On the Convexity of Stability or Non Bifurcation Domains in Elasticity and in Plasticity

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The determination of the domain of stability and non-bifurcation is discussed for elastic and plastic structures under multi-factor loads. If conservative loads and small pre-buckling deformation are assumed, it is shown first that in elasticity the stability and non-bifurcation domain is convex in the stress space and in the load parameter space. In plasticity, convexity cannot be established in the load parameter space because of residual stresses and loading history dependence. Their elimination leads to the definition of the safest estimate.

Secondly, the numerical computation of these domains is discussed. Efficient methods to obtain the critical boundary in the load parameter space are considered. In particular, an iterative method is proposed.

1 - INTRODUCTION

In this paper, we are interested in the determination of the domain of stability or non-bifurcation of an elastic or elastic-plastic body, submitted to a combination of conservative loads $\lambda^i P_i$, $i = 1,N$. Classical criteria of stability and bifurcation lead to the definition of critical loads $\Lambda_c(\lambda^1,\dots\lambda^N)_c$ and associated critical stresses σ_c. The critical surface defined in the load parameter space Λ is rather complex in the general case.Therefore, the following assumptions will be admitted here :

- linear and conservative loads : the external force potentiel is a linear function of displacement.
- small pre-deformations : the geometry modification before buckling can be neglected.

In the first part, the convexity of these domains are considered. In elasticity, it is well-known that convexity can be deduced from the criterion of second variation. In plasticity, Hill's criteria of stability and non-bifurcation can be used but the problem is more complicated because of the influence of initial stresses and of the loading history upon the actual stress distribution. Various estimates in the safety sense will be introduced.

In the second part, the numerical determination of the critical surface is discussed. A critical load Λ_c is the solution of and eigenvalue problem ; a point wise construction of this surface is possible but time-consuming. More efficient methods must be introduced. We propose here an iterative procedure based upon the power method.

2 - STABILITY AND NON-BIFURCATION DOMAINS IN ELASTICITY

Let us assume for simplicity that the combined load $\lambda^i P_i$ is defined on a part S_t of the boundary. On the complementary part S_u, no displacement is allowed. We recall that an equilibrium is stable if and only if the potential energy Φ is a local minimum. This condition gives after linearization, the second variation criterion :

$$\text{Stability} \Longleftrightarrow \Phi''v^2 \geqslant 0 \quad \forall\, v \in V_{ad} = \left\{ v \mid v = 0 \quad \text{on} \quad S_u \right\} \tag{1}$$

with :

$$\Phi'' . v^2 = a(v,v) - b_\sigma(v,v)$$

$$a(v,v) = \int_\Omega E_{ijkl} V_{i,j} V_{k,l} \, d\Omega$$

$$b_\sigma(v,v) = \int_\Omega - \sigma_{ij} V_{k,i} V_{k,j} \, d\Omega$$

E_{ijkl} denote the elastic coefficients and σ_{ij} the equilibrium stress. If the following Rayleigh quotient is defined :

$$r(\sigma) = \operatorname*{Sup}_{v \in V_{ad}} \frac{b_\sigma(v,v)}{a(v,v)} \tag{2}$$

then from (1), we obtain :

$$
\begin{array}{lll}
\text{if} & r(\sigma) < 1 & \text{Stable Equilibrium} \\
\text{if} & r(\sigma) > 1 & \text{Unstable Equilibrium} \\
r(\sigma) & = 1 & \text{corresponds to a critical state of stress } \sigma \, .
\end{array}
\qquad (3)
$$

In the stress space, the stability domain is defined by $r(\sigma) \leqslant 1$. In the load parameter space, the stability domain is defined by $R(\Lambda) = r(\sigma(\Lambda)) \leqslant 1$. Since the function $r(\sigma)$ is convex in the sense that $r(\alpha\sigma_o + (1-\alpha)\sigma_1) \leqslant \alpha\, r(\sigma_o) + (1-\alpha)\, r(\sigma_1)$ for whatever σ_o, σ_1 and $0 \leqslant \alpha \leqslant 1$ and the function $\sigma(\Lambda)$ is linear, the function $R(\Lambda)$ is convex. The domain of stability, in the load parameter space, $D = \{\Lambda \mid R(\Lambda) \leqslant 1\}$ is thus convex. This domain is also that of non-bifurcation because in elasticity, stability and non-bifurcation are synonymous [5].

