REDUCED BASIS METHOD FOR FLOW CONTROL *

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Abstract. This article presents a reduced order method for simulation and control of fluid flows. The major advantage of this method over others such as finite element, finite difference or spectral method is that it has fewer degrees of freedom. The present methodology's feasibility for flow control is demonstrated on two boundary control problems. The first one is a velocity tracking problem in cavity flow and the second one is a vorticity control problem in channel flow. We cast the control problems as constrained minimization problem and compute the control by applying Newton like methods to the necessary conditions of optimality. Our computational experiments seem to indicate the proposed reduced order model's promise.

Key words. reduced basis method, Navier-Stokes equations, finite element, optimal control

AMS subject classifications. 93B40, 49M05, 76D05, 49K20, 65H10

1. Introduction.

Control problems that involve partial differential equations as state equations are formidable problems to solve. One such situation arises in control of fluid dynamical systems in which the state equations are the Navier-Stokes equations, the geometry is often complex and the time interval involved is often very large. If one were to solve such problems using standard finite element or finite difference method the resulting system is prohibitively large.

We in this article discuss a reduction type method which over comes this difficulty. This method hereafter we call reduced basis method uses functions as basis functions which are closely related to the problem that is being solved. This is in contrast to the traditional numerical methods such as finite difference method which uses grid functions as basis functions or finite elements method which uses piecewise polynomials for this purpose.

There are several approaches available for the selection of basis functions. One such approach is Taylor approach in which one uses solutions at a point along with their derivatives as basis functions. Another approach which we call Lagrange approach uses solutions of the problem at various parameter values as basis functions. Finally the Hermite approach is a hybrid of Lagrange and Taylor approaches.

Our goal here is to show the feasibility of reduced basis method for control problems in fluid flows. We will demonstrate this by performing computations on cavity and forward-facing-step channel flow.

1.1. Choices of Reduced Basis Subspaces.

In order to illustrate the reduced basis element approach, we assume for ease in exposition that we are dealing with nonlinear dynamics about the stable equilibrium points. Consider the the parameterized stationary

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problem

$$\mathcal{E}(y,\mu) = 0 \quad \text{for } \mu \in \mathbb{R}\,,\ y \in X, \tag{1.1}$$

where μ represents some physical parameter, for example, Reynolds number or viscosity, about which we choose to interpolate to obtain a reduced finite dimensional set of basis elements. In standard finite element approximations, one approximates X with a piecewise polynomial space X_R. However, the choices for the reduced basis method are different.

The Taylor Subspace. In this choice, one assumes at some value of μ, say μ^*, the solution is known and it has M derivatives then the reduced basis subspace X_R is defined as

$$X_R = \text{span}\{y^j | y^j = \frac{\partial^j y}{\partial \mu^j}|_{\mu=\mu^*}, j = 1, ..., M\},$$

where y^j is obtained from successive differentiation of (1.1), i.e.

$$\mathcal{E}_y(y_0,\mu_0)y_j = \mathcal{F}_j(u_0, u_1, ..., u_{j-1}, \lambda_0). \tag{1.2}$$

For example, u_1 satisfies the equation

$$\mathcal{E}_y(y_0,\mu_0)y_1 = -\mathcal{E}_\mu(u_0,\mu_0).$$

We note here that each y^j can be obtained from its predecessors by solving a linear system with the same coefficient matrix $\mathcal{E}_y(y_0,\mu_0)$. However, one cannot continue to use the same basis vectors generated at fixed parameter μ^* to compute solutions when the parameter of interest is significantly away from it. From time to time, reduced basis vectors have to be updated and the solution is sought in the new reduced basis space. Moreover, generating the right hand side of (1.2) could be quite complicated in certain problems. This choice has been extensively used in the literature, see for e.g [7], [6] for structural analysis problems and [8] for high Reynolds number steady state fluid flow calculations.

The Lagrange Subspace. In this case, the basis vectors are solutions of the nonlinear problem under study at various parameter values μ_j. The reduced subspace is given by

$$X_R = \text{span}\{y^j | y^j = y(\mu_j), j = 1, ..., M\}.$$

This kind of subspace was used to study structural problems in [1]. A possible advantage in this choice is that updating the basis vectors can be done one basis vector at a time instead of generating the whole space.

