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	0	7	8	9	1	3	5	2	4	6
	6	1	7	8	9	2	4	3	5	0
	5	0	2	7	8	9	3	4	6	1
	4	6	1	3	7	8	9	5	0	2
0	6	5	4	9	8	7	1	2	3	8
7	1	0	6	5	9	8	2	3	4	6
8	7	2	1	0	6	9	3	4	5	1
9	8	7	3	2	1	0	4	5	6	3
1	9	8	7	4	3	2	5	6	0	7
3	2	9	8	7	5	4	6	0	1	2
5	4	3	9	8	7	6	0	1	2	8
2	3	4	5	6	0	1	7	8	9	4
4	5	6	0	1	2	3	8	9	7	5
6	0	1	2	3	4	5	9	7	8	6

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CONSTRUCTION OF RULED SURFACES IN 5-DIMENSIONAL FINITE  
PROJECTIVE GEOMETRIES\*

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1. Introduction. I want to describe, as briefly as possible, the methods for constructing one of my ruled surfaces in  $PG(5,q)$ . Here  $q$  is any prime-power. However, the theory becomes trivial (and exceptional) for  $q=2$ . So assume

$$(1.1) \quad q \geq 3 \quad (q \text{ a prime-power}).$$

2. Preliminary. Nothing is said in my paper on ruled surfaces (the Luxembourg paper) about reguli. I need the following as background:

Lemma 2.1. Let  $S$  be a surface in  $\Sigma = PG(5,q)$  ruled by 3 systems of planes, belonging, say, to classes I, II and III. Let  $L$  be a line lying in a ruling plane of (say) class I. Let  $R$  be the set of  $q+1$  ruling planes of (say) class II each of which contains exactly one point of  $L$ .

Then: (i)  $R$  is a regulus of planes of  $\Sigma$ .

(ii) Each of the  $q^2$  and  $q+1$  ruling planes of class I meets the  $q+1$  planes of  $R$  in the  $q+1$  points of a line -- a transversal line to  $R$ .

(iii) Each of the  $q^2$  and  $q+1$  ruling planes of class III meets the  $q+1$  planes of  $R$  in the  $q+1$  points of a conic.

Proof. It is to be understood that  $S$  is constructed as in my Luxembourg paper. Set

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$$F = GF(q), \quad K = GF(q^3),$$

and let  $V = K \times K$  be the 6-dimensional vector space over  $F$  consisting of ordered pairs  $(x,y)$ ,  $x,y \in K$ , with

$$\left. \begin{aligned} (x,y) + (x',y') &= (x+x', y+y') \\ f(x,y) &= (fx, fy) \end{aligned} \right\} \begin{aligned} \forall x,y,x',y' \in K \\ \forall f \in F. \end{aligned}$$

Then  $S$  consists of all points  $\langle(x,y)\rangle$  such that

$$N(x) = N(y);$$

equivalently

$$x^{q^2+q+1} = y^{q^2+q+1}.$$

The classes (I), (II), (III) are defined as the set of planes of the following sorts:

$$\left. \begin{aligned} \text{(I)} \quad y &= kx \\ \text{(II)} \quad y &= kx^q \\ \text{(III)} \quad y &= kx^{q^2} \end{aligned} \right\} \begin{aligned} \text{For any fixed } k \text{ such that} \\ N(k) = k^{q^2+q+1} = 1. \end{aligned}$$

For the proof, we may assume that  $L$  is the 2-dimensional vector space

$$L = \langle(1,1), (a,a)\rangle$$

for some fixed  $a$ ,

$$a \in K-F.$$

Then the  $q+1$  points of  $L$  are the following: the point

$$\langle(1, 1)\rangle$$

and  $q$  points of form

$$\langle f(1,1) + (a,a) \rangle = \langle (f+a, f+a) \rangle, \quad f \in F.$$

The plane of class (II) through  $\langle (1,1) \rangle$  is

$$\Pi_{\infty}: y = x^q.$$

The plane of class (II) through  $\langle (f+a, f+a) \rangle$  is

$$\Pi_f: y = \frac{f+a}{f+a^q} x^q.$$

We may check that, for each  $k \in K$ ,  $k \neq 0$ , the line

$$L_k = \langle (k, k^q), (ka, k^q a) \rangle$$

meets  $\Pi_{\infty}$  in  $\langle (k, k^q) \rangle$  and  $\Pi_f$  in  $\langle (k(a+f), k^q(a^q+f)) \rangle$ , and hence is a transversal to

