TWO NEW COMBINATORIAL CORRESPONDENCES

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1. Introduction. Let $f = (x_1, x_2, \ldots, x_n)$ be an arbitrary sequence of real numbers ($n > 0$) and C the set of all sequences that can be formed from f by permutations.

If $g = (x_{i_1}, x_{i_2}, \ldots, x_{i_n})$ is in C, we set

$N(g)$ = number of indices j $(1 \leq j \leq n)$ such that

$$x_{i_1} + x_{i_2} + \ldots + x_{i_j} > 0$$

$L(g) = 0$ if $N(g) = 0$

= smallest index j $(1 \leq j \leq n)$ for which

$$x_{i_1} + x_{i_2} + \ldots + x_{i_j} = \max \{x_{i_1} + x_{i_2} + \ldots + x_{i_k} : 1 \leq k \leq n\}$$

otherwise.

The equivalence principle in the theory of fluctuations of sums of random variables asserts that for any nonnegative integer $m \geq 0$ there are in C as many sequences g such that $N(g) = m$ as sequences g' such that $L(g') = m$. This result was proved by Sparre Andersen (1), then by Feller [6]. Ian Richards (cited in [3]) gave an explicit proof of this result by constructing a permutation φ of C such that

(1) $$N(g) = L(\varphi(g)) \quad \text{for all} \quad g \in C .$$

This construction was then extended by Sparre Andersen [2] to a more general case.

Our purpose here is to present two new equivalence principles on sequences of numbers, i.e. for two other pairs of functions (S, T) and (U, V) defined on C to give the construction of two permutations ψ and θ of C such that

(2) $$S(g) = T(\psi(g))$$

(3) $$U(g) = V(\theta(g))$$

identically.

All the definitions appear in the next section. An example is given in section 3 to illustrate our results. Finally in sections 4 and 5 the constructions of ψ and θ are given.

2. Definitions. In what follows it is assumed that $n > 1$ and the sequence $f = (x_1, x_2, \ldots, x_n)$ be fixed once for all. Let $b : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $b(x, x') = 0$ if $x \leq x'$ and $h : \mathbb{R}^n \to \mathbb{R}$ a symmetric function (i.e. $h(u_{w_1}, u_{w_2}, \ldots, u_{w_n}) = h(u_1, u_2, \ldots, u_n)$ for any permutation $(w_1, w_2, \ldots, w_n)$ of $[1, n]$). Then set for $g = (x_{i_1}, x_{i_2}, \ldots, x_{i_n}) \in C$

$$S(g) = h(b(x_{i_1}, x_1), b(x_{i_2}, x_2), \ldots, b(x_{i_n}, x_n))$$

$$T(g) = h(b(x_{i_1}, x_{i_2}), b(x_{i_2}, x_{i_3}), \ldots, b(x_{i_{n-1}}, x_{i_n}), 0)$$

Note that besides the condition $b(x, x') = 0$ for all pairs (x, x') such that $x \leq x'$ and the fact that h is symmetric, no other assumption is made.

Finally the (U, V)-pair is so defined:

for $g = (x_{i_1}, x_{i_2}, \ldots, x_{i_n}) \in C$, $U(g)$ is the sum of all indices j such that $1 \leq j \leq n-1$ and $x_{i_j} > x_{i_{j+1}}$ and $V(g)$ is the number of couples (j, k) such that $1 \leq j < k \leq n$ and $x_{i_j} > x_{i_k}$.

3. Example. In the following example let us take

$$b(x, x') = (x - x')^+ \quad (= x - x' \text{ if } x - x' > 0 \text{ and } 0 \text{ otherwise})$$

$$h(u_1, u_2, \ldots, u_n) = u_1 + u_2 + \ldots + u_n .$$

Thus we have for $g = (x_{i_1}, x_{i_2}, \ldots, x_{i_n})$

$$S(g) = \sum_{k=1}^{n} (x_{i_k} - x_k)^+ \quad \text{and} \quad T(g) = \sum_{k=1}^{n-1} (x_{i_k} - x_{i_{k+1}})^+$$

Moreover the fixed sequence f is

$$f = (x_1, x_2, x_3, x_4) = (-2, -1, 1, 1)$$

We now list the 12 sequences g that can be formed from f by permutations and the values of the six functions N, L, S, T, U and V taken by each sequence.

	x_{i_1} , x_{i_2} , x_{i_3} , x_{i_4}	N	L	S	T	U	V
f =	-2 , -1 , 1 , 1	0	0	0	0	0	0
	-2 , 1 , -1 , 1	0	0	2	2	2	1
	-2 , 1 , 1 , -1	0	0	2	2	3	2
	-1 , -2 , 1 , 1	0	0	1	1	1	1
	-1 , 1 , -2 , 1	0	0	3	3	2	2
	-1 , 1 , 1 , -2	1	3	3	3	3	3
	1 , -2 , -1 , 1	1	1	3	3	1	2
	1 , -2 , 1 , -1	1	1	3	5	4	3
	1 , -1 , -2 , 1	1	1	3	3	3	3
	1 , -1 , 1 , -2	2	1	3	5	4	4
	1 , 1 , -2 , -1	2	2	5	3	2	4
	1 , 1 , -1 , -2	3	2	5	3	5	5

It is readily seen that every real number r occurs the same number of times in columns N and L (resp. S and T , U and V).