The Hermite Subspace. This is a hybrid of the Lagrange and Taylor approach. The basis vectors are solutions and their first derivatives at various parameter values μ_j. The reduced subspace is given by

$$X_R = \text{span}\{y^j, y'^j | y^j = y(\mu_j), y'^j = \tfrac{\partial y}{\partial \mu}|_{\mu=\mu_j}, j = 1, ..., M\}.$$

2. The Reduced Basis Method for Navier-Stokes Flows. Let us formulate the reduced basis method for steady viscous incompressible flows governed by the Navier-Stokes equations. The Navier-Stokes equations, when written in primitive variables, are

$$-\nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{2.1}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega \tag{2.2}$$

and

$$\mathbf{u} = \mathbf{b} \quad \text{on } \Gamma, \tag{2.3}$$

where $\mathbf{u}(\mathbf{x})$ and $p(\mathbf{x})$ denote the velocity and pressure, respectively, $\mathbf{f}(\mathbf{x})$ the body force per unit mass, ν the kinematic viscosity. Furthermore $\mathbf{b}$ is the boundary velocity, $\mathbf{u}_0(\mathbf{x})$ is a given function and Ω is a bounded region in $\mathbb{R}^2$ whose boundary is Γ.

We choose variational formulation and finite element method to approximate (2.1)-(2.4) but other methods can also be used with the reduced basis method. Casting (2.1)-(2.4) in appropriate variational form requires introduction of some notations.

2.1. Notations. We denote by $L^2(\Omega)$ the collection of square-integrable functions defined on Ω. Let $H^1 = \left\{v \in L^2(\Omega) : \frac{\partial v}{\partial x_i} \in L^2(\Omega) \text{ for } i = 1,2\right\}$; $H^1_0(\Omega) = \{v \in H^1 : v|_{\partial\Omega} = 0\}$; $L^2_0(\Omega) = \{q \in L^2(\Omega) : \int_\Omega q\, d\Omega = 0\}$; and $H^m = \left\{v \in L^2(\Omega) : \frac{\partial^{|\alpha|}v}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}} \in L^2(\Omega), \text{ for all } \alpha = (\alpha_1,\alpha_2) \text{ with } |\alpha| \le m\right\}$. Vector-valued counterparts of these spaces are denoted by bold-face symbols, e.g., $\mathbf{H}^1 = [H^1]^2$. We define the following standard bilinear and trilinear forms associated with the Navier–Stokes problem

$$a(\mathbf{u},\mathbf{v}) = \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\, d\Omega \qquad \forall\ \mathbf{u},\mathbf{v} \in \mathbf{H}^1(\Omega),$$

$$b(\mathbf{u},q) = -\int_\Omega q\,\nabla\cdot\mathbf{u}\, d\Omega \qquad \forall\ \mathbf{u} \in \mathbf{H}^1(\Omega), \forall\ q \in L^2(\Omega)$$

and

$$c(\mathbf{u},\mathbf{v},\mathbf{w}) = \tfrac{1}{2}\textstyle\int_\Omega\{(\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w} - (\mathbf{u}\cdot\nabla)\mathbf{w}\cdot\mathbf{v}\}\, d\Omega \qquad \forall\ \mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{H}^1(\Omega).$$

For given $\mathbf{b} \in \mathbf{H}^{\frac{1}{2}}(\Gamma)$ and the boundary condition

$$\mathbf{u} = \mathbf{b} \text{ on } \Gamma \text{ with } \int_\Gamma \mathbf{b}\cdot\mathbf{n}\, d\Gamma = 0$$

we define

$$\mathbf{V}_\mathbf{b} = \{\mathbf{u} \in \mathbf{H}^1(\Gamma) :\ \mathbf{u} = \mathbf{b} \text{ on } \Gamma, \quad \mathbf{b} \in \mathbf{H}^{\frac{1}{2}}(\Gamma)\}.$$

We now summarize some properties of these linear forms. We have the coercivity relations associated with $a(\cdot,\cdot)$:

$$a(\mathbf{u},\mathbf{u}) = \|\nabla \mathbf{u}\|_0^2 \geq C_0 \, \|\mathbf{u}\|_1^2 \qquad \forall \; \mathbf{u} \in \mathbf{H}_0^1(\Omega)$$

which is a direct consequence of Poincaré inequality. The forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ and $c(\cdot,\cdot,\cdot)$ are all continuous; in particular, we have

$$\Big| c(\mathbf{u},\mathbf{v},\mathbf{w}) \Big| \leq C_1 \, \|\mathbf{u}\|_1 \, \|\mathbf{v}\|_1 \, \|\mathbf{w}\|_1 \, .$$

