λ and $LRC(\lambda)$.

For each row numbering, construct a $(\lambda_i - \lambda_{i+1})$-th degree polynomial of the form

$$\sum_{j=0}^{\lambda_1 - \lambda_{i+1}} t^j.$$

Clearly, this assignment yields the *t-analogue*

$$[\lambda_1 - \lambda_{i+1} + 1]_t.$$

Theorem 3.3.3. $T(\lambda) \cong \prod_{i=1}^{l} [\lambda_1 - \lambda_{i+1} + 1]_t.$

Proof. We proceed inductively to prove symmetry.

Remark 3.3.4. *As a notational convenience, we reverse the ordering of the terms of the t-Analogues so that the resulting multiplication matches the ordering of the terms of the T-polynomials.*

Case 1: Assume $l = 1$. Then $LRC(\lambda) = \emptyset$. Thus, $T(\lambda) \cong [n - 0 + 1] = t^n + t^{n-1} + \cdots + t + 1$. Then, for length one partitions, $T(\lambda)$ is isomorphic to the product of t-analogues.
Case 2: Assume $l = 2$. We break this into two cases:

- Assume $\lambda_1 = \lambda_2$. Then $[\lambda_1 - \lambda_2 + 1]_t = t^0 = 1$. Thus,

$$[\lambda_1 - \lambda_2 + 1]_t[\lambda_2 - 0 + 1]_t = 1[\lambda_2 + 1]_t = t^{\frac{n}{2}} + t^{\frac{n}{2}-1} + \cdots + t + 1.$$

- Assume $\lambda_1 \neq \lambda_2$. Let $|\lambda_1 - \lambda_2| = p$. Then

$$[\lambda_1 - \lambda_2 + 1]_t[\lambda_2 - 0 + 1]_t = 1[p+1]_t[\lambda_2 + 1]_t = (t^p + t^{p-1} + \cdots + t + 1)(t^{\lambda_2} + t^{\lambda_2 - 1} + \cdots + t + 1).$$

  If we record this product in the form

$$
\begin{array}{ccccccccccccc}
t^{p+\lambda_2} & + & t^{p+(\lambda_2-1)} & + & \cdots & & + & t^{p+1} & + & t^p & & & \\
 & + & t^{(p-1)+\lambda_2} & + & t^{(p-1)+(\lambda_2-1)} & & + & \cdots & + & t^p & + & t^{p-1} & \\
 & & \ddots & + & \ddots & & + & \cdots & + & \ddots & + & \ddots & \\
 & & & + & t^{\lambda_2} & & + & t^{\lambda_2-1} & + & \cdots & + & t & + \; 1 \\
\hline
\end{array}
$$

we see that the result is exactly $(p+1)$ shifted sums of $[\lambda_2+1]_t$. Thus, since $[\lambda_2+1]_t \cong t^{\lambda_2}+t^{\lambda_2-1}+\cdots+t+1$,

$$[\lambda_1-\lambda_2+1]_t[\lambda_2+1]_t \cong T(\lambda).$$

Then, for length two partitions, $T(\lambda)$ is isomorphic to the product of t-analogues.

Assume for all $l \leq r$, $T(\lambda) \cong \prod_{i=1}^{l}[\lambda_i-\lambda_{i+1}+1]_t$.

Case 3: Consider $l = r+1$. We break this into two cases:

- Assume $\lambda_1 = \lambda_2$. Then

$$[\lambda_1-\lambda_2+1]_t = [0+1]_t = [1]_t = t^0 = 1.$$

  Thus,

$$\prod_{i=1}^{r+1}[\lambda_i-\lambda_{i+1}+1]_t = \prod_{i=1}^{r}[\lambda_i-\lambda_{i+1}+1]_t \cong T(\lambda/\lambda_1)$$

  by assumption.

- Assume $\lambda_1 \neq \lambda_2$. Let $|\lambda_1-\lambda_2| = p$. Then

$$
\begin{aligned}
[\lambda_1-\lambda_2+1]_t[\lambda_2-\lambda_3+1]_t\cdots[\lambda_{r+1}-0+1]_t &= [p+1]_t\prod_{i=2}^{r+1}[\lambda_i-\lambda_{i+1}]_t \\
&= [p+1]_t\prod_{i=1}^{r}[\lambda_i-\lambda_{i+1}+1]_t
\end{aligned}
$$

  after reindexing. Thus, by assumption, we have $(p+1)$ shifted sums of $\prod_{i=1}^{r}[\lambda_i-\lambda_{i+1}+1]_t$ which is isomorphic to $(p+1)$ shifted sums of $T(\lambda/\lambda_1)$ which equals $T(\lambda)$. Then, for length $(p+1)$ partitions, $T(\lambda)$ is isomorphic to the product of t-analogues.

