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12.* ALMOST SURE INVARIANCE PRINCIPLES FOR CONTINUOUS PARAMETER STOCHASTIC PROCESSES. **

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12.* Almost Sure Invariance Principles for Continuous Parameter Stochastic Processes. **

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12.1 Introduction

In Philipp and Stout (1974a) almost sure invariance principles are established for certain discrete parameter stochastic processes. This work is summarized in Philipp and Stout (1974b). The results of Philipp and Stout (1974a) can be extended in a straightforward fashion to continuous parameter stochastic processes. Here we illustrate this on processes whose increments are either jointly Gaussian or ϕ-mixing.

Let $\{S(t), t \geq 0\}$ denote the process under consideration and let, as usual, $\{X(t), t \geq 0\}$ denote standard Brownian motion. We shall establish the

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almost sure invariance principle

$$S(t) - X(t) \ll t^{\frac{1}{2}-\eta} \quad \text{a.s.}$$

as $t \to \infty$ for some $\eta > 0$. Since, usually

$$S([t]) - X(t) \ll t^{\frac{1}{2}-\eta} \quad \text{a.s.} \tag{12.1.1}$$

is an immediate consequence of a discrete parameter almost sure invariance principle it remains to show, by means of an appropriate maximal inequality, that

$$S(t) - S([t]) \ll t^{\frac{1}{2}-\eta} \quad \text{a.s.} \tag{12.1.2}$$

as $t \to \infty$.

12.2 Processes with Gaussian Increments

Throughout this section we assume that $\{S(t), t \geq 0\}$ is a separable Gaussian process centered at expectations with continuous covariance function

$$R(s,t) = E\{S(s)S(t)\}, \quad 0 \leq s \leq t < \infty,$$

and $S(0) = 0$.

Theorem 12.1: Suppose that, uniformly in s,

$$E\{S(s + t) - S(s)\}^2 = \sigma^2 t + O(t^{1-\varepsilon}) \tag{12.2.1}$$

as $t \to \infty$ for some $\varepsilon > 0$. Suppose that $\sigma^2 > 0$, assuming $\sigma^2 = 1$ without further loss of generality. Moreover, assume that, uniformly in s

$$E\{(S(s+t+1)-S(s+t))(S(s+1)-S(s))\} \ll t^{-2} \tag{12.2.2}$$

as $t \to \infty$. Finally, suppose that

$$E\{S(s+t)-S(s)\}^2 \leq \psi^2(t) \;, \quad 0 \leq t \leq 1 \;, \quad s \geq 0 \;,$$

where ψ is an increasing function satisfying

$$\int_1^\infty \psi(p^{-u^2})\,du < \infty \tag{12.2.3}$$

for some integer $p > 1$. Then without changing the distribution of the process $\{S(t)\}$, we can redefine $\{S(t)\}$ on a richer probability space together with standard Brownian motion $\{X(t)\}$ such that

$$S(t) - X(t) \ll t^{\frac{1}{2}-\eta} \quad \text{a.s.}$$

as $t \to \infty$ for each $\eta < \min(1/60,\ 4\varepsilon/15)$.

Remark: If $R(\cdot,\cdot)$ is differentiable, then the hypothesis (12.2.2) may be replaced by the hypothesis that, uniformly in s,

$$\frac{\partial^2}{\partial u \partial v} R(u,v)\Big|_{s,s+t} \ll t^{-2} \tag{12.2.2a}$$

as $t \to \infty$.

Proof: By (12.2.1) and (12.2.2) it follows from Theorem 5.1 of Philipp and Stout (1974a) (Theorem 7 of Philipp and Stout (1974b)) that (12.1.1) holds on some probability space for each $\eta < \min(1/60,\ 4\varepsilon/15)$. Note that $\{S([t]), S(t)\}$ and $\{S(t),\ X(t)\}$ are perhaps defined on different probability spaces. By choosing $\{S(t)\}$ and $\{X(t)\}$ conditionally independent given

$\{S([t])\}$ on still another probability space the joint distributions of $\{S([t])\}$ and $\{X(t)\}$ as well as of $\{S([t])\}$ and $\{S(t)\}$ are preserved.

By Fernique's lemma (1964) (see also Marcus (1972))

$$P\left\{\sup_{n\le t\le n+1} |S(t) - S(n)| > n^{\frac{1}{4}}\right\} \ll \int_{n^{\frac{1}{4}}}^{\infty} \exp\left(-\tfrac{1}{2}u^2\right)du \ll \exp\left(-n^{\frac{1}{2}}\right) ,$$

which implies (12.1.2) in view of the Borel Cantelli lemma.