The bilinear form $b(\cdot,\cdot)$ satisfies the following inf-sup conditions:

$$\inf_{q \in L_0^2(\Omega)} \; \sup_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)} \int_\Omega q \, \nabla \cdot \mathbf{v} \, d\Omega \geq C_2 \, .$$

2.2. Variational Formulation and Finite Element Approximation. We derive a variational formulation of the problem (2.1)–(2.3) by multiplying both sides of (2.1) and (2.2) by $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$ and $q \in L^2(\Omega)$, respectively, and applying divergence theorem. We obtain

Find $\mathbf{u} \in \mathbf{V_b}$ *and* $p \in L_0^2(\Omega)$ *such that*

$$\tfrac{1}{Re} a(\mathbf{u},\mathbf{v}) + c(\mathbf{u},\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) = (\mathbf{f},\mathbf{v}) \quad \textit{for all } \mathbf{v} \in \mathbf{H}_0^1(\Omega) \tag{2.4}$$

and

$$b(\mathbf{u},q) = 0 \quad \textit{for all } q \in L_0^2(\Omega). \tag{2.5}$$

A typical finite element approximation of (2.4)-(2.5) is to seek solutions $\mathbf{u}^h \in \mathbf{V}_\mathbf{b}^h \subset \mathbf{V_b}$ and $p^h \in S_0^h \subset L_0^2(\Omega)$

$$\tfrac{1}{Re} a(\mathbf{u}^h,\mathbf{v}^h) + c(\mathbf{u}^h,\mathbf{u}^h,\mathbf{v}^h) + b(\mathbf{v}^h,p^h) = (\mathbf{f},\mathbf{v}^h) \quad \text{for all } \mathbf{v}^h \in \mathbf{V}_0^h \tag{2.6}$$

and

$$b(\mathbf{u}^h,q^h) = 0 \quad \text{for all } q^h \in S_0^h, \tag{2.7}$$

where $\mathbf{V}_0^h \subset \mathbf{H}_0^1(\Omega)$ and $S_0^h \subset L_0^2(\Omega)$ are approximating finite element subspaces, and we are setting $Re = \frac{1}{\nu}$ which is the Reynolds number, i.e., the variables are appropriately nondimensionalized.

2.3. The reduced basis method and reduced order model. One set of reduced basisovectors is determined by solving (2.6)-(2.7) for different values of Reynelds number Re. Thus, given a set of values for the Reynolds number $\{Re_i : i = 0,1,2,3,....M\}$, we solve (2.6)–(2.7) $M+1$ times to determine the set $\{\widehat{\mathbf{u}}_m, \widehat{p}_m : m = 0,1,....,M\}$, where $\widehat{\mathbf{u}}_m = \mathbf{u}^h(Re_i)$ and $\widehat{p}_m = p^h(Re_i)$ for $m = 0,1,....,M$. We then set $\mathbf{V}_L^M = \text{span}\{\widehat{\mathbf{u}}_m : m = 0,1,....,M\} \subset \mathbf{V}_0^h$ and $S_L^M = \text{span}\{\widehat{p}_m : m = 0,1,....,M\} \subset S_0^h$. The basis vector generated this way is called *Lagrange reduced basis functions.*

Once we have a reduced basis functions we write the *reduced order model* in the form: seek $\mathbf{u}_R^M(Re^*) \in \mathbf{V}_T^M = \text{span}\{\mathbf{u}_m : m = 0, 1, ..., M\} \subset \mathbf{V}_0^h$ and $p_R^M(Re^*) \in S_T^M = \text{span}\{p_m : m = 0, 1, ..., M\} \subset S_0^h$ such that

$$\tfrac{1}{Re^*} a(\mathbf{u}_R^M, \mathbf{v}_R^M) + c(\mathbf{u}_R^M, \mathbf{u}_R^M, \mathbf{v}_R^M) + b(\mathbf{v}_R^M, p_R^M) = (\mathbf{f}, \mathbf{v}_R^M) \quad \text{for all } \mathbf{v}_R^M \in \mathbf{V}_T^M \tag{2.8}$$

and

$$b(\mathbf{u}_R^M, q_R^M) = 0 \quad \text{for all } q^h \in S_T^M, \tag{2.9}$$

Actually, by construction $\mathbf{u}_R^M(Re^*, \cdot)$ automatically satisfies (2.9) and the system reduces to

$$\tfrac{1}{Re^*} a(\mathbf{u}_R^M, \mathbf{v}_R^M) + c(\mathbf{u}_R^M, \mathbf{u}_R^M, \mathbf{v}_R^M) = (\mathbf{f}, \mathbf{v}_R^M) \quad \text{for all } \mathbf{v}_R^M \in \mathbf{V}_T^M \tag{2.10}$$