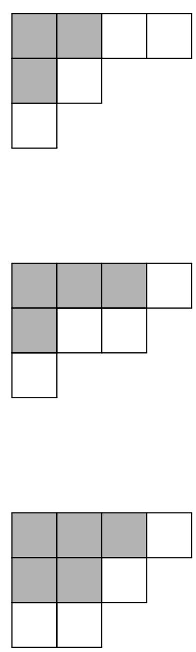
Therefore, for all partitions λ with $l(\lambda) \geq 1$, the $T(\lambda)$ is isomorphic to the product of t-analogues. □

This result allows us to determine the symmetry of the LRT lattice without actually having to construct the LRT lattice for a partition. This result also explains why the map

$\phi : LRT(\lambda) \to \mathbb{Z}[t]$, where $T \in LRT(\lambda) \longmapsto t^{|\lambda_T|-|LRC(\lambda)|}$ and λ_T is the partition corresponding to the tableau T, does not have an inverse. Specifically, since a T-polynomial is the product of polynomial factors, any permuting of those factors produces the same T-polynomial. However, as each polynomial factor results from the difference between consecutive rows of a Littlewood-Richardson tableau, any permutation of the polynomial factors of the T-polynomial must result in a distinct Littlewood-Richardson tableau.

In the next example, up to the insertion of rows i having the same length as row $(i+1)$ into a Littlewood-Richardson tableau, we provide every unique partition having λ such that $T(\lambda) = t^4 + 3t^3 + 4t^2 + 3t + 1$.

Example 3.3.5. Consider the partitions $\lambda_1 = (4, 2, 1)$, $\bar{\lambda} = (4, 3, 1)$, and $\hat{\lambda} = (4, 3, 2)$. By constructing the t-analogues for each row of the partitions in Figure 3.14 we verify the result of Example 3.2.12 that $T((4, 2, 1)) = T((4, 3, 1)) = t^4 + 3t^3 + 4t^2 + 3t + 1$ and prove that $T((4, 3, 2))$ is also $t^4 + 3t^3 + 4t^2 + 3t + 1$.



$$(t + 1)^2(t^2 + t + 1) = t^4 + 3t^3 + 4t^2 + 3t + 1$$

Figure 3.14: Products of t-analogues for $\lambda = (4, 2, 1)$, $\bar{\lambda} = (4, 3, 1)$, and $\hat{\lambda} = (4, 3, 2)$.

3.4 Shifted Littlewood-Richardson Tableaux and Lattices

To highlight additional LRT lattice symmetries and combinatorial identities obscured by the shapes of the varying LRT present within the lattice, we introduce the following tableaux normalizing algorithm.

Definition 3.4.1. For a given partition λ with LRT lattice, we define the *Shifted Littlewood-Richardson Tableaux, SLRT, lattice algorithm* as follows:

1. Recall that the $LRT(\lambda)$ lattice consists of $(\lambda_1 - \lambda_2 + 1)$ diagonally shifted bands of the LRT lattice for $LRC(\lambda)$. Within each band, arrange the tableaux so that the first entry of level j is directly below the last entry of level i for all $i < j$, i.e. within each band draw the tableaux such that the first tableau of a partition of weight k is directly below the last tableau of a partition of weight $(k+1)$. Observe this does not change the ordering or tableaux containment of the $LRT(\lambda)$ lattice.

2. Replace each tableau with $\boxed{\bullet}$.

In the next example, we continue to denote those tableaux contained in a band with a black line between them and the connections between bands of tableaux with red lines. The $SLRT$ lattice has also been drawn so that the $\boxed{\bullet}$ are evenly spaced vertically and horizontally, i.e. the tableau of the partition of greatest weight within each band begins a new column of tableaux.

Example 3.4.2. Consider the partitions $\lambda = (4, 2, 1)$ and $\bar{\lambda} = (4, 3, 1)$. Using Definition 3.4.1, the $SLRT$ lattice of $\lambda = (4, 2, 1)$ is shown in Figure 3.15.

Counting the $\boxed{\bullet}$ appearing in each column in Figure 3.15 and recording the results produces the integer sequence $2-4-4-2$ which is symmetric but not monic.

