Imperfection Sensitivity of Elastic Structures a Numerical Postbuckling Approach

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Abstract

From a total potential energy functional of an elastic structure the incremental equations of a geometric nonlinear problem are derived and presented in the context of shell stability. Within the method of finite elements an algorithm is shown to detect and to enter bifurcation buckling paths. Over and above that a direct postbuckling approach is transformed into a FE-formulation and, as well as the complete nonlinear computation, applied to several examples.

1. Introduction

Structures, sensitive to instability phenomena, demand for a careful design and for efficient methods to precalculate their real load bearing capacity. Within the two basic types of elastic buckling phenomena, namely the snap-through and the bifurcation buckling, especially the last one may be connected with a drastic decrease of the critical load, when geometric imperfections are involved. By transformation of the general energy formulation two finite element procedures are presented to treat postbuckling behavior:

- complete nonlinear FE analysis,
- asymptotic FE analysis in bifurcation point.

2. Energy formulation of buckling and post-buckling behavior

Considering a nonlinear strain-displacement and a linear elastic stress-strain relationship within a structure and a dead-loading condition, the total potential energy $\Pi[u; \lambda]$ can be formed dependent only on the deformation u and a load intensity scalar λ. Points of equilibrium fulfill the stationarity condition of Π :

$$\delta\Pi = 0 = \Pi'[u; \lambda]\, \delta u \tag{2.1}$$

where $(\ )'$ denotes the Gateaux derivative with respect to u. Refering to an initial state of deformation a Total-Lagrangean formulation of the energy is adopted and the deformation therefore decomposed in a fundamental $(\bar{u})$ and an incremental part $(\dot{u})$.

$$u = \bar{u} + \overset{+}{u} \tag{2.2}$$

Consider the fundamental state to be an equilibrium one for the load value λ , we look for an adjacent state $\bar{u} + \overset{+}{u}$ to fulfill the equilibrium condition (2.1) for the load value $\lambda = \bar{\lambda} + \Delta\lambda$.

By a Taylor expansion of (2.1) in $\bar{u}$ with a variation due to $\overset{+}{u}$ we obtain:

$$\Pi'[\bar{u} + \overset{+}{u};\ \lambda]\delta\overset{+}{u} = 0 = \Pi'[\bar{u};\ \lambda]\delta\overset{+}{u} + \Pi''[\bar{u};\ \lambda]u\ \delta\overset{+}{u} + 1/2\ \Pi'''[\bar{u};\ \lambda]\overset{+}{u}^2\ \delta\overset{+}{u} + \dots , \tag{2.3}$$

and by linearisation due to the unknown deformation $\overset{+}{u}$:

$$\Pi'[\bar{u};\ \lambda]\delta\overset{+}{u} + \Pi''[\bar{u};\ \lambda]\overset{+}{u}\delta\overset{+}{u} + E(\overset{+}{u};\lambda) = 0 , \tag{2.4}$$

where $E(\overset{+}{u};\ \lambda)$ denotes the linearisation error.

It should be pointed out that $\hat{u}$ represents an equilibrium state only for $\hat{\lambda}$, but not for $\lambda = \bar{\lambda} + \Delta\lambda$.

The question, whether an equilibrium state is stable or unstable can usually be proved by the second variation of Π, being positive for stable states, and therefore representing an isolated energy-minimum. On the other hand we know, that an equilibrium state u_c is critical, if for an existent infinite small adjacent state $\overset{+}{u}$, there is no change in the load value, i.e. $\Delta\lambda = 0$. Let $\bar{\lambda} = \lambda_c$ we obtain from (2.3):

$$\Pi'[u_c;\ \bar{\lambda}]\delta\overset{+}{u} + \Pi''[u_c;\ \lambda_c]\overset{+}{u}\delta\overset{+}{u} + 1/2\ \Pi'''[u_c;\ \lambda_c]\overset{+}{u}^2\delta\overset{+}{u} + \dots = 0 \tag{2.5}$$

While the first term dops out by (2.1), the nonlinear terms can be eliminated by the assumption

$$\lim_{\Delta\lambda\to 0} \overset{+}{u} = 0 \quad ; \qquad u_1 = \lim_{\Delta\lambda\to 0} \overset{+}{u}/\|\hat{u}\| , \tag{2.6}$$

where $\|\overset{+}{u}\|$ denotes a suitable norm for the deformation. Thus we obtain the critical stability condition

$$\Pi''[u_c;\ \lambda_c]u_1\ \delta u = 0 , \tag{2.7}$$

where λ_c represents the critical load value and u_1 the according eigenmode(s).