Therefore the computation of velocity uncouples from that of the pressure. Note that, due to the global support of the reduced basis elements, the system (2.10) is equivalent to a dense nonlinear system of equations as opposed to the system (2.6)-(2.7) which is a sparse nonlinear system due to the local support of the basis. Thus the reduced basis technique can be effective only when the number basis vectors necessary is small. Our computational experiments and the computations reported for structural problems in the references mentioned earlier seem to indicate that an accurate approximation can be obtained for large range of parameter values using 5 to 10 basis elements. Therefore, although the resulting reduced order model is dense, they are small compared to the sparse but large system that result from the standard basis functions. One of the reasons why the reduced order method requires so fewer elements is that the reduced basis elements quickly become linearly dependent and adding more elements does not really improve the approximation.

3. Control of Reduced Order Model. Let us first formulate an optimal control problem in steady viscous incompressible flow.

$$\text{Minimize} \quad \mathcal{J}(\mathbf{u}, g) \tag{3.1}$$

subject to

$$-\nu\Delta\mathbf{u} + \mathbf{u} \cdot \nabla\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{3.2}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \tag{3.3}$$

$$\mathbf{u}|_{\Gamma_1} = \mathbf{b} \tag{3.4}$$

and

$$\mathbf{u}|_{\Gamma_2} = g\boldsymbol{\chi}. \tag{3.5}$$

We discuss the boundary control problem and thus the body force is $\mathbf{f}$ is fixed. The function g is the control input that influences the flow through the movement of part of the boundary Γ_2, the function $\mathbf{b}$ is a fixed boundary value on Γ_1 and $\boldsymbol{\chi}$ is a characteristic function whose explicit forms will become clear later. We note here that this control mechanism is nondistructive in the sense that no mass is added into the system. We will study two control problems that are cast in the framework of (3.1)–(3.5):

C1 The cavity control problem with the cost function

$$\mathcal{J}(\mathbf{u}, g) = \int_\Omega |\mathbf{u} - \mathbf{u}_d|^2 \, d\Omega.$$

C2 The channel control problem with the cost function

$$\mathcal{J}(\mathbf{u}, g) = \int_\Omega |\nabla \times \mathbf{u}|^2 \, d\Omega.$$

Regarding the set of admissible controls g, we assume
Hypothesis 3.1.1 The set of admissible control g is closed and bounded in $\mathbb{R}$. One can take without loss of generality, $g \in \mathcal{U} = [-1, 1]$.

Defining the set

$$S = \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \, g \in \mathcal{U}, \, \mathbf{u} \text{ satisfies } (3.2)\text{–}(3.5)\}.$$

We have the following lemma whose proof can be given by first defining appropriate extensions $\mathbf{u}^1$ and $\mathbf{u}^2$ to the boundary values and redefining (3.2)–(3.5) with a change of variable $\mathbf{u} = \mathbf{u}^0 + g\mathbf{u}^1 + \mathbf{u}^2$ such that the velocity now satisfies homogeneous boundary values. Then estimating the terms in the variational form of (3.2)–(3.5) using the coercivity and continuity properties of the bilinear and trilinear forms and the antisymmetry property of the trilinear form. We have

LEMMA 3.1. The set S is bounded in $\mathbf{H}^1(\Omega)$. In fact

$$\|\mathbf{u}\|_1 \leq C.$$

Let us state an existence result regarding the control problems **C1** and **C2**.

THEOREM 3.2. Suppose the Hypothesis 3.1.1 holds. Then the control problems **C1** and **C2** have solutions.