For $\bar{\lambda} = (4, 3, 1)$, the $SLRT$ lattice is shown in Figure 3.16.

Counting the $\boxed{\bullet}$ appearing in each column in Figure 3.16 and recording the results produces the integer sequence $2-4-4-2$ which is again symmetric but not monic.

We make one final modification to the $SLRT(\lambda)$ lattice.

Definition 3.4.3. Define the *Condensed SLRT*(λ)*, CSLRT*(λ)*, lattice* in the following way:

- Remove all horizontal and vertical spacing between $\boxed{\bullet}$ in the lattice.
- Denote the bands of tableaux with a solid black outline, i.e form polygons.

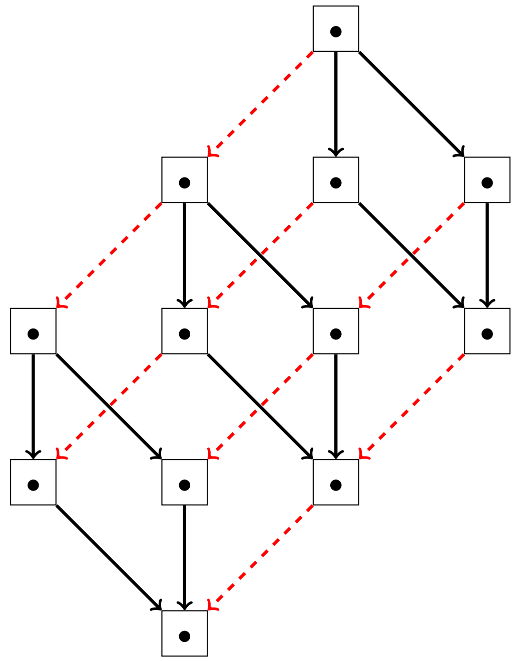


Figure 3.15: $SLRT$ lattice for $\lambda = (4, 2, 1)$

Remark 3.4.4. *Observe that, under Definition 3.4.3, a tableau $T_{\bar{\lambda}}$ is contained within the tableau T_λ if $T_{\bar{\lambda}}$ appears in a strictly lower level of a shared polygon or if the upper right corner of $T_{\bar{\lambda}}$ is the same as the lower left corner of T_λ.*

Remark 3.4.5. *Definition 3.4.3 records bands of tableaux as ribbons and thus the $CSLRT(\lambda)$ is the shifted sum of ribbons by Theorem 3.2.7 and Corollary 3.2.9.*

Example 3.4.6. Consider the partitions $\lambda = (4, 2, 1)$ and $\bar{\lambda} = (4, 3, 1)$. Using Definition 3.4.3, the condensed $SLRT$ lattice of $\lambda = (4, 2, 1)$ is shown in Figure 3.17.

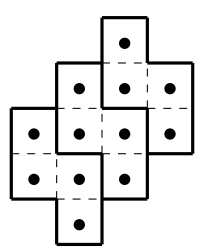


Figure 3.17: Condensed $SLRT$ lattice for $\lambda = (4, 2, 1)$

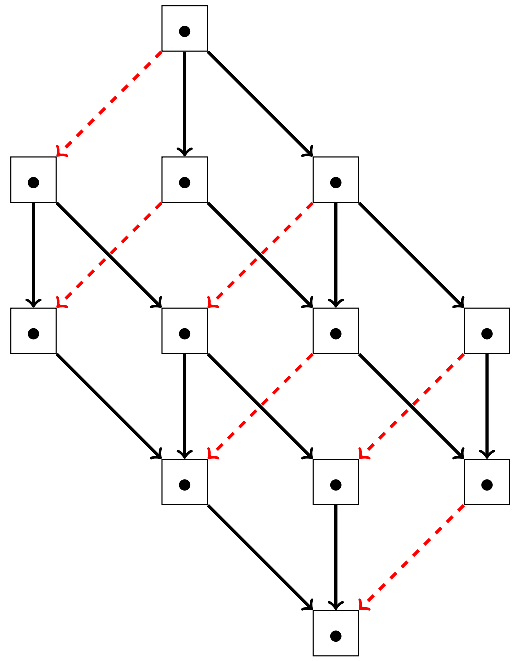


Figure 3.16: $SLRT$ lattice for $\lambda = (4, 3, 1)$

For $\bar{\lambda} = (4, 3, 1)$, the condensed $SLRT$ lattice is shown in Figure 3.18.