3 - STABILITY OR BIFURCATION DOMAINS IN PLASTICITY

In plasticity, an equilibrium is stable if Hill's stability criterion is verified [2] :

$$
\text{Stability} \iff a^*_\sigma(v,v) - b_\sigma(v,v) > 0 \quad \forall\, v \in V_{ad} \qquad (4)
$$

with :

$$
a^*_\sigma(v,v) = \int_\Omega 2\Phi_\sigma(\varepsilon(v))\,d\Omega
$$

$$
\Phi_\sigma(\varepsilon) = \begin{cases} \frac{1}{2} E_{ijkl}\varepsilon_{ij}\varepsilon_{kl} & \text{if } f(\sigma, A) < 0 \\[2ex] \frac{1}{2} E_{ijkl}\varepsilon_{ij}\varepsilon_{kl} - \frac{1}{2}\dfrac{<\frac{\partial f}{\partial \sigma} \cdot E \cdot \varepsilon>^2}{h + \frac{\partial f}{\partial \sigma} \cdot E \cdot \frac{\partial f}{\partial \sigma}} & \text{if } f(\sigma, A) = 0 \end{cases} \qquad (5)
$$

where h is the hardening modulus, and A is the hardening state. We introduce again the Rayleigh quotient :

$$
r_p(\sigma) = \sup_{v \in V_{ad}} \frac{b_\sigma(v,v)}{a^*_\sigma(v,v)} \; . \qquad (6)
$$

To establish the convexity of $r_p(\sigma)$, it is necessary to specify the evolution of the hardening state. For generalized standard materials i.e. for usual models of plasticity, the proof is based upon the convexity of $f(\sigma, A)$ in the force space $\sigma \times A$ [1]. We obtain thus the convexity of the stability domain in the stress space.

The correspondence $\Lambda \to \sigma(\Lambda)$ is now very complex because the stress distribution σ depends on the loading history $\{\Lambda(\tau), 0 \leqslant \tau \leqslant t\}$ and the initial stress $\sigma(o)$.

In these conditions, the stability domain in the load parameter-space cannot be properly defined. But some estimate in the safety sense can be obtained :

- to eliminate the history dependence and the initial stress, the safest coefficient $\overline{r}_p$ can be introduced as :

$$\overline{r}_p(\Lambda) = \sup_{\substack{f(\sigma,A) \leqslant 0 \\ \sigma \in S(\Lambda)}} r_p(\sigma) \tag{7}$$

i.e. we compute the maximum of $r_p(\sigma)$ among statically admissible (with load Λ) and plastically admissible stress distributions. For example if there is kinetic or isotropic hardening, then we obtain simply :

$$\overline{r}_p(\Lambda) = \sup_{\sigma \in S(\Lambda)} \overline{r}_p(\sigma) \tag{8}$$

where $\overline{r}_p(\sigma)$ is computed from (5) and (6) with $\overline{h} = \text{Min } h$ and $f = 0$ everywhere. This means that the body is assumed to be completely plastic with the smallest tangent modulus.

- Another solution to eliminate the history dependence and the initial stress consists of adopting the deformation theory in stability analysis, since it is widely accepted that deformation theory of plasticity usually gives good estimates of experimental critical values [3].

We can note that $\overline{r}_p(\sigma)$ and $\overline{r}_p(\Lambda)$ are convex functions. It follows that the stability domain D_o defined as $\overline{r}_p(\Lambda) \leqslant 1$ is convex.

But plastic buckling is a bifurcation problem and it has been pointed out that bifurcation does not imply loss of stability [3]. Hill's criterion of non-bifurcation can be written as :

$$a^*_\sigma(v_1 - v_2, v_1 - v_2) - b(v_1 - v_2, v_1 - v_2) \geqslant 0 \quad \forall\, v_1 \neq v_2 \in V_{ad} \quad . \tag{9}$$

Again, loading history and initial stress cannot be negleted and a domain of non-bifurcation cannot be properly defined. But it is not difficult to prove that the Rayleigh quotient $\overline{r}_p$ gives an estimate of the non-bifurcation domain in the safety sense.

4 - NUMERICAL COMPUTATION OF THE CRITICAL SURFACE IN ELASTICITY

Let us consider here the case of two parameters. Formally, the critical loads $\Lambda^1_c = (\lambda^1_c, 0)$, $\Lambda^2_c = (0, \lambda^2_c)$ are the smallest positive eigen values of the spectral problem :

$$Au = \lambda^1 B_1 u$$

$$Au = \lambda^2 B_2 u$$

B_1 and B_2 are associated respectively with the initial stress σ_1 and σ_2 . A_1B_1, B_2 are symmetrical and linear operators, B_1, B_2 are bounded and A^{-1} is compact. The critical surface consists of loading vectors $\Lambda(\lambda\alpha, \lambda(1 - \alpha))$such that λ is the smallest positive eigen value of the spectral problem

$$Au = \lambda\,[\alpha B_1 + (1-\alpha)B_2]\,u \quad . \tag{10}$$

It can be defined point by point, by the resolution of (10) for different values of α. Since such a procedure is relatively expensive, we give here a method to define the critical surface starting from one point by perturbation. Equation (10) may be written as a perturbation

$$Au = \lambda\,[B + \alpha B']\,u\ , \qquad 0 \leqslant \alpha \leqslant 1 \tag{11}$$

of the eigen value problem

$$Au = \lambda\,Bu \tag{12}$$

We take $B = B_2$ and $B' = B_1 - B_2$ if, for example, we wish to obtain the critical surface near Λ_c^2 .