4. The pair of functions (S, T). The combinatorial theorem involving the pair (S, T) is the following

THEOREM ([7] p. 156). Let $f = (x_1, x_2, \ldots, x_n)$ be a sequence of n $(n > 1)$ real numbers and C the set of all sequences that can be formed by permutations. One can construct a permutation ψ of C such that to each $g = (x_{i_1}, x_{i_2}, \ldots, x_{i_n}) \in C$ corresponds a sequence $\psi(g) = (x_{v_1}, x_{v_2}, \ldots, x_{v_n}) \in C$ satisfying the following condition

to each $k \in [1, n]$ for which $x_{i_k} > x_k$

corresponds in a one-to-one manner as index $k' \in [1, n-1]$ such that

$$x_{v_{k'}} = x_{i_k} > x_k = x_{v_{k'+1}} .$$

Identity (2) is satisfied by ψ for if $g = (x_{i_1}, x_{i_2}, \ldots, x_{i_n})$ is an element of C, let us designate by $(k_1, k_2, \ldots, k_m)$ the increasing sequence of indices k such that $x_{i_k} > x_k$ and by $k'_1, k'_2, \ldots, k'_m$ the corresponding indices k' such that $x_{v_{k'}} > x_{v_{k'+1}}$ in the sequence $\psi(g) = (x_{v_1}, x_{v_2}, \ldots, x_{v_n})$.

Then we get, since $b(x, x') = 0$ if $x \leq x'$ and h is a symmetric function,

$$\begin{aligned} S(g) &= h(b(x_{i_1}, x_1), b(x_{i_2}, x_1), b(x_{i_2}, x_2), \ldots, b(x_{i_n}, x_n)) \\ &= h(b(x_{i_{k_1}}, x_{k_2}), b(x_{i_{k_2}}, x_{k_2}), \ldots, b(x_{i_{k_m}}, x_{k_m}), 0, \ldots, 0) \\ &= h(b(x_{v_{k'_1}}, x_{v_{k'_1+1}}), b(x_{v_{k'_2}}, x_{v_{k'_2+1}}), \ldots, b(x_{v_{k'_m}}, x_{v_{k'_{m+1}}}), \ldots, 0, \ldots, 0) \\ &= h(b(x_{v_1}, x_{v_2}), b(x_{v_2}, x_{v_3}), \ldots, b(x_{v_{n-1}} x_{v_n}), 0) \\ &= T(\psi(g)). \end{aligned}$$

Hence identity (2) is satisfied.

If the set C contains $n!$ elements exactly, i.e. if all the x_k 's $(1 \leq k \leq n)$ are distinct, the construction of ψ is easy and is connected with the one used by Spitzer [15] to establish his famous identity.

Let us sketch the construction in this case. Let $g = (x_{i_1}, x_{i_2}, \ldots, x_{i_n})$ be in C. Since all the x_k 's $(1 \leq k \leq n)$ are distinct, the map

$$\bar{g} : x_{i_k} \longrightarrow x_k \quad (1 \leq k \leq n)$$

defined on the set $A = \{x_1, x_2, \ldots, x_n\}$ is a permutation of A that can be written as a product of disjoint cycles

$$\bar{g} = (x_{v_1} x_{v_2} \ldots x_{v_{k_1}})(x_{v_{k_1+1}} \ldots x_{v_{k_2}}) \ldots (x_{v_{k_{s-1}+1}} \ldots x_{v_{k_s}})$$

As each cycle is defined up to a cyclic permutation of its elements, we can assume that for each $p \in [1, s]$

$$x_{v_{k_{p-1}+1}} = \max \left\{ x_{v_{k_{p-1}+1}}, \ x_{v_{k_{p-1}+2}}, \ldots, x_{v_{k_p}} \right\} .$$

Moreover $\bar{g}$ being defined up to a permutation of its cycles, we can assume that the sequence of the first elements of each cycle, i.e.

$$x_{v_1}, x_{v_{k_1+1}}, x_{v_{k_2+1}}, \ldots, x_{v_{k_{s-1}+1}}$$

is increasing. It can be verified that to each $g \in C$ corresponds in a one-to-one manner a permutation $\bar{g}$ that can be written as above.

Then we set

$$\psi(g) = (x_{v_1}, x_{v_2}, \ldots, x_{v_n}).$$

It is easily proved that the conclusions of theorem 1 are satisfied.

Now if the x_k 's $(1 \leq k \leq n)$ are not distinct, one has to make use of a result proved by Schützenberger [14] generalizing the works by Meier-Wunderli [12] , Hall [9] and Chen, Fox & Lyndon [4] on the factorisation of free monoids. It would be too long to reproduce here that construction. The reader will find all the details in [7] (chapter 8).