Concentrating to bifurcation problems the postbuckling behavior can be calculated generally also by use of the stationarity condition (2.1). Beside the possibility to compute postbuckling equilibrium states in an incremental way, a direct analysis approaches the postbuckling behavior by two asymptotic expansions:

$$\begin{aligned} u &= u_0(\lambda) + \xi u_1 + \xi^2 u_2 + \dots \\ \lambda &= \lambda_c + \lambda_1\xi + \lambda_2\xi^2 + \dots \end{aligned} \tag{2.8}$$

Refering back to KOITER /6/ and BUDIANSKY /2/ these expansions are put into a Taylor series of (2.5) with respect to λ . The polynominal coefficients of ξ represent the variational principles for the higher order deformation modes $u_2, u_3, \dots$, and equations for the parameters $\lambda_1, \lambda_2, \dots$. (Fig. 2.1).

The big advantage of such an asymptotic approach must be seen in the single valued (λ_1, λ_2)

examination of the postbuckling behaviour. Over and above that the load decrease due to imperfections can be approximated very efficiently /2, 5/ (Fig. 2.1).

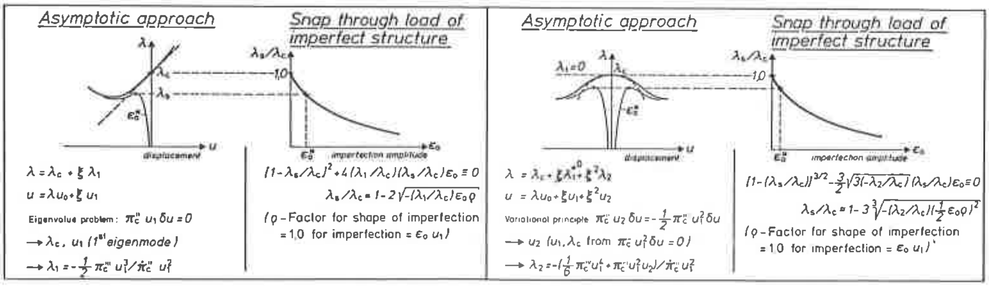

Fig. 2.1 Asymptotic postbuckling analysis

In fig. 2.1 the postbuckling behavior is divided into an asymmetric, $\lambda_1 \neq 0$, and a symmetric case, $\lambda_1 = 0$, where the symmetric one is unstable for $\lambda_2 < 0$.

3. Finite element application to shell problems

In accordance with most nonlinear problems the geometric strain-displacement relationship is reduced to:

$$\underline{\varepsilon} = \underline{D}_L \ (u) + 1/2 \ \underline{D}_{NL} \ (u, u) \tag{3.1}$$

where $\underline{D}_L$ denotes a linear operator and $\underline{D}_{NL}$ a symmetric nonlinear operator in u . The application to shell structures, being an example for highly nonlinear problems, succeed simply by expressing the operators in form of a strain tensor formulation due to the shell theory.

For a DONNELL/MARGUERRE theory, under the assumptions of a KIRCHHOFF/LOVE hypothesis and small strains and rotations, we end up with /1/:

$$\underline{\varepsilon} = \begin{bmatrix} \alpha_{(\alpha\beta)} \\ \beta_{(\alpha\beta)} \end{bmatrix} \quad ; \quad \alpha_{(\alpha\beta)} = 1/2 \ (u_{\alpha|\beta} + u_{\beta|\alpha} - 2 \ u_3 \ b_{\alpha\beta}) + \underline{1/2 \ u_{3|\alpha} \ u_{3|\beta}} \qquad \beta_{(\alpha\beta)} = -u_{3|\alpha\beta} \tag{3.2}$$

where the underlined terms represent the nonlinear part of the membrane strains $\alpha_{(\alpha\beta)}$. The curvature strains $\beta_{(\alpha\beta)}$ contain only linear terms in u .

Using a Hookean constitutive relationship the total potential energy can be written in the common form:

$$\Pi = 1/2 \int_{\overset{\circ}{F}} \underline{\varepsilon}^T \underline{E} \underline{\varepsilon} \ d\overset{\circ}{F} - \int_{\overset{\circ}{F}} \underline{u}^T \underline{p} \ d\overset{\circ}{F} - \int_{\overset{\circ}{B}} \underline{r}^T \overset{\circ}{\underline{t}} \ ds \tag{3.3}$$

where ^{0} denotes the initial state, F the shell surface, $\underline{p}$ the surface forces, $\underline{t}$ the boundary forces and $\underline{r}$ the kinematic boundary quantities.