<u>Corollary 12.1.</u> Let $\{x(t)\}$ be a measurable stationary Gaussian process, centered at expectations. Denote the covariance function by $r(t - s) = r(s,t)$. Suppose that r is integrable and that

$$\sigma^2 \overset{\text{def.}}{=} 2\int_0^{\infty} r(t)dt > 0 ,$$

assuming $\sigma^2 = 1$ without further loss of generality. Moreover, suppose that

$$r(t) \ll t^{-2} \tag{12.2.4}$$

as $t \to \infty$. Define $\{S(t), t \ge 0\}$ by

$$S(t) = \int_0^t x(s)ds .$$

Then, without changing the distribution of $\{S(t), t \ge 0\}$ we can redefine the process $\{S(t)\}$ on a richer probability space together with standard Brownian motion $\{X(t)\}$ such that

$$S(t) - X(t) \ll t^{\frac{1}{2}-\eta} \quad \text{a.s.}$$

as $t \to \infty$ for each $\eta < 1/60$.

This is an immediate consequence of Theorem 12.1. Indeed, by (12.2.4)

$$E\{S(s+t) - S(s)\}^2 = 2\int_0^t \int_0^v r(u)dudv$$

$$= 2t\int_0^\infty r(u)du - 2t\int_t^\infty r(u)du - 2\int_0^t \int_v^t r(u)dudv$$

$$= t + O(\log t)$$

establishing (12.2.1). Similarly, using the stationarity of $\{x(t)\}$

$$E\{S(s+t) - S(s)\}^2 \le 2r(0)\int_0^t \int_0^v dudv = r(0) \cdot t^2 = \psi^2(t) \quad , \quad \text{say.}$$

Hence (12.2.3) is satisfied. Finally, using (12.2.4)

$$|E\{(S(s+t+1) - S(s+t))(S(s+1) - S(s))\}|$$

$$\le \int_s^{s+1} \int_{s+t}^{s+t+1} |E\, x(u)x(v)|dudv \ll t^{-2} .$$

This proves (12.2.2) and thus the corollary.

Oodaira (1973) shows that the integral

$$S(t) = \int_0^t x(s)ds \tag{12.2.5}$$

of the Ornstein-Uhlenbeck process $\{x(t)\}$ obeys Strassen's functional law of the iterated logarithm. (The Ornstein-Uhlenbeck process has covariance function $r(t-s) = \frac{1}{2}\gamma \exp(-\gamma(t-s))$ for $s \le t$.) It follows from Corollary 12.1 that for $\{S(t)\}$ an almost sure invariance principle holds. In his paper Oodaira considers some more examples of $x(t)$ whose integral $S(t)$ obeys the functional law of the iterated logarithm, such as processes with spectral density $f(x) = A[(\lambda^2 - \beta^2)^2 + \alpha^2\lambda^2]^{-1}$ where $A > 0$ and α, β are real numbers, (the spectral density of the solution to the second order stochast differential equation

$$\frac{d^2x(t)}{dt^2} + \frac{\alpha dx(t)}{dt} + \beta^2 x(t) = AX'(t) ,$$

the Ornstein-Uhlenbeck process being a special case). Corollary 12.1 shows that an almost sure invariance principle holds for the corresponding processes $S(t)$.

It should be remarked that various authors obtain laws of the iterated logarithm and related upper and lower class results for stationary Gaussian processes $\{x(t), t \geq 0\}$ (as contrasted with our study of $S(t)$ given by (12.2.5). See the papers by Oodaira (1972), (1973), by Lai (1973) and by Pathak and Qualls (1973).

12.2 Processes with Mixing Increments

We will consider only processes with ϕ-mixing increments, since it will be obvious how to handle other kinds of mixing. For a given stochastic process, let F_a^b be the σ-field generated by $\{S(t) - S(a), a \leq t \leq b\}$. Then $\{S(t), t \geq 0\}$ is said to have ϕ-mixing increments if there exists a non-decreasing function $\phi(t) \downarrow 0$ such that for $s \geq 0$, $t \geq 0$, $A \in F_0^s$, $B \in F_{s+t}^\infty$ we have

$$|P(AB) - P(A)P(B)| \leq \phi(t)P(A) . \qquad (12.3.1)$$

We assume throughout that $\{S(t)\}$ has ϕ-mixing increments with

$$\int_1^\infty \phi^{\frac{1}{2}}(t)dt < \infty . \tag{12.3.2}$$

We will say that $\{S(t)\}$ has strictly stationary increments if the process $\{S(s+t) - S(s), t \geq 0\}$ has the same distribution for each $s \geq 0$. Suppose throughout that $S(0) = 0$, $ES(t) = 0$, $t \geq 0$ and that for some $\delta > 0$

$$E \sup_{0\leq t\leq 1} |S(t)|^{2+\delta} < \infty . \tag{12.3.3}$$

Now (12.3.2) is easily seen to imply the existence of the limit

$$\sigma^2 = \lim_{n\to\infty} n^{-1} E\, S^2(n) \tag{12.3.4}$$

(see Lemma 4.2.2 of Philipp and Stout (1974a)). We assume throughout that $\sigma^2 > 0$ so that without loss of generality we can assume that $\sigma^2 = 1$.