For simplicity, we assume that B is positive. We recall [4] that the eigen values $\lambda_i(\alpha) \geqslant 0$ and the eigen vectors $u_i(\alpha)$ of (11) converge respectively to the eigen values and eigen vectors of the unperturbed problem (12) when $\alpha \to 0$. If λ_1 is simple then $\lambda_1(\alpha)$ is also simple and :

$$\begin{aligned} \lambda_1(\alpha) &= \lambda_1 + \lambda_1^{(1)}\alpha + \lambda_1^{(2)}\alpha^2 + \dots \\ u_1(\alpha) &= u_1 + u_1^{(1)}\alpha + u_1^{(2)}\alpha^2 + \dots \end{aligned} \tag{13}$$

For normed vectors $(u_1, u_1) = 1$, $(u_1(\alpha), u_1) = 1 \quad \forall\,\alpha$, we obtain from (11) and (13) :

$$\begin{aligned} &\bullet\ (A - \lambda_1 B)u_1 = 0 \\ &\bullet\ (A - \lambda_1 B)u_1^{(1)} = (\lambda_1^{(1)}B + \lambda_1 B')u_1 \\ &\quad (A - \lambda_1 B)u_1^{(2)} = (\lambda_1^{(2)}B + \lambda_1^{(1)}B')u_1 + (\lambda_1^{(1)}B + \lambda_1 B')u_1^{(1)} \\ &\quad \text{-------} \\ &\quad (A - \lambda_1 B)u_1^{(n)} = (\lambda_1^{(n)}B + \lambda_1^{(n-1)}B')u_1 + \dots + (\lambda_1^{(1)}B + \lambda_1 B')u_1^{(n-1)} \end{aligned} \tag{14}$$

and :

$$\begin{aligned} \lambda_1^{(1)} &= -\frac{\lambda_1(B'u_1\ ,\ u_1)}{(Bu_1,\ u_1)} \\ \lambda_1^{(n)} &= -\frac{\lambda_1^{(n-1)}(B'u_1,\ u_1) + \dots + ((\lambda_1^{(n-p)}B + \lambda_1^{(n-p-1)}B')u_1^{(p)}\ ,\ u_1) + \dots}{(Bu_1,\ u_1)} \end{aligned} \tag{15}$$

An iterative method to compute $u_1^{(n)}$ is given by (14) :

$$\begin{aligned} &(A - \lambda_1 B)u = f^{(n)} \\ &f^{(n)} = (\lambda_1^{(n)}B + \lambda_1^{(n-1)}B')u_1 + \dots + (\ \lambda_1^{(n-p)}B + \lambda_1^{(n-p-1)}B')u_1 \\ &(u, u_1) = 0 \end{aligned} \tag{16}$$

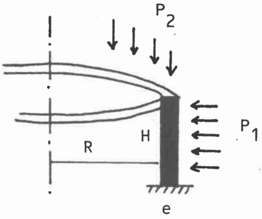
Pratically, we obtain $u_1^{(n)}$ according to the sequence

$$\begin{aligned} Ay^{(m+1)} &= \lambda_1 BV^{(m)} + f^{(n)} \\ V^{(m+1)} &= y^{(m+1)} - (u_1, y^{(m+1)})u_1 \\ \lim_{m \to \infty} V^{(m)} &= u_1^{(n)} \end{aligned} \tag{17}$$

and $\lambda_1^{(n)}$ by the formulae (15).

Example : Cylinder under axial and lateral pressure.

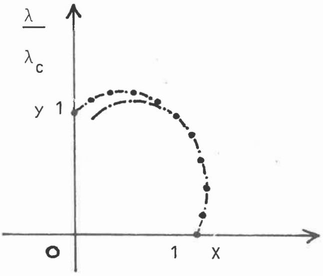
A short cylinder is submitted to axial pressure P_1 and lateral pressure P_2 (fig.1). Only axi-symetric displacement has been considered.



e = 10 mm
R = 12000 mm
H = 500 mm
E = 19.500 hb
ν = 0.3

Figure 1 - Cylinder under axial and lateral pressure

The critical pressures are P_c^1 = 250 hb, P_c^2 = 53.5 hb. The critical surface has been computed by the proposed method starting form X using 9 terms of the series (13) and starting from Y using 8 terms.



● Direct computation

-.-.Iterative method

Figure 2 - Stability domain

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