$$R = \{\Pi_{\infty}\} \cup \{\Pi_f \mid f \in F\}.$$

Furthermore,  $L_k$  lies in the following plane of class I,

$$y = k^{q-1}x.$$

We also note that

$$L_k = L_{k'}, \iff \frac{k'}{k} \in F,$$

so that the total number of distinct lines  $L_k$  is

$$\frac{q^3-1}{q-1} = q^2 + q + 1.$$

Thus:  $R$  is a regulus of (skew) planes of  $\Sigma:PG(5,q)$ , with the  $q^2+q+1$  lines  $L_k$  as its transversal lines.

Next consider a typical ruling plane of class III, say

$$\alpha: y = kx^{q^2},$$

where  $k$  is some fixed element of  $K$  with

$$k^{q^2+q+1} = 1.$$

We can choose  $b$  (in  $q-1$  ways) so that

$$k = b^{q-1}.$$

Thus the equation of  $\alpha$  becomes

$$\alpha: y = b^{q-1}x^{q^2}.$$

Then we may check that

$$\Pi_{\infty} \cap \alpha = \langle (b^{-q^2}, b^{-1}) \rangle$$

$$\Pi_f \cap \alpha = \langle (b^{-q^2}(f+a)^{-q^2}, b^{-1}(f+a)^{-1}) \rangle, \quad \forall f \in F.$$

To complete the proof, we must show that the  $q+1$  points so obtained lie on a conic (in  $\alpha$ ). If  $q$  is odd, it is enough to show that no three of the points lie on a line. Consider the special case of these points

$$\Pi_{f_i} \cap \alpha, \quad i = 1, 2, 3,$$

where  $f_1, f_2, f_3$  are distinct elements of  $F$ . If these points lie on a line, we must have

$$\begin{aligned}
& u(b^{-q} (f_1+a)^{-q^2}, b^{-1}(f_1+a)^{-1}) + v(b^{-q} (f_2+a)^{-q^2}, b^{-1}(f_2+a)^{-1}) \\
& \quad + w(b^{-q} (f_3+a)^{-q^2}, b^{-1}(f_3+a)^{-1}) \\
& = 0
\end{aligned}$$

for elements  $u, v, w$  of  $F$ , not all zero. Considering components, we get two equations for  $u, v, w$  which (after cancelling some non-zero factors) become

$$(1) \quad (f_1+a)^{-q^2} u + (f_2+a)^{-q^2} v + (f_3+a)^{-q^2} w = 0$$

$$(2) \quad (f_1+a)^{-1} u + (f_2+a)^{-1} v + (f_3+a)^{-1} w = 0.$$

We note that (2) implies (1). However, (2) (with  $u, v, w$  in  $F$ , not all zero) means that  $a$  satisfies a quadratic equation with coefficients in  $F$ . Since  $a \in K-F$ , and  $K$  is three dimensional over  $F$ , this is false.

Similarly, three points  $\Pi_\infty \cap \alpha, \Pi_{f_1} \cap \alpha, \Pi_{f_2} \cap \alpha$ , cannot be collinear.

We leave the proof of Lemma 2.1 at this point.

### 3. Construction of surfaces $S$ . (Some algebraic details.)

We suppose given a regulus  $R$  of planes of  $\Sigma = PG(5, q)$ , consisting of  $q+1$  skew planes  $\Pi$ ;

$$R = \{ \Pi_i \mid i = 1, 2, \dots, q+1 \}$$

such that each of the  $q^2+q+1$  transversal lines to  $\Pi_1, \Pi_2, \Pi_3$  meets every  $\Pi_i$  in a point. We also suppose given one more plane,  $\Pi$ , disjoint from the  $q+1$  planes  $\Pi_i$ .

