On Mathematical Modelling of Inelastic Deformation Behavior of Anisotropic Media

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The reliability estimation of high loaded structural elements crystallized as unidirectional cubes or constructed of single crystals, or of composite materials, requires rheological modelling of anisotropic media. Some reasons for a material properties symmetry enable one to receive rather simple and physically determined relations for the material behavior description in elastic as well as inelastic conditions (plasticity, creep). These relations correspond to two types of symmetry: orthoisotropic (for briefness from now on it will be denoted as M1) and transversely isotropic (M2).

1.Elastic properties. According to Hook"s law defining the linear connection of stress tensor σ and strain tensor ε

$$\sigma = C \cdot\cdot \varepsilon$$

(dotes denote scalar multiplication) the fourth-rank elasticity tensor C in the general case has to obtain three modes of symmetry: $C = C^{1T2} = C^{3T4} = C^{1T3,2T4}$, where T denotes tensor transposition according to the given suffixes.

The space symmetry of a specific anisotropic media imposes the additional boundaries on the tensor C.

In the case of M1 (cubic monocrystal) the properties obtained in the main crystallographic axes directions <100> (denoted be low by e_1, e_2, e_3) are identical but differ from those obtained in other directions. Thus, while constructing tensor C apart from the second-rank isotropic tensor I ($I \cdot x = x$ for any x) vectors e_1, e_2, e_3 may be used.

There appears only three independent fourth-rank tensors corresponding to the above properties of symmetry:

$$A_1 = II; \quad U = A_2 = (A_1^{2T3} + A_1^{2T3,1T2} + A_1^{2T3,3T4} + A_1^{2T3,1T3,2T4})/4; \; A_3 = e_1e_1e_1e_1 + e_2e_2e_2e_2 + e_3e_3e_3e_3 \quad (1)$$

So C may be represented as a linear combination of three tensors only: $C = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3$, and the orthoisotropic materials will differ in the values of three elasticity moduli α_i. U is the fourth-rank unit tensor, employing for the second-rank symmetrical tensors ($U \cdot\cdot \varepsilon = \varepsilon \cdot\cdot U = \varepsilon$).

In the case of M2 there is a single preferred vector t, denoting infinite order of symmetry axis. From the necessary symmetry and linear independence condition three more tensors including vector t may be formed apart from the isotropic tensors A_1 and U:

$$A_4 = tttt; \quad A_5 = (ttI + Itt)/2; \quad A_6 = (tIt + tIt^{1T2} + tIt^{3T4} + tIt^{1T3,2T4})/4. \quad (2)$$

The M2 elasticity properties are determined by 5 constants.

Tensors (1),(2) form the basis for the construction of tensors C of the materials mentioned. More acceptable variants of decomposition in each case

$$C = \beta_i B_i \quad (3)$$

(where B_i - the linear combination of the shown tensors) take place if B_i are projecting tensors

$$B_i \cdot\cdot B_j = \delta_{ij} B_j \quad , \qquad \sum_i B_i = U \quad ; \qquad \sigma^{(i)} = \sigma \cdot\cdot B_i \, , \tag{4}$$

δ_{ij} is the Kronecker delta. Then, tensors σ and ε are decomposed into mutual orthogonal ($\sigma^{(i)} \cdot\cdot \sigma^{(j)} = 0$ if $i \neq j$ due to (4)) components, and Hook"s law is represented in the simplest form: $\sigma^{(i)} = \beta_i \, \varepsilon^{(i)}$. This expression is easily transversed, and the expression for internal forces work is quite simple.

In the case of M1 the orthogonal set of tensors is as follows:

$$B_1 = A_1/3 \; , \quad B_2 = A_3 - B_1 \; , \quad B_3 = U - A_3 . \tag{5}$$

The stress tensor components are $\sigma^{(1)} = \sigma_0 I$ - the common isotropic stress tensor (characterized by one scalar value $\sigma_0 = \sigma \cdot\cdot I/3$); $\sigma^{(2)}$ - the "main" deviator (no shear stresses on the planes e_i) is determined by two scalar values, e.g. S_{11} , S_{22} ; $\sigma^{(3)}$ - the "shear" deviator (no normal stresses on the planes e_i) depends on three scalars σ_{12} , σ_{13}, σ_{23} . A sum $S = \sigma - \sigma_0 I = \sigma^{(2)} + \sigma^{(3)}$ is a common stress deviator.

The physical meaning of the elasticity constants β_i :

$$\beta_1 = 3K \; , \qquad \beta_2 = 2G \quad , \quad \beta_3 = 2G'$$

consists in the fact that K - is the volume compression modulus, G and G' - shear moduli on the planes e_i and on those equally inclined to them.