A proof of this result follows from the observation that the cost functionals are weakly sequentially lower semicontinuous and bounded below by zero, the solution set S is bounded in in a Hilbert space $\mathbf{H}^1(\Omega)$, the set $\mathcal{U}$ is compact and $\mathbf{H}^1(\Omega)$ is compactly imbedded in $\mathbf{L}^4(\Omega)$. Then if we take a minimizing sequence $(\mathbf{u}_n, g_n) \in S \times \mathcal{U}$, there is a limit $(\mathbf{u}^*, g^*)$ to this sequence and the limit is in fact a minimum to the control problem.

Although these control problems can also be solved by minimization techniques such as conjugate gradient or augmented Lagrangian technique [2] and [3], we in this article use unconstrained minimization techniques based on the necessary condition of optimality. Let us first derive the necessary conditions of optimality for our control problems. To facilitate the forthcoming discussion we cast the control problems in the following abstract setting: *For* $(\mathbf{u}, g) \in \mathbf{H}_0^1(\Omega) \times \mathcal{U}$

$$\textit{Minimize} \quad \mathcal{J}(\mathbf{u}, g)$$

$$\textit{subject to} \quad \mathcal{G}(\mathbf{u}, g) = 0 \textit{ and } \mathcal{H}(\mathbf{u}) = 0.$$

where $\mathcal{G}(\mathbf{u}, g) = 0$ now represents the Navier-Stokes equation and $\mathcal{H}(\mathbf{u}) = 0$ the divergence free condition. Then the Lagrangian formulation can be written as

$$\text{Minimize} \quad \mathcal{J}(\mathbf{u}, g) + < \boldsymbol{\lambda}, \mathcal{G}(\mathbf{u}, g) > + < \sigma, \mathcal{H}(\mathbf{u}) >,$$

where $\boldsymbol{\lambda}$ and σ are Lagrange multipliers whose existence is guaranteed by the regular point condition, i.e. the linearized constraint is surjective. Before discussing the regular point condition further, let us define the variational form of the gradient of the constraints. Given $(\mathbf{v}, q, h) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \mathbb{R}$,

$$\begin{aligned} < \boldsymbol{\Psi}, \mathcal{G}'(\mathbf{u}, g) > + < r, \mathcal{H}'(\mathbf{u}) > = {} & a(\mathbf{v}, \boldsymbol{\Psi}) + h\, a(\mathbf{u}_1, \boldsymbol{\Psi}) + c(\mathbf{v}, \mathbf{u}, \boldsymbol{\Psi}) \\ & + h\, c(\mathbf{u}_1, \mathbf{u}, \boldsymbol{\Psi}) + c(\mathbf{u}, \mathbf{v}, \boldsymbol{\Psi}) + h\, c(\mathbf{u}, \mathbf{u}_1, \boldsymbol{\Psi}) \\ & + b(q, \boldsymbol{\Psi}) + b(\mathbf{v}, r) + h\, b(\mathbf{u}_1, r) \quad \forall (\boldsymbol{\Psi}, r) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega). \end{aligned}$$

We then have the following equivalent solvability condition for the regular point condition:

Setting $\boldsymbol{\Phi} = \mathbf{v} + h\mathbf{u}_1$ the solvability condition can be written as: given $\mathbf{r} \in \mathbf{H}^{-1}(\Omega)$ find $\Phi \in \mathbf{H}^1(\Omega)$, $\nabla \cdot \Phi = 0$ and $r \in L_0^2(\Omega)$ such that

$$\begin{aligned} a(\boldsymbol{\Phi}, \boldsymbol{\Psi}) + c(\boldsymbol{\Phi}, \mathbf{u}, \boldsymbol{\Psi}) + c(\mathbf{u}, \boldsymbol{\Phi}, \boldsymbol{\Psi}) + b(\boldsymbol{\Psi}, r) = < \mathbf{r}, \boldsymbol{\Psi} > \\ \forall \boldsymbol{\Psi} \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \end{aligned}$$

and

$$b(\boldsymbol{\Psi}, q) = 0 \quad \forall q \in L_0^2(\Omega).$$

The solvability of this system can be shown at least when data are small. Next as a result of the regular point condition [5], we have