We consider the problem of constructing a triply-ruled surface  $S$  of  $\Sigma$  such that the set

$$R \cup \{\Pi\}$$

of  $q+2$  skew planes forms part of one of the three systems of  $q^2+q+1$  ruling planes of  $S$ .

In the light of Lemma 2.1, we may assume that the planes of  $R \cup \{\Pi\}$  belong to Class II, and that every plane of Class I contains a (unique) transversal line to  $R$ .

We may suppose that  $\Sigma = PG(5, q)$  is given by a six-dimensional vector space  $V$  over  $F = GF(q)$  with a basis  $l_1, l_2, l_3, l'_1, l'_2, l'_3$  chosen so that

$$\Pi_1 = J(\infty) \text{ has basis } l_1, l_2, l_3$$

$$\Pi_2 = J(0) \text{ has basis } l'_1, l'_2, l'_3,$$

$$\Pi_3 = J(I) \text{ has basis } l_1+l'_1, l_2+l'_2, l_3+l'_3.$$

Then every plane skew to  $\Pi_1=J(\infty)$  has form  $J(X)$  for a unique  $3 \times 3$  matrix  $X=(x_i)$  over  $F=GF(q)$ , where

$$J(X) \text{ has basis } x_{11}l_1 + x_{12}l_2 + x_{13}l_3 + l'_1,$$

$$x_{21}l_1 + x_{22}l_2 + x_{23}l_3 + l'_2,$$

$$x_{31}l_1 + x_{32}l_2 + x_{33}l_3 + l'_3.$$

In particular,  $R$  consists of  $J(\infty)$  and the  $J(fI)$ ,  $f \in F$ , where

$$fI = \begin{pmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & f \end{pmatrix}.$$

Also

$$\Pi = J(U)$$

for some irreducible  $3 \times 3$  matrix  $U$ .

Every vector  $a$  in  $J(\infty)$  has form

$$a = a_1 l_1 + a_2 l_2 + a_3 l_3$$

for unique elements  $a_1, a_2, a_3$  in  $F$ . We define

$$\begin{aligned} a^X = & (a_1 x_{11} + a_2 x_{21} + a_3 x_{31}) l_1 + (a_1 x_{12} + a_2 x_{22} + a_3 x_{32}) l_2 \\ & + (a_1 x_{13} + a_2 x_{23} + a_3 x_{33}) l_3 \end{aligned}$$

for every  $3 \times 3$  matrix  $X$ . In this notation

$$J(X) \text{ has basis } l_1^X + l'_1, l_2^X + l'_2, l_3^X + l'_3.$$

Let  $a$  be a non-zero vector in  $J(\infty)$ , so that  $\langle a \rangle$  is a point of the plane  $J(\infty)$ . The transversal line,  $L_a$ , to  $R$  through  $\langle a \rangle$  is the two-dimensional vector space

$$L_a = \langle a, a' \rangle$$

where we define

$$a' = a_1 l'_1 + a_2 l'_2 + a_3 l'_3.$$

A ruling plane, of our proposed surface  $S$ , which has Class (I) and contains  $L_a$  must (in particular) meet  $\Pi = J(U)$  in a point, say in

$$\langle b^U + b' \rangle,$$

where  $b$  is some non-zero vector in  $J(\infty)$ . Then this ruling plane is (say)

$$\alpha = \langle a, a', b^U + b' \rangle .$$



The ruling plane  $\alpha$  should meet each plane of  $\mathcal{R}$  in a point, namely in a point of  $L_a$ . This puts some restrictions on the point  $\langle b^U + b' \rangle$  of  $\Pi$  or, equivalently, the point  $\langle b \rangle$  of  $J(\infty)$ . To see this, we note that each point of  $\alpha$  has form  $\langle v \rangle$  where  $v$  is a non-zero vector of form