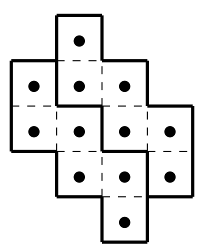


Figure 3.18: Condensed $SLRT$ lattice for $\lambda = (4, 3, 1)$

We now define a new polynomial in $\mathbb{Z}[t]$ from the $CSLRT$ lattice of a partition which is always symmetric and proves that the $CSLRT(\lambda)$ lattice is also symmetric if you count the number of tableau appearing in each vertical column.

Definition 3.4.7. Given a partition λ, define the map

$$\theta : CSLRT(\lambda) \to \mathbb{Z}[t]$$

$$\text{by } \boxed{\bullet} \in CSLRT(\lambda) \longmapsto t^k$$

where k is the k-th vertical column of $CSLRT(\lambda)$ from the left and

$$1 \leq k \leq \Big(\sum_{k=0}^{\lambda_1} \Big(C_m - 1 \Big) + (\lambda_1 - \lambda_2 + 1) \Big)$$

where C_m is the number of tableaux of weight $|LRC(\lambda)| + m$ where m is the number of boxes added to the $LRC(\lambda)$.

Define the *T'-polynomial*, $T'(\lambda)$, of λ by

$$T'(\lambda) = \sum_{k=1}^{\lambda_1} D_k t^k$$

where D_k is the number of tableaux in the k-th vertical column $CSLRT(\lambda)$ lattice.

Example 3.4.8. Consider the partition $\lambda = (4, 2, 1)$. Using Definition 3.4.7, the T'-polynomial of $\lambda = (4, 2, 1)$ is shown below the condensed $SLRT$ lattice in Figure 3.19. To highlight the advantage of drawing the LRT-tableaux in condensed shifted fashion, note that reading across the horizontal levels of the $CSLRT$ lattice preserves the T-polynomial of λ.

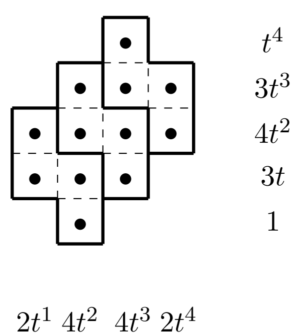


Figure 3.19: T and T'-polynomials for $\lambda = (4, 2, 1)$

Theorem 3.4.9. *The map $\theta : SLRT \to \mathbb{Z}[t]$ does not have an inverse.*

Proof. This proof is similar to the proof that the T-polynomial map does not have an inverse.

We saw in Example 3.4.8 that $T'(4,2,1) = 2t^1 + 4t^2 + 4t^3 + 2t^4$ and observe from Figure 3.18, the $T'(4,3,1) = 2t^1 + 4t^2 + 4t^3 + 2t^4$. $\square$

As with T-polynomials, our next result proves that T'-polynomials and, thus, $CSLRT$ lattices are symmetric.

Theorem 3.4.10. *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ be a partition of n. The column weights of the (condensed) $SLRT(\lambda)$ lattice form a symmetric polynomial, $T'(\lambda)$, consisting of copies of $T'(LRC(\lambda))$.*

Proof. We proceed inductively to proof symmetry.

Case 1: Assume $l(\lambda) = 1$. Since the Young diagram of λ is a horizontal strip, $LRC(\lambda) = \emptyset$ and the $SLRT(\emptyset)$ lattice consists of a single $\boxed{\bullet}$. Then since, $T(\lambda) = x^n + x^{n-1} + \cdots + 1$, the $SLRT(\lambda)$ lattice consists of $(n+1)$ $\boxed{\bullet}$ in a single column. Thus, $T'(\lambda) = nt^1$ which is symmetric.

Case 2: Assume $l(\lambda) = 2$. We break this into two cases:

- Assume $\lambda_1 = \lambda_2$. Then $T(\lambda) = T(LRC(\lambda))$. Since $LRC(\lambda)$ is a horizontal strip, by Case 1) we have that $SLRT(LRC(\lambda))$ is a single vertical column consisting of $(\lambda_2 + 1)$ $\boxed{\bullet}$. This means that $SLRT(\lambda)$ is also a single vertical column consisting of (λ_2+1) $\boxed{\bullet}$. Thus, $T'(\lambda) = \binom{\frac{n}{2}}{}t^1$ which is symmetric.