In the case of M2 (the initial basis of tensors is U , A_1, A_4 , A_5, A_6) the tensor σ can not be decomposed into five orthogonal components. Two components $\sigma^{(3)}$ and $\sigma^{(4)}$ can be easily determined due to symmetry: i.e. the shear on the plane t and the deviator without stress on the plane t . The fourth-rank tensors forming these components are the following:

$$B_3 = 2(A_6 - A_4) \quad , \quad B_4 = U + A_5 + (A_4 - 3A_1)/2 - 2A_6 . \tag{6}$$

The remained part of the tensor $\sigma - \sigma^{(3)} - \sigma^{(4)}$ is characterized by two scalar values. So only two independent tensor components can be found though we have the initial basis of three tensors A_1, A_4, A_5 orthogonal to B_3, B_4 . The projecting tensors B_1 , B_2 :

$$B_1 = \frac{1}{1+\kappa^2}\Big[\frac{3}{2}A_1 + \frac{(\sqrt{2}\kappa - 1)^2}{2}A_4 + (\sqrt{2}\kappa + 1)A_5\Big];$$
$$B_2 = \frac{1}{1+\kappa^2}\Big[\frac{3\kappa^2}{2}A_1 + \frac{(\kappa + \sqrt{2})^2}{2}A_4 - \kappa(\kappa + \sqrt{2})A_5\Big]; \tag{7}$$

κ - the material constant that determines stress states $\sigma^{(i)}$ (i =1,2), leading to similar strains $\varepsilon^{(i)}$.

The formation creation of tensor $\sigma_0 I$ for the case of M2 does not have physical sense as such a stress state does not lead to the similar strain state.

<u>2.The elasticity moduli determination for M1.</u> The dependence of Young"s modulus E_n and the shear modulus G_{mn} on the direction of the specimen axis n relatively is easily determined on the basis of (5) (m -the unit vector, orthogonal to the specimen axis, $m \cdot n = 0$)

$$\frac{1}{E_n} = (\beta_1^{-1} - \beta_2^{-1} + 3\beta_3^{-1})/3 + (\beta_2^{-1} - \beta_3^{-1}) \sum_i n_i^4 \quad , \qquad n_i = n \cdot e_i \, ;$$
$$\frac{1}{G_{mn}} = \frac{1}{G'} + \Big(\frac{2}{G} - \frac{2}{G'}\Big) \sum_i m_i^2 n_i^2 \, , \qquad m_i = m \cdot e_i \, .$$

The analysis shows that there exist only two directions n (100) and (111) for which the shear strain at twisting is not followed by the linear one; on the contrary, the shear strain does not occur at tensile. Thus, the constants β_i may be determined on the basis of the first form frequency values under flexural and torsional resonant vibrations for specimen s oriented in these directions (in this case the vibrations may be initiated separately).

3. The generalization of Ilyushin"s isotropy postulate. While representing the stress and strain tensors as the vectors $\bar{\sigma}$ and $\bar{\varepsilon}$ of six-dimensional linear space L_6 (in which $\sigma \cdot\cdot \varepsilon = \bar{\sigma}\cdot\bar{\varepsilon}$), the decomposition $\sigma = \sum_i \sigma^{(i)}$ corresponds to $\bar{\sigma} = \sum_i \bar{\sigma}^{(i)}$. Due to orthogonality of the basis B_i subspaces $L^{(i)}$ (to which the vectors $\bar{\sigma}^{(i)}$ belong) are mutually orthogonal. Hook"s law in the space L_6 takes the form $\bar{\sigma}^{(i)} = \beta_i \varepsilon^{(i)}$ that is the elasticity properties in subspaces $L^{(i)}$ are isotropic. The directions preferred are absent in them. Any component $\bar{\sigma}^{(i)}$ depends on the corresponding component $\bar{\varepsilon}^{(i)}$ only.

For the isotropic body Ilyushin made the assumption that the preferred direction does not occur out of the elasticity limit as well. If two loading histories are characterized by the congruent stress trajectory

$$\underset{2}{\bar{S}}(t) = \tilde{R}\cdot \underset{1}{\bar{S}}(t) \tag{8}$$

($\bar{S}$ - stress vector in the deviatoric space, t - time, $\tilde{R}$ - any second-rank tensor of that space, determining rotation and(or) mirror transversation in the deviatoric space), then the corresponding strain deviator histories posses the analoques congruent property