$$v = fa + ga' + h(b^U + b'),$$

where  $f, g, h$  are elements of  $F$ , not all zero. Equivalently,

$$v = (fa + hb^U) + (ga + hb)'$$

The point  $\langle v \rangle$  will be in  $J(\infty)$  if and only if

$$ga + hb = 0.$$

We wish this condition to imply  $h = 0$ . Thus we want

$$(1) \quad \langle b \rangle \neq \langle 0 \rangle.$$

If  $t \in F$ , so that  $J(tI)$  is in  $\mathcal{R}$ , the point  $\langle v \rangle$  will be in  $J(tI)$  if and only if

$$fa + hb^U = t(ga + hb)$$

or

$$h(b^U - tb) = (-f + tg)a$$

or

$$h \cdot b^{U-tI} = (-f + tg)a.$$

We want this equation to imply  $h=0$ . Thus we want

$$\langle b^{U-tI} \rangle \neq \langle a \rangle$$

or, equivalently,

$$(2) \quad \langle b \rangle \neq \langle a^{(U-tI)^{-1}} \rangle, \quad \forall t \in F.$$

It may be shown that the point  $\langle a \rangle$  of  $J(\infty)$  together with the  $q$  points

$$\langle a^{(U-tI)^{-1}} \rangle, \quad t \in F$$

of  $J(\infty)$  form a conic of  $J(\infty)$ . Hence, also, the corresponding points of  $\Pi$ , namely

$$\langle a^U + a' \rangle \quad \text{and} \quad \langle a^{(U-tI)^{-1}U} + (a^{(U-tI)^{-1}})' \rangle, \quad t \in F,$$

form a conic of  $\Pi$ . Also the point  $\langle b^U + b' \rangle$  must avoid the  $q+1$  points of the latter conic. This gives

$$(q^2 + q + 1) - (q + 1) = q^2$$

choices of the point  $\langle b^U + b' \rangle$  so that

$$\alpha = \langle a, a', b^U + b' \rangle$$

meets each of the  $q+2$  planes in  $R\cup\{\Pi\}$  in (exactly) a point.

It is easy to check that the conditions on  $b$  are equivalent to the following:

$$(3) \quad a, b, b^U \quad \text{form a basis of } J(\infty) \quad \text{over } F = GF(q).$$

However, since  $U$  is irreducible and  $b \neq 0$ , the vectors  $b, b^U, b^{U^2}$  form a basis of  $J(\infty)$  over  $F = GF(q)$ . Hence (3) means that

$$\alpha = fb + gb^U + hb^{U^2}$$

for some  $f, g, h$  in  $F$  with  $h \neq 0$ . Equivalently (the case  $h=1$ )

$$(3') \quad \langle a \rangle = \langle f_0 b + g_0 b^U + b^{U^2} \rangle$$

for some  $f_0, g_0$  in  $F$ . (Note that the ordered pair  $f_0, g_0$  can be chosen in  $q^2$  ways.)

Assuming (3'), we can write the plane  $\alpha$  as the plane  $\alpha(b)$  where

$$(4) \quad \alpha(b) = \langle f_0 b + g_0 b^U + b^{U^2}, f_0 b' + g_0 (b^U)' + (b^{U^2})', b^U + b' \rangle.$$

Here  $b$  can be any non-zero vector in  $J(\infty)$ . If, now, for fixed  $f_0, g_0$  in  $F$ , we consider all the planes  $\alpha(b)$ , we check easily that they form a set of

$$q^2 + q + 1$$

distinct, mutually skew, planes, each of which meets every plane of  $R \cup \{\Pi\}$  in exactly a point, and each of which contains a unique transversal line to  $R$ , namely the line

$$\langle f_0 b + g_0 b^U + b^{U^2}, f_0 b' + g_0 (b^U)' + (b^{U^2})' \rangle.$$

Although we are not supplying a proof here, these  $q^2 + q + 1$  planes  $\alpha(b)$  constitute the desired (complete) collection of planes of Class I.