- Assume $\lambda_1 \neq \lambda_2$. Let $|\lambda_1 - \lambda_2| = p$. By Theorem 3.2.7,

$$T(\lambda) = \sum_{i=0}^{\lambda_1 - \lambda_2} T_i(LRC(\lambda)).$$

  Hence, the $SLRT(\lambda)$ lattice consists of $(p+1)$ shifted vertical columns of height $(\lambda_2 + 1)$. Thus, $T'(\lambda)$ is symmetric.

  Therefore, every length two partition has a symmetric T'-polynomial.

Case 3: Assume $l(\lambda) = 3$. We break this into three cases:

- Assume $\lambda_1 = \lambda_2$. As we have seen previously, $T(\lambda) = T(LRC(\lambda))$. Since every T'-polynomial of a length two partition is symmetric, $T'(\lambda)$ must also be symmetric.

- Assume $\lambda_1 \neq \lambda_2$ and $\lambda_2 = \lambda_3$. Let $|\lambda_1 - \lambda_2| = p$. As before, the $SLRT(\lambda)$ lattice consists of $(p+1)$ shifted vertical columns of height $(\lambda_2 + 1)$. Thus, $T'(\lambda)$ is symmetric.

- Assume $\lambda_1 \neq \lambda_2$ and $\lambda_2 \neq \lambda_3$. Let $|\lambda_1 - \lambda_2| = p$. Clearly, the $SLRT(\lambda)$ lattice consists of $(p+1)$ shifted copies of the $SLRT((\lambda_2, \lambda_2, \lambda_3))$ lattice which is the same as $(p+1)$ shifted copies of the $SLRT(LRC(\lambda))$ lattice.

  As the $SLRT(LRC(\lambda))$ is not a vertical strip, we need to verify that the $SLRT(\lambda)$ lattice is still symmetric.

  Since $SLRT(LRC(\lambda))$ is symmetric by Case 2), $T'(LRC(\lambda))$ is symmetric.

  Let $T'(LRC(\lambda))$ have length r. There are two further cases to consider:

  i. Assume r is odd. Then $T'(LRC(\lambda))$ is of the form

$$T'(LRC(\lambda)) = c_1t^1 + c_2t^2 + \cdots + c_{\lceil \frac{r}{2} \rceil - 1}t^{\lceil \frac{r}{2} \rceil - 1} + c_{\lceil \frac{r}{2} \rceil}t^{\lceil \frac{r}{2} \rceil} + c_{\lceil \frac{r}{2} \rceil - 1}t^{\lceil \frac{r}{2} \rceil + 1} + \cdots + c_2t^{r-1} + c_1^r$$

  where c_i is the height of the i-th column from the left (or right) and $1 \leq i \leq \lceil \frac{r}{2} \rceil$.

  As the $SLRT(\lambda)$ lattice is the result of diagonally shifting $(p+1)$ copies of (condensed) $SLRT(LRC(\lambda))$ lattice, which are each ribbons, we need only consider the action of the coefficients of $T'(LRC(\lambda))$.

$$p+1 \left\{ \begin{array}{cccccccccccc}
 & & & c_1 & c_2 & \cdots & c_{\lceil \frac{r}{2} \rceil - 1} & c_{\lceil \frac{r}{2} \rceil} & c_{\lceil \frac{r}{2} \rceil - 1} & \cdots & c_2 & c_1 \\
 & & \iddots & \iddots & \cdots & \iddots & \iddots & \iddots & \cdots & \iddots & \iddots & \\
 & c_1 & c_2 & \cdots & c_{\lceil \frac{r}{2} \rceil - 1} & c_{\lceil \frac{r}{2} \rceil} & c_{\lceil \frac{r}{2} \rceil - 1} & \cdots & c_2 & c_1 & & \\
c_1 & c_2 & \cdots & c_{\lceil \frac{r}{2} \rceil - 1} & c_{\lceil \frac{r}{2} \rceil} & c_{\lceil \frac{r}{2} \rceil - 1} & \cdots & c_2 & c_1 & & & \\
\hline
\end{array} \right.$$

  which is a symmetric polynomial of degree $(r + p + 1)$.

  ii. Assume r is even. Then $T'(LRC(\lambda))$ is of the form

$$T'(LRC(\lambda)) = c_1t^1 + c_2t^2 + \cdots + c_{\frac{r}{2}}t^{\frac{r}{2}} + c_{\frac{r}{2}}t^{\frac{r}{2}+1} + \cdots + c_2t^{r-1} + c_1^r$$

  where c_i is the height of the i-th column from the left (or right) and $1 \leq i \leq \frac{r}{2}$.