Substituting the shell theory into the operator equation (3.1) and applying the derived formulas of chapter 2 to the total potential energy (3.3) we obtain an incremental energy relation, a critical stability equation and a postbuckling approach for general shell strutures /5/.
The discretization of the deformation u leads to a finite element displacement model

$$\underline{u}^p = \underline{\Phi}\ \underline{v}^p$$
$$\underline{\varepsilon}^p = \underline{D}_L\ (\underline{u}^p) + 1/2\ \underline{D}_{NL}\ (\underline{u}^p, \underline{u}^p) \qquad (3.4)$$
$$= \underline{D}_L\ (\underline{\Phi})\ \underline{v}^p + 1/2\ (\underline{v}^p)^T\ \underline{D}_{NL}\ (\underline{\Phi}, \underline{\Phi})\ \underline{v}^p \quad ,$$

where $\underline{\Phi}$ denotes the basic functions within an element p . The variational principle of equilibrium in an adjacent incremental state can be transformed in finite element matrix notation:

$$(2.4) \rightarrow (\underline{K}_e + \underline{K}_{gL} + \underline{K}_{gN} + \underline{K}_{uL} + \underline{K}_{uN})\ \underline{v} = \underline{p}_e - \underline{p}_i \quad . \qquad (3.5)$$

The tangent stiffness matrix is decomposed into the usual parts: elastic, linear and nonlinear initial stress, linear and nonlinear initial displacement matrices, whose calculation has been discussed in former papers /3, 4/.
The error term in (2.4) is represented by the difference of external and internal load vectors on the right side of equation (3.5).
The solution of (2.3) requires an efficient iterative procedure to reduce the error to an arbitrarily small value. The use of a sophisticated arc-length algorithm, as the RIKS/WEMPNER/WESSEL method, enables us to compute states of equilibrium, being stable or unstable. This can be checked by watching the diagonal elements of the decomposed tangential stiffness matrix /3, 7/.
As the numerical procedure above is valid for general nonlinear load-displacement problems, it can be applied also to snap-through buckling.
In bifurcation problems the questions, how to detect the critical point and how to enter the intersecting path within the numerical procedure, have to be answered. From equation (2.7) we obtain a quadratic algebraic eigenvalue problem:

$$(2.7) \rightarrow ((\underline{K}_e + \lambda_c\ (\underline{K}_{gL} + \underline{K}_{uL}) + \lambda_c^2\ (\underline{K}_{gN} + \underline{K}_{uN}))\ \underline{v}_1 = 0 \quad , \qquad (3.6)$$

which has got a linear solution for the eigenvalue $\lambda_c = 1$. $\underline{v}_1$ in the sense of (2.7) represents the eigenmode of the lowest critical eigenvalue λ_c .
The nonlinear path and the bifurcation points within are calculated by the combination of incremental/iterative methods and the solution of the linearized eigenvalue problem $(\lambda^2 \sim \lambda)$. The analysis of postbuckling behavior gives the hint for using a perturbation into the direction of the buckling mode $\underline{v} = \underline{v}_c + \xi\ \underline{v}_1$. This enables to enter the postbuckling path, on which the equilibrium iteration continues.
With a linear elastic norm

$$\|\underline{v}\| = \langle\underline{v}, \underline{v}\rangle^{1/2} = \underline{v}^T\ \underline{K}_e\ \underline{v} = 1 \qquad (3.7)$$

the postbuckling parameters are computed to:

$$\lambda_1/\lambda_c = \underline{v}_1^T \, \underline{K}_{gL} \, (\underline{v}_1) \, \underline{v}_1$$
$$\lambda_2/\lambda_c = - \underline{v}_1^T \, \underline{K}_{gN} \, (\underline{v}_1, \underline{v}_1) \, \underline{v}_1 + \underline{v}_1^T \, \underline{K}_{gL} \, (\underline{v}_2) \, \underline{v}_1 + 2 \, \underline{v}_1^T \, \underline{K}_{gL} \, (\underline{v}_1) \, \underline{v}_2 \quad ; \lambda_1 = 0 \qquad (3.8)$$

where $\underline{v}_2$ denotes an orthogonal displacement to $\underline{v}_1$: $\langle \underline{v}_1, \underline{v}_2 \rangle = 0$, computed by an inhomogenous variational principle (Fig. 2.1) /5/. The equations above are valid for the case of a linear prebuckling path. They yield a quick but effective examination of postbuckling behavior. For imperfections in the shape of the buckling modes, $\varepsilon \cdot \underline{v}_1$, the theory delivers efficient formulas to approximate the snap-through buckling load in comparison to the bifurcation load of the perfect structure. The following examples demonstrate complete nonlinear and postbuckling calculations for shells of several types.