$$\underset{2}{\bar{e}}(t) = \tilde{R}\cdot \underset{1}{\bar{e}}(t). \tag{9}$$

In the case of M1 it is quite resonable to use the assumptions on the absence of the isotropic stress tensor influence on the rheological properties and on the isotropy properties extention in spaces $L^{(2)}$, $L^{(3)}$ outside the elasticity range. Let us assume that for loadings (8) the expression (9) is true. But on account that the isotropy is valid only for subspaces $L^{(3)}$, $L^{(2)}$ taking separately, let the operator $\tilde{R}$ to be represented as $\tilde{R} = \tilde{R}_2 + \tilde{R}_3$ where each component characterizes the transformation in its subspace only. Thus, at the proportional loading in the subspace $L^{(2)}$ deformation is proportional and belongs to $L^{(2)}$; vectors $\bar{\sigma}$ and $\bar{\varepsilon}$ are collinear; the same refers to the subspace $L^{(3)}$. Yet at proportional loading with nonzero components $\bar{\sigma}^{(2)}$ and $\bar{\sigma}^{(3)}$ the trajectory $\bar{\varepsilon}(t)$ is not necessarily proportional.

In the case of M2 the six-dimensional space need not to be divided into spheric and deviator subspaces; it is devided into four subspaces: one-dimensional for $\sigma^{(1)}$, $\sigma^{(2)}$ and two-dimensional for $\sigma^{(3)}$, $\sigma^{(4)}$. The latter are isotropic in the elasticity range. Let us assume that their isotropy is valid outside the above range as well. The isotropy postulate is written in the mentioned form (8),(9) where $\tilde{R}$ is the sum of four components: $\tilde{R} = \pm g_1 g_1 \pm g_2 g_2 + \tilde{R}_3 + \tilde{R}_4$; g_1 , g_2 are unity vectors of two one-dimensional subspaces, $\tilde{R}_3$, $\tilde{R}_4$ - tensors of mirror transformation and(or) rotation in the two-dimensional subspaces.

4. Deformational plasticity and ideal viscousity theories. The generalisation of the isotropy postulate for the anisotropic media enables one to construct the simplest rheological models.

In the case of M1, on account of the assumption for elastic compressibility and noninfluence of σ_0 on the rheological properties, the deformational theory equations may be written as follows:

$$\sigma_0 = 3K\varepsilon_0 ; \qquad \bar{r} = [f_2(\varepsilon_э)\tilde{U}^{(2)} + f_3(\varepsilon_э)\tilde{U}^{(3)}]\cdot\bar{\varepsilon}, \tag{10}$$

where $\bar{r} = \tilde{C}^{-1}\cdot\bar{\sigma}$ - elastic strain, $\tilde{C} = \beta_2\tilde{U}^{(2)} + \beta_3\tilde{U}^{(3)}$; $\tilde{U}^{(2)}$, $\tilde{U}^{(3)}$ - are identical transformation tensors in subspaces $L^{(2)}$ and $L^{(3)}$. The arguments of scalar functions f_2 and f_3 are invariant characteristics of strain vectors $\bar{\varepsilon}^{(2)}$, $\bar{\varepsilon}^{(3)}$ only. For this purpose it is suggested in (10) to use the "equivalent" strain

$$\varepsilon_э = \sqrt{\bar{\varepsilon}\cdot\tilde{Q}\cdot\bar{\varepsilon}}, \tag{11}$$

$$\tilde{Q} = \tilde{U}^{(2)} + \lambda^2\tilde{U}^{(3)}. \tag{12}$$

The deformational anisotropy parameter λ characterizes the difference of deformation properties in subspaces $L^{(2)}$ and $L^{(3)}$.

The equations take the simplest form on assuming that $f_2 = f_3 = f$:

$$\bar{\tau} = f(\varepsilon_э)\bar{\varepsilon}. \tag{13}$$

In the form this equation does not differ from the deformational theory equation for the isotropic body. The elastic and total strain collinearity (but not tne stress) retains outside the elasticity range as well. However the equipotential surfaces $\tau_э = \sqrt{\bar{\tau} \cdot \widehat{Q} \cdot \bar{\tau}}$ =const are ellipsoidal (not spherical as in the isotropic body), for each of them the axes in subspaces $L^{(2)}$ and $L^{(3)}$ ratio is determined by the parameter λ.

In the case of M2 the deformational theory is constructed on the analagy. In its simplified form the state equation coincides with (13) but in the "equivalent" strain expression (11) the tensor $\widehat{Q}$ is written as a combination of four tensors of identical transformation $\widetilde{U}^{(i)}$ in $L^{(i)}$:

$$\widehat{Q} = \lambda_1^2 \widetilde{U}^{(1)} + \lambda_2^2 \widetilde{U}^{(2)} + \lambda_3^2 \widetilde{U}^{(3)} + \widetilde{U}^{(4)} \tag{14}$$

The deformation properties anisotropy is determined by three scalar parameters λ_i.