  As the $SLRT(\lambda)$ lattice is the result of diagonally shifting $(p+1)$ copies of (condensed) $SLRT(LRC(\lambda))$ lattice, which are each ribbons, we need only consider the action of the coefficients of $T'(LRC(\lambda))$.

$$
p+1 \left\{ \begin{array}{ccccccccccc}
 & & & c_1 & c_2 & \cdots & c_{\frac{r}{2}} & c_{\frac{r}{2}} & \cdots & c_2 & c_1 \\
 & & \iddots & \iddots & \cdots & \iddots & \iddots & \cdots & \iddots & \iddots & \\
 & c_1 & c_2 & \cdots & c_{\frac{r}{2}} & c_{\frac{r}{2}} & \cdots & c_2 & c_1 & & \\
c_1 & c_2 & \cdots & c_{\frac{r}{2}} & c_{\frac{r}{2}} & \cdots & c_2 & c_1 & & & \\
\hline
\end{array} \right.
$$

which is a symmetric polynomial of degree $(r+p+1)$.

Therefore, every length three partition has a symmetric T'-polynomial.

Case 4: Assume $l(\lambda) > 3$ and that $T'(LRC(\lambda))$ is symmetric for all $l(LRC(\lambda)) \leq n$. Consider λ such that $l(\lambda) \geq n+1 > 3$. We break this into two cases:

- Assume $\lambda_1 = \lambda_2$. Then $T'(\lambda) = T'(LRC(\lambda))$ and the $SLRT(\lambda)$ lattice is symmetric.
- Assume $\lambda_1 \neq \lambda_2$. Let $|\lambda_1 - \lambda_2| = p$. Then the $SLRT(\lambda)$ lattice consists of $(p+1)$ shifted copies of the $SLRT(LRC(\lambda))$ lattice. Then since the $SLRT(LRC(\lambda))$ is symmetric, the $SLRT(\lambda)$ is also symmetric.

Therefore, the T'-polynomial is symmetric for all partitions λ with $l(\lambda) \geq 1$. □

Even though the map $\theta : SLRT \to \mathbb{Z}[t]$ does not have an inverse, we observe that exists a finite number of unique shapes of condensed $SLRT$ lattices which can occur given a T-polynomial and these shapes are uniquely determined by the value $(\lambda_1 - \lambda_2)$. This is to say that given a T-polynomial there exists a finite number of unique T'-polynomials which can be constructed and they are uniquely determined by the t-analogue associated to $(\lambda_1 - \lambda_2)$.

Theorem 3.4.11. *Let $|D_\lambda|$ be the number of unique, non-constant, polynomial factors of a T-polynomial. There exists $|D_\lambda|$ T'-polynomials which can be constructed via the condensed $SLRT$ lattice algorithm. Furthermore, if no pair of consecutive entries are equal, there exists*

$$\sum_{i=1}^{|D_\lambda|} \binom{l-1}{m_1, m_2, \ldots, m_i - 1, \ldots, m_{|D_\lambda|}},$$

unique partitions having the desired T-polynomial where m_i is the multiplicity of the i-th unique factor of the T-polynomial.

Proof. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ be a partition with T-polynomial given by $T(\lambda)$ and Littlewood-Richardson tableau T_λ. For each entry λ_i such that $\lambda_i \neq \lambda_{i+1}$, record the degree of the corresponding t-analogue as defined in Theorem 3.3.3 as d_i for $1 \leq i \leq l$. Define m_i to be the multiplicity of d_i.

Define the sets

$$D_\lambda = \{d_i | 1 \leq i \leq l\} \quad \text{and}$$
$$M_\lambda = \{m_i | 1 \leq i \leq l\}.$$

If there exists j, $1 \leq i < j \leq l$, such that $d_i = d_j$, remove d_j from D_λ and increment m_i in M_λ by one. Repeat this process until $d_i \neq d_j$ in D_λ. Re-index the d_i in D_λ and the m_i in M_λ such that $i < k$ for all $1 \leq i < k \leq |D_\lambda|$.

Since a T-polynomial is isomorphic to the product of t-analogues of the rows of its LRT, if unique partitions λ and $\bar{\lambda}$ have $D_\lambda = D_{\bar{\lambda}}$ and $M_\lambda = M_{\bar{\lambda}}$, $T(\lambda) = T(\bar{\lambda})$.