We still have to supply the

$$q^2 + q + 1 - (q+2) = q^2 - 1$$

missing planes of Class II, and the  $q^2 + q + 1$  planes of Class III.

4. Geometric approach. The material of Section 3 may be summarized as follows:

Suppose given a regulus,  $R$ , of  $q+1$  skew planes of  $\Sigma = PG(5, q)$  and a plane  $\Pi$  disjoint from each of the planes in  $R$ . Then, in precisely  $q^2$  distinct ways, we can set up a one-to-one correspondence

$$(4.1) \quad L \leftrightarrow P$$

between the  $q^2+q+1$  transversal lines  $L$  to  $R$  are the  $q^2+q+1$  points  $P$  of  $\Pi$  such that:

(i) If  $L, P$  are a corresponding pair, the plane  $L+P$  intersects  $\Pi$  in  $P$  and intersects each plane  $\Pi_i$  of  $R$  in the point  $L \cap \Pi_i$ .

(ii) The  $q^2+q+1$  planes  $L+P$  obtained by letting  $L, P$  range over all corresponding pairs are structurally skew.

This is essentially all we need to complete the construction of the surface  $S$ .

We assume that some fixed correspondence (4.1) has been chosen. Consider any corresponding pair  $L, P$ . We note first that the projective space

$$(4.2) \quad H_P = L + \Pi$$

is four-dimensional and hence is a hyperplane of  $\Sigma = PG(5, q)$ . Next we construct a projective 3-space,

$$T_P,$$

in the following manner: Let  $\Pi_1, \Pi_2$  be two distinct planes of  $R$  and let  $M$  be the unique transversal line through  $P$  to  $\Pi_1, \Pi_2$ , meeting  $\Pi_1, \Pi_2$  in points  $P_1, P_2$  respectively. Let  $L_1, L_2$  be the transversal lines to  $R$  through  $P_1, P_2$  respectively. Note that  $L_1 \neq L_2$ , else the point  $P$  of  $\Pi$  would lie on a transversal to  $R$  and hence on one of the planes of  $R$ , a contradiction. Since  $L_1 \neq L_2$ , then the space

$$(4.3) \quad T_P = L_1 + L_2$$

is a projective 3-space. Since  $M$  contains  $P, P_1, P_2$  and since  $P_1$  is on  $L_1$ ,  $P$  is on  $L_1$ , then  $T_P$  contains  $P$  and  $M$  is the transversal through  $P$  to  $L_1, L_2$  in  $T_P$ .

We need to note the following:

(a) The 3-space  $T_p$  depends only on  $R$  and  $P$ , not on  $\Pi$  or on the choice of the planes  $\Pi_1, \Pi_2$  of  $R$ .

(b)  $T_p$  meets each plane of  $R$  in a line, giving  $q+1$  such lines, and contains precisely  $q+1$  transversals to  $R$ , namely the transversals to the first set of lines. These two sets of  $q+1$  lines form the two sets of rulings of a doubly-ruled quadric

$$Q_p$$

of  $T_p$ ; and  $Q_p$  depends only on  $P$  and  $R$ .

(c)  $P$  lies on exactly  $q+1$  planes to  $Q_p$  in  $T_p$ . Each of these contains exactly one transversal line to  $R$  and meets exactly one plane of  $R$  in a line. Each of the remaining  $q^2$  planes of  $T_p$  through  $P$  meets  $Q_p$  in a conic, contains no transversal line to  $R$ , and meets no plane of  $R$  in a line.

(d)  $T_p \cap \Pi = P$ .

Next we need the following:

(4.4)  $T_p \cap L$  is the empty point-set.

(4.5)  $\text{III}_p = T_p \cap H_p$  is a plane (disjoint from  $L$ ).