In the steady creep theory (ideal viscosity)

$$\bar{\varepsilon} = \bar{\tau} + \bar{p}, \qquad \dot{\bar{p}} = \bar{\Phi}(\bar{\sigma}),$$

the function $\bar{\Phi}(\bar{\sigma})$ is assumed to be potential, that is we assume the existance of the potential $\Psi(\bar{\sigma})$, the gradient of which is the function $\bar{\Phi}$.

The simplest variant for the scalar function $\Psi(\bar{\sigma})$ remaining the same anisotropy as in the elasticity range, may be represented with the help of the "equivalent" stress $\sigma_э = \sqrt{\bar{\sigma} \cdot \widehat{Q} \cdot \bar{\sigma}}$, tentsor $\widehat{Q}$ is determined by expressions (12) and (14) for materials M1 and M2 respectively.

Then, the inelastic strain rate equation takes the form

$$\dot{\bar{p}} = \Psi'(\sigma_э)\, \widehat{Q} \cdot \bar{\sigma}. \tag{15}$$

5.Structural model of anisotropic media. A more adequate description of rheological properties is achieved by making use of the structural model which formally represents the material micro nonhomogeneity. Each body element is supposed to work as a set of ideal viscous subelements having equal strains. In the case of M1 each element properties are expressed in the following:

$$\varepsilon = \sum_i \varepsilon^{(i)} = \varepsilon_K = \sum_i \varepsilon_K^{(i)} = \rho_K + \vartheta_K + \pi_K = \sum_i \rho_K^{(i)} + \sum_i \vartheta_K^{(i)} + \sum_i \pi_K^{(i)};$$

$$\sigma = \sum_i \sigma^{(i)} = \sum_{K=1}^{N} \sigma_K g_K, \quad \sigma_K = \sum_i \sigma_K^{(i)}, \quad \sigma^{(i)} = \sum_K \sigma_K^{(i)} g_K;$$

$$\rho_K^{(i)} = \beta_i^{-1} \sigma_K^{(i)}, \qquad \vartheta_K^{(2)} = \vartheta_K^{(3)} = 0, \qquad \vartheta_K = \vartheta_K^{(1)} = \alpha T I;$$

$$\pi_K^{(1)} = 0; \qquad \dot{\pi}_K = \dot{\pi}_K^{(2)} + \dot{\pi}^{(3)} = \Phi(\sigma_{эK}/z_K, T)(\sigma_K^{(2)} + \lambda^2 \sigma_K^{(3)})/\sigma_{эK};$$

$$\sigma_{эK}^2 = \sigma_K^{(2)} \cdot\cdot\, \sigma^{(2)} + \lambda^2 \sigma_K^{(3)} \cdot\cdot\, \sigma_K^{(3)};$$

where K - the subelement index, N - the number of subelements, ρ, ϑ, π - elastic, thermal and inelastic components of strain, T - temperature, α - the thermal expansion coefficient, z_K - the subelement constant, g_K - subelement weight ($\sum_{K=1}^{N} g_K = 1$).

To identify the model it is necessary to determine the constants α, β_i, the parameters z_K, g_K, (using the stress-strain curve) the function $\Phi(x,T)$ in actual temperature range (using the isothermal stress-strain and creep curves) and the parameter λ characterizing the rheological properties anisotropy. The latter equals to the similarity coefficient of diagrams $\tau_э \sim \varepsilon_э$ obtained at proportional loading in subspaces $L^{(2)}$ and $L^{(3)}$. Rheological functions $\Phi(\sigma_э)$ for the considered cases are similar relatively the axis $\dot{p}_э$ with the same coefficient.

It is conventient to use tensile-compression and torsion tests on tubular specimens which axes coincide with the direction of any axis e_i. It is sufficient to carry out only tensile-compression tests but specimens with the axis direction (111) will be required as well.

The model suggested at loading in any subspace $\mathcal{L}^{(2)}$ or $\mathcal{L}^{(3)}$ describes nonlinear strain hardening, unsteady creep, mutual influence of plasticity and creep, nonisothermal and nonproportional loading effects (nonparametric temperature influence, scalar and vector delay). At proportional loading in the space $\mathcal{L}^{(2)} \oplus \mathcal{L}^{(3)}$ the deformation is nonproportional, though as in the case of the isotropic material, the Masing principle is valid. At an initial quick nonisothermal loading there is a thermomechanical surface: at loading after the reverse the analogous surface exists too which is similar to the initial one. At hold-time periods creep strain trajectory is nonlinear. Asymmetric cyclic loading leads to cyclic creep.

In the case of M2, the simplest structural model is constructed on the analogy. The rheological properties anisotropy (for the deformation and ideal viscousity theories) is determined by three anisotropy parameters. To find them tests under four types of stress state are necessary. The model structure remains unchanged on the whole.