Without loss of generality, let λ_i be the first entry such that $(\lambda_i - \lambda_{i+1}) = d_1$. Then there exists $(m_1 - 1)$ entries λ_i in λ, such that $(\lambda_i - \lambda_{i+1}) = d_1$, m_2 entries λ_i in λ such that $(\lambda_i - \lambda_{i+1}) = d_2$, m_3 entries λ_i in λ such that $(\lambda_i - \lambda_{i+1}) = d_3$, $\ldots$ in the remaining $(l-1)$ entries of λ. As these differences correspond to the polynomial factors of $T(\lambda)$, any arrangement of the rows of $T(\lambda)$ will produce the same T-polynomial. Hence, every partition such that the first entry λ_i with $(\lambda_i - \lambda_{i+1}) = d_1$ has the same T-polynomial.

Therefore, since T'-polynomials are the shifted sums of T-polynomials, every such partition also has the same T'-polynomial and condensed $SLRT$ lattice shape. As there are exactly $|D_\lambda|$ unique, non-constant, polynomial factors of $T(\lambda)$, there are $|D_\lambda|$ T'-polynomials which can be constructed via the condensed $SLRT$ lattice algorithm.

If $\lambda_i \neq \lambda_{i+1}$ for all $1 \leq i \leq l$, then there exists $\binom{l-1}{m_1, m_2, \ldots, m_i - 1, \ldots, m_{|D_\lambda|}}$ unique partitions such that $\lambda_1 - \lambda_2 = d_i$. Hence, there exists

$$\sum_{i=1}^{|D_\lambda|} \binom{l-1}{m_1, m_2, \ldots, m_i - 1, \ldots, m_{|D_\lambda|}}$$

unique partitions having the desired T-polynomial where m_i is the multiplicity of the i-th unique factor of the T-polynomial. □

Example 3.4.12. Consider the T-polynomial $1 + 4t + 7t^2 + 7t^3 + 4t^4 + t^5 = (1+t)^3(1+t+t^2)$. For partitions $\lambda = (5, 3, 2, 1)$, $\bar{\lambda} = (5, 4, 2, 1)$, and $\hat{\lambda} = (5^2, 3, 2, 1)$, by Theorem 3.3.3, $T(\lambda) = T(\bar{\lambda}) = T(\hat{\lambda}) = 1 + 4t + 7t^2 + 7t^3 + 4t^4 + t^5$. However, by Theorem 3.4.11, $T'(\lambda)$, $T'(\bar{\lambda})$, and $T'(\hat{\lambda})$ are uniquely determined by the difference between the number of cells contained in the first and second rows of their LRT and thus their T'-polynomials and respective condensed $SLRT$ lattices are pairwise non-equal. We consider the condensed $SLRT$ lattices of each partition individually.

Let $\lambda = (5, 3, 2, 1)$ as shown in Figure 3.20. Then $D_\lambda = \{2, 1\}$ and $M_\lambda = \{1, 3\}$ and $(\lambda_1 - \lambda_2) = 2$.

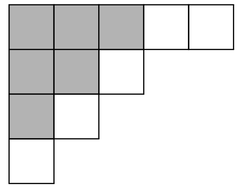


Figure 3.20: LRT for $\lambda = (5, 3, 2, 1)$

This means that the remaining three consecutive rows of $LRT(\lambda)$ must differ in length by exactly one cell and hence, there exists $\binom{3}{3} = 1$ unique partition, specifically $(5, 3, 2, 1)$, up to insertion of an entry $\lambda_i = \lambda_{i+1}$ into λ for $i \neq 1$, where $T'(\lambda)$ is the shifted sum of three copies of $T((3, 2, 1))$.

Therefore, if a partition has the T-polynomial $1+4t+7t^2+7t^3+4t^4+t^5 = (1+t)^3(1+t+t^2)$ and $(\lambda_1 - \lambda_2) = 2$, it must have the condensed $SLRT$ lattice shown in Figure 3.21 and there is only one such unique partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ where $\lambda_i \neq \lambda_{i+1}$ for $1 \leq i \leq l$.

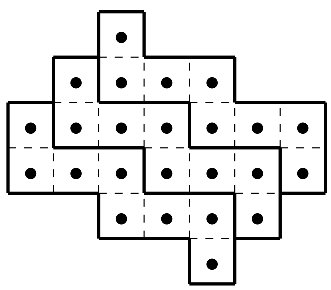


Figure 3.21: Condensed $SLRT$ lattice for $\lambda = (5, 3, 2, 1)$

Let $\bar{\lambda} = (5, 4, 2, 1)$ as shown in Figure 3.22. Then $D_{\bar{\lambda}} = \{1, 2\}$ and $M_{\bar{\lambda}} = \{3, 1\}$ and $(\bar{\lambda_1} - \bar{\lambda_2}) = 1$.