4. Examples

Nonlinear postbuckling and asymptotic calculations are compared in the appended figures for a cylindrical, an ellipsoidal and a hyperbolic shell, being all examples for symmetric postbuckling behavior. The first two examples show critical symmetric behavior, while the chosen cooling tower behaves stable symmetric. The asymptotic approach proves true in all cases and also the load decrease due to imperfections fits well.

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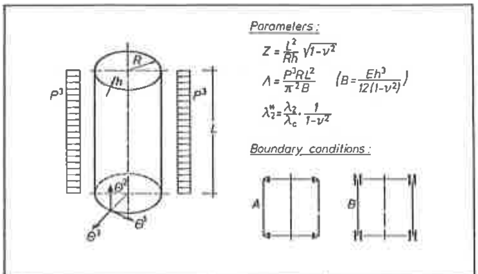
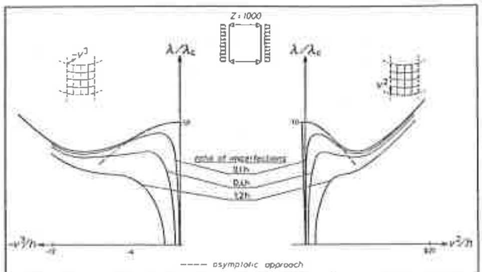
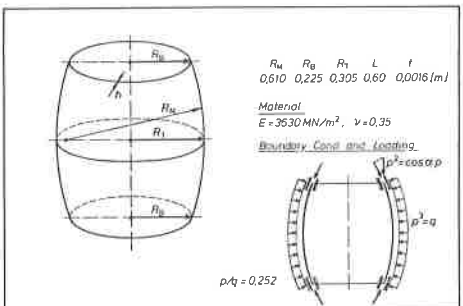


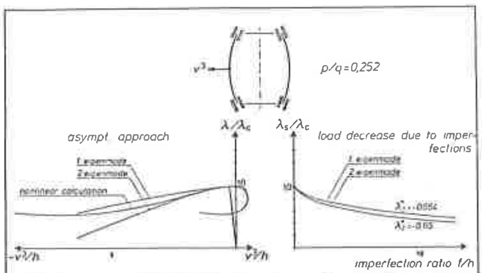
Cylindrical shell under radial pressure — Ruhr-Universität Bochum KIB III - FS 85 - 311



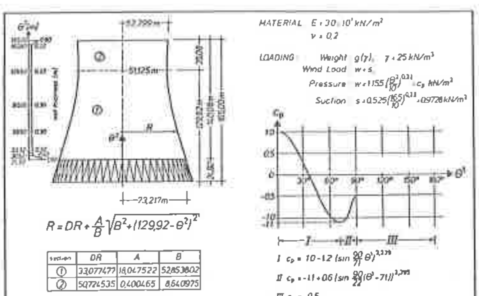
Postbuckling and imperfection sensitivity of a cylindrical shell under pressure load — Ruhr-Universität Bochum KIB III - FS 85 - 312



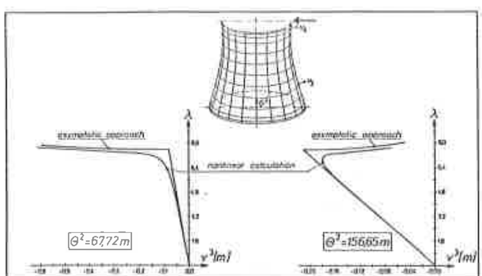
Ellipsoid under combined loading — Ruhr-Universität Bochum KIB III - FS 85 - 313



Ellipsoid under combined loading Postbuckling behavior and imperfection-sensitivity — Ruhr-Universität Bochum KIB III - FS 85 - 314



Geometry and loading conditions of a cooling tower shell — Ruhr-Universität Bochum KIB III - FS 85 - 315



Cooling tower, wind loaded stable symmetric postbuckling behavior — Ruhr-Universität Bochum KIB III - FS 85 - 316

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