As a conclusion to this section we mention that the particular case where b is the function equal to 1 if $x > x'$ and 0 otherwise and where h is the function $(u_1 , u_2 , \dots , u_n) \to u_1 + u_2 + \dots + u_n$, has already been considered by MacMahon ([10] p. 186). For this particular couple of functions he proved that for any integer m there are as many elements g in C such that $S(g) = m$ as elements g' such that $T(g') = m$. However no explicit one-to-one correspondence was given.

5. The (U, V)-couple. The functions U and V are defined in section 2. The function U appears in MacMahon ([10] p. 135) in the study of ordered partitions. The function V seems to have been introduced by Netto [13] and rediscovered many times in statistics. $V(g)$ is then called the inversion number of the sequence g and is used in the two-sample problem under the name of Wilcoxon-Mann-Whitney statistic (see e.g. [5]) .

MacMahon [11] showed that the generating functions of U and V have the same expression. In other words if $r_1, r_2, \dots , r_d$ are d $(d > 0)$ real numbers and if the fixed sequence $f = (x_1, x_2, \dots , x_n)$ contains n_1 times the element r_1 exactly, ... , n_d times the element n_d exactly with $n_1 + n_2 + \dots + n_d = n$ and if q is a real or complex variable, then he showed

$$\sum \left\{ q^{U(g)} : g \in C \right\} = \sum \left\{ q^{V(g)} : g \in C \right\} = \frac{\prod_{k=1}^{n} (1 - q^k)}{\prod_{k=1}^{n_1} (1 - q^k) \cdot \prod_{k=1}^{n_2} (1 - q^k) \cdot \cdots \cdot \prod_{k=1}^{n_d} (1 - q^k)} .$$

Hence the fact that for any nonnegative integer m there are in C as many sequences g such that $U(g) = m$ as sequences g' such that $V(g') = m$.

Our purpose is to construct a permutation θ of C that will expressly state this result, i.e. satisfying the identity

(3) $\qquad U(g) = V(\ \theta(g)\ )$ for all $g \in C$.

The properties of this function are not given here. The reader will find them in [8].

To construct θ it will be convenient to consider the set $F(\underset{\sim}{R})$ of all non-empty finite sequences of real numbers. In particular $C \subset F(\underset{\sim}{R})$. For each $r \in \underset{\sim}{R}$ we first define a bijection α_r of $F(\underset{\sim}{R})$ onto itself in the following manner:

$$\text{let } g = (x_{j_1}, x_{j_2} \ldots , x_{j_m}) \in F(\underset{\sim}{R}) ;$$

if $x_{j_m} > r$ (resp. $x_{j_m} \leq r$), we designate by $t_1, t_2, \ldots , t_s$ the increasing sequences of all indices t $(1 \leq t \leq m)$ such that $x_{j_t} > r$ (resp. $x_{j_t} \leq r$).

Then we set

$$\alpha_r (g) = (x_{w_1}, x_{w_2}, \dots, x_{w_m})$$

where first

$$w_{t_1} = t_1 - 1 \qquad \text{if } t_1 > 1$$

$$= t_1 = 1 \qquad \text{if } t_1 = 1$$

$$w_t = t_2 - 1 \qquad \text{if } t_2 - t_1 > 1$$

$$= t_2 \qquad \text{otherwise}$$

......

$$w_{t_s} = t_s - 1 \qquad \text{if } t_s - t_{s-1} > 1$$

$$= t_s \qquad \text{otherwise;}$$

then

$$w_1 = t_1 , w_{t_1+1} = t_2, \dots , w_{t_{s-1}+1} = t_s ;$$

finally for each k different from $1, t_1, t_1+1, t_2, \dots , t_{s-1} + 1,$ t_s , where $w_k = k-1$.

It can be verified that for every $r \in \underset{\sim}{R}$, α_r is a permutation of $F(\underset{\sim}{R})$.

We then define a permutation of ρ of $F(\underset{\sim}{R})$ by induction on the length n of the sequence $g = (x_{i_1}, x_{i_2}, \dots , x_{i_n})$.

If $n = 1$ we set $\rho(g) = g$.

If $n > 1$, if $\rho(x_{i_1}, x_{i_2}, \dots , x_{i_{n-1}}) = (x_{j_1}, x_{j_2}, \dots, x_{j_{n-1}})$

and if $\alpha_{x_{i_n}}(x_{j_1}, x_{j_2}, \ldots, x_{j_{n-1}}) = (x_{u_1}, x_{u_2}, \ldots, x_{u_{n-1}})$

then we set

$$\rho\,(g) = (x_{u_1}, x_{u_2}, \ldots , x_{u_{n-1}} , x_{i_n}) .$$

It can be proved that the restriction θ of ρ on C is a permutation that satisfies identity (3) .

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