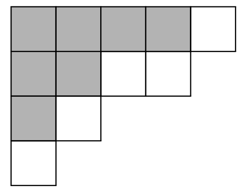


Figure 3.22: LRT for $\bar{\lambda} = (5,4,2,1)$

This means that two of the remaining three consecutive rows of $LRT(\bar{\lambda})$ must differ in length by exactly one cell and the other set of consecutive rows must differ in length by exactly two cells. As it does not matter the order in which the lengths of the rows differ, there exists $\binom{3}{2,1} = 3$ unique partitions, specifically $(5,4,2,1)$, $(5,4,3,1)$, and $(5,4,3,2)$, up to insertion of an entry $\lambda_i = \lambda_{i+1}$ into λ for $i \neq 1$, where $T'(\lambda)$ is the shifted sum of two copies of $T((4,2,1))$.

Therefore, if a partition has the T-polynomial $1+4t+7t^2+7t^3+4t^4+t^5 = (1+t)^3(1+t+t^2)$ and $(\lambda_1 - \lambda_2) = 1$, it must have the condensed $SLRT$ lattice shown in Figure 3.23 and there are three such unique partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ where $\lambda_i \neq \lambda_{i+1}$ for $1 \leq i \leq l$.

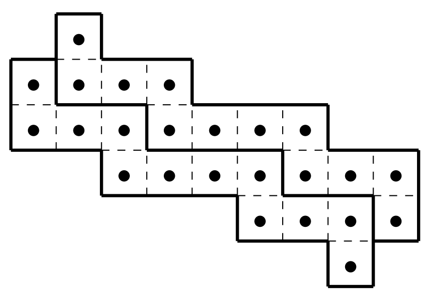


Figure 3.23: Condensed $SLRT$ lattice for $\bar{\lambda} = (5,4,2,1)$

Let $\hat{\lambda} = (5^2,3,2,1)$ as shown in Figure 3.24. Then $D_{\hat{\lambda}} = \{2,1\}$ and $M_{\hat{\lambda}} = \{1,3\}$ and $(\hat{\lambda_1} - \hat{\lambda_2}) = 0$.

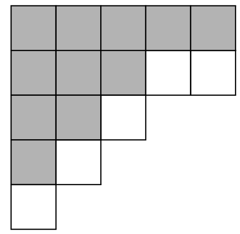


Figure 3.24: LRT for $\hat{\lambda} = (5^2, 3, 2, 1)$

This means that three of the remaining four consecutive rows of $LRT(\hat{\lambda})$ must differ in length by exactly one cell and the other set of consecutive rows must differ in length by exactly two cells. As it does not matter the order in which the lengths of the rows differ, $\hat{\lambda}$ is the unique partition, up to insertion of an entry $\lambda_i = \lambda_{i+1}$ into λ for $i \neq 1$, where $T'(\lambda)$ is the shifted sum of one copy of $T((5, 3, 2, 1))$.

Therefore, if a partition has the T-polynomial $1+4t+7t^2+7t^3+4t^4+t^5 = (1+t)^3(1+t+t^2)$ and $(\lambda_1 - \lambda_2) = 0$, it must have the condensed $SLRT$ lattice shown in Figure 3.25.

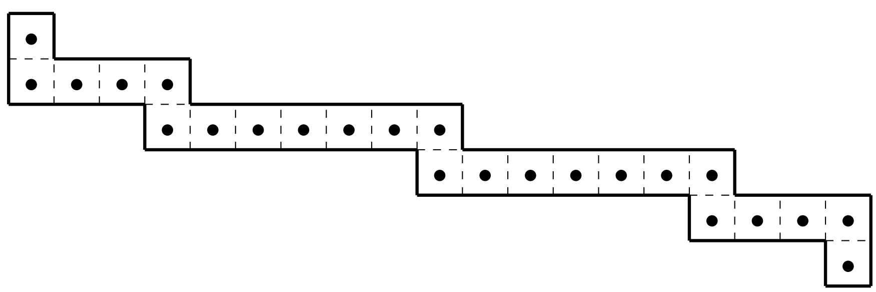


Figure 3.25: Condensed $SLRT$ lattice for $\hat{\lambda} = (5^2, 3, 2, 1)$

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