Theorem 12.2. Let $\{S(t)\}$ be a separable process, with ϕ-mixing and strictly stationary increments, centered at expectations and satisfying (12.3.2), (12.3.3) and (12.3.4) with $\sigma^2 = 1$. Then, on an appropriate probability space,

$$S(t) - X(t) \ll t^{\frac{1}{2}-\eta} \quad \text{a.s.}$$

as $t \to \infty$ for each $\eta < \delta/(12 + 6\delta)$.

Proof: By Theorem 4.1 of Philipp and Stout (1974a) (Theorem 2 of Philipp and Stout (1974b)), relation (12.1.1) holds on some probability space. Separability and stationarity imply that $\sup_{0\leq t\leq 1} |S(t)|$ and $\sup_{n\leq t\leq n+1} |S(t) - S(n)|$ have the same distribution. Thus by (12.3.3)

$$P\left\{\sup_{n\leq t\leq n+1} |S(t) - S(n)| > n^{\frac{1}{2}-\eta}\right\} = P\left\{\sup_{0\leq t\leq 1} |S(t)| > n^{\frac{1}{2}-\eta}\right\}$$

$$\leq E \sup_{0\leq t\leq 1} |S(t)|^{2+\delta}\, n^{-(2+\delta)\left(\frac{1}{2}-\eta\right)} \ll n^{-1-\delta/3} .$$

(12.1.2) follows now from the Borel-Cantelli lemma. The theorem is proved.

For a given stochastic process $\{x(t)\}$ let G_a^b be the σ-field generated by $\{x(t),\ a \le t \le b\}$. Then $\{x(t)\}$ is said to be ϕ-mixing if there exists a nonincreasing function $\phi(t) \downarrow 0$ such that for all $s \ge 0$, $t \ge 0$, $A \in G_0^s$ and $B \in G_{s+t}^\infty$

$$|P(AB) - P(A)P(B)| \le \phi(t)P(A) .$$

We assume throughout that $\{x(t)\}$ is measurable and that for all $t > 0$

$$\int_0^t E|x(s)|^{2+\delta}\, ds < \infty . \qquad (12.3.5)$$

Here $\delta > 0$ is a fixed number. Write

$$S(t) = \int_0^t x(s)ds . \qquad (12.3.6)$$

It is easy to see that if $\{x(s)\}$ is a ϕ-mixing process, $\{S(t)\}$ has ϕ-mixing increments with the same ϕ.

Corollary 12.2. Let $\{x(t)\}$ be strictly stationary, measurable, centered at expectations and ϕ-mixing satisfying (12.3.2) and (12.3.5). Let $S(t)$ be defined by (12.3.6). Suppose $\sigma^2 = 1$ in (12.3.4). Then on an appropriate probability space

$$S(t) - X(t) \ll t^{\frac{1}{2}-\eta} \quad \text{a.s.}$$

as $t \to \infty$ for some $\eta > 0$.

This corollary is an immediate consequence of Theorem 12.2, noting that (12.3.5) implies (12.3.3).

12.4 Markov Processes Satisfying Doeblin's Condition

As an application of the results of the previous section we briefly discuss functionals of Markov processes satisfying Doeblin's condition. Let $(S,\mathcal{B})$ be an abstract state space, where $\mathcal{B}$ is the σ-field of subsets of S. Assume stationary transition probabilities for a Markov process defined on $(S,\mathcal{B})$. Under the Doeblin condition, the Markov transition function $\left(p(t,x,B),\ t \geq 0,\ x \in S,\ B \in \mathcal{B}\right)$ has a unique stationary distribution π. Moreover (see Doob, Stochastic Processes, p. 256) convergence to this distribution is exponentially fast:

$$|p(t,x,B) - \pi(B)| \leq \gamma\rho^{t} \tag{12.4.1}$$

for some $\gamma > 0$, $\rho < 1$, uniformly in $x \in S$ and $B \in \mathcal{B}$. With this in mind, we define a Doeblin process to be a stationary Markov process $\{x(t)\}$ such that (12.4.1) holds. It is easily shown that discrete parameter Doeblin processes are ϕ-mixing with $\phi(n) = C\rho^{n}$, $\rho < 1$.

Theorem 12.3. Let $\{y(t)\}$ be a Doeblin process with abstract state space $(S,\mathcal{B})$. Let f be a real valued function defined on $(S,\mathcal{B})$. Suppose that

$$\{x(t)\} \overset{\text{def.}}{=} \{f(y(t))\}$$

is measurable with

$$E\,x(0) = 0 \quad \text{and} \quad E|x(0)|^{2+\delta} < \infty \tag{12.4.2}$$

for some $\delta > 0$. Let

$$S(t) = \int_0^t x(s)ds$$

define $\{S(t)\}$. Suppose that $\sigma^2 = 1$ in (12.3.4). Then, without changing the distribution of the process $\{S(t)\}$, we can define $\{S(t)\}$ on a richer probability space together with standard Brownian motion such that

$$S(t) - X(t) \ll t^{\frac{1}{2}-n} \quad \text{a.s.} \quad (t \to \infty) \tag{12.4.3}$$

holds for some $n > 0$.

This theorem is an immediate consequence of Corollary 12.2.

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