To prove (4.4), first suppose that  $L$  is contained in  $T_p$ . Then  $P+L$  is a tangent plane to  $Q_p$  and hence meets some plane of  $R$  in a line. This is a contradiction. Hence  $L$  is not in  $T_p$ . Next suppose that  $L$  meets  $T_p$  in a point  $P'$ . (Since  $P$  is not on  $L$ , necessarily  $P' \neq P$ .) Since  $P'$  is on  $L$ , then  $P'$  is on some plane, say  $\Pi_i$ , of  $R$ . Since  $P'$  is in  $\Pi_i \cap T_p$ , the  $P'$  is on  $Q_p$ . In particular, the transversal line through  $P'$  to  $R$  is a ruling of  $Q_p$ , and is in  $T_p$ . But this transversal, being the unique transversal line through  $P'$  to

$R$ , must be  $L$ . Hence  $L$  is in  $T_p$ , a contradiction. Therefore, (4.4) must be true.

In view of (4.4), the projective space  $T_p + L$  has dimension  $3+1-(-1) = 5$ . Therefore,

$$(4.6) \quad T_p + L = \Sigma.$$

Since  $H_p$  contains  $L$ , we see from (4.6) that

$$T_p + H_p = \Sigma.$$

Hence

$$\dim(\text{III}_p) = \dim T_p + \dim H_p - \dim \Sigma = 3 + 4 - 5 = 2.$$

Thus  $\text{III}_p$  is a plane. Moreover

$$\text{III}_p \cap L = T_p \cap H_p \cap L$$

is empty by (4.4). This proves (4.5). From (4.3) we get

$$(4.7) \quad \text{III}_p + L = H_p.$$

Indeed, the left-hand side of (4.7) is contained in the right-hand side, and both sides have projective dimension 4. From (4.7) and the fact that  $P$  is in  $\text{III}_p$  we get

$$\text{III}_p + (P+L) = H_p$$

and hence

$$(4.8) \quad \text{III}_p \cap (P+L) = P.$$

Next we need

(4.9) If  $\Sigma_4$  is a projective 4-space of  $\Sigma$  containing  $\Pi$ , then  $\Sigma_4$  contains one and only one transversal line to  $R$ .

To see this, we note that there are precisely  $q^2+q+1$  distinct transversal lines to  $R$  and precisely  $q^2+q+1$  distinct projective 4-spaces of  $\Sigma$  containing  $\Pi$ . If (4.9) is false, there must be a  $\Sigma_4$  which contains  $\Pi$  but contains no transversal line to  $R$ . Let  $\Sigma_4$  be such a 4-space, and let  $\Pi_1$  be one of the planes of  $R$ . Then  $\Sigma_4$  intersects  $\Pi_1$  in a line, say  $M$ . The  $q+1$  transversal lines to  $R$  through the points of  $M$ , each meet  $\Sigma_4$  in a single point, namely a point of  $M_1$ . The same process, carried out for the  $q+1$  distinct planes of  $R$ , shows that there must be at least

$$(q+1)^2 > q^2+q+1$$

distinct transversal lines to  $R$ . This is a contradiction. Hence (4.9) is true. As a special case of (4.9),

(4.10)  $L$  is the only transversal line to  $R$  contained in  $H_p = L + \Pi$ .

By (4.10) and (4.4), (4.5),

(4.11) The plane  $III_p$  contains no transversal line to  $R$ .

In view of (4.11),  $III_p$  meets the  $q+1$  planes of  $R$  in the points of a conic. (The conic lies on  $Q_p$ .)

The  $q^2+q+1$  planes  $III_p$ , one for each point  $P$  of  $\Pi$ , are our candidates for the ruling planes of Class III. (Compare Lemma 2.1.) We omit the proof that every two of these are disjoint.

Of course, the planes of class I are the planes

$$I_p = P + L$$

where  $L, P$  is a corresponding pair in the sense of (4.1). It has to be shown that every plane of class I meets every plane of class III in a point.

In addition, we have in class II only the set

$$R \cup \{\Pi\}$$

of  $q+2$  planes. But it should be clear at this point that the missing  $q^2-1$  planes are uniquely determined by the planes of Classes I and III. We have only to consider Lemma 2.1 with classes I, II, III replaced (for example) by classes III, I, II respectively.

I will stop here. Note that one must prove that the construction can actually be completed consistently. (This is, in fact, true.)