THEOREM 3.3. Suppose the regular point condition is satisfied. Then we obtain the first order necessary condition for $(\mathbf{u}, g, \sigma, \boldsymbol{\lambda}) \in \mathbf{H}^1(\Omega) \times \mathbb{R} \times L_0^2(\Omega) \times \mathbf{H}^{-1}(\Omega)$

$$\begin{aligned} a(\boldsymbol{\lambda}, \mathbf{v}) + c(\mathbf{v}, \mathbf{u}, \boldsymbol{\lambda}) + c(\mathbf{u}, \mathbf{v}, \boldsymbol{\lambda}) + b(\mathbf{v}, \sigma) + < \mathcal{J}'(\mathbf{u}), \mathbf{v} > = 0 \\ \forall \boldsymbol{\Psi} \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \end{aligned} \tag{3.6}$$

$$a(\mathbf{u}_1, \boldsymbol{\lambda}) + c(\mathbf{u}, \mathbf{u}_1, \boldsymbol{\lambda}) + c(\mathbf{u}_1, \mathbf{u}, \boldsymbol{\lambda}) + b(\mathbf{u}_1, \sigma) + < \mathcal{J}'(\mathbf{u}), \mathbf{u}_1 > = 0 \tag{3.7}$$

and

$$b(\boldsymbol{\lambda}, q) = 0 \quad \forall q \in L_0^2(\Omega). \tag{3.8}$$

The necessary conditions for the minimization problem (3.1)–(3.5) is given by the system (3.2)–(3.8) which characterizes the optimal control and optimal states and we call this optimality system.

3.1. Control of driven cavity flow. In this section we formulate and numerically solve a control problem in driven cavity using reduced basis technique. It is that of finding the bottom surface velocity g such that the fluid velocity $\mathbf{u}$ is driven to a desired state $\mathbf{u}_d$. This control problem can be cast as a minimization problem with the cost function

$$\mathcal{J}(\mathbf{u}) = \int_\Omega |\mathbf{u} - \mathbf{u}_d|^2 \, d\Omega,$$

where $\mathbf{u}_d$ is the desired velocity field and subject to the constraint that the fluid obeys the equation of motion.

Replacing the cost function in **C1** in the abstract problem, the control problem for the driven cavity is written as:

$$\text{Minimize} \quad \mathcal{J}(\mathbf{u}) = \int_\Omega |\mathbf{u} - \mathbf{u}_d|^2 \, d\Omega$$

subject to

$$\tfrac{1}{Re} a(\mathbf{u}, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}) \quad \text{for all } \mathbf{v} \in \mathbf{H}_0^1(\Omega)$$

and

$$b(\mathbf{u}, q) = 0 \quad \text{for all } q \in L_0^2(\Omega),$$

$$\mathbf{u}|_{\Gamma_{\text{top}}} = (\mathrm{U}, 0), \quad \mathbf{u}|_{\Gamma_{\text{bot}}} = (\mathrm{g}, 0),$$

$$\text{and} \quad \mathbf{u}|_{\Gamma_{\text{side}}} = (0, 0),$$

where U and g are top and bottom surface velocities respectively. We wish to find the control input g such that the flow matches as close as possible to a desired flow $\mathbf{u}_d$. The top velocity is fixed throughout the problem. Figure 1 gives the physical domain and the boundary conditions.

Using the reduced basis technique, we now consider the approximate control problem for driven cavity in the following form:

$$\text{Minimize} \quad \mathcal{J}(\mathbf{u}_R^M, g)$$

subject to

$$\tfrac{1}{Re} a(\mathbf{u}_R^M, \mathbf{v}_R^M) + c(\mathbf{u}_R^M, \mathbf{u}_R^M, \mathbf{v}_R^M) = (\mathbf{f}, \mathbf{v}_R^M) \quad \text{for all } \mathbf{v}_R^M \in \mathbf{V}_T^M.$$

Basis vectors are computed with the boundary conditions described in Table 1. Then we seek the solution as $\mathbf{u}_R^M = \mathbf{u}_0 + g\mathbf{u}_3 + \sum_1^2 \alpha_i \boldsymbol{\phi}_i$ where g is the control (tangential velocity at the boundary) and, $\boldsymbol{\phi}_1 = \mathbf{u}_1 - \mathbf{u}_0 - \mathbf{u}_3$ and $\boldsymbol{\phi}_2 = \mathbf{u}_2 - \mathbf{u}_0 + \mathbf{u}_3$.

The computation of optimal control is carried out in two steps: First the necessary conditions of optimality system (3.2)–(3.8) is derived for this problem. Then this system is solved by applying Newton's method. The computations for this problem were done with 29×29 nonuniform mesh and the Reynolds number was 500 ($\nu = 1/500$)). The top wall velocity $U = 1$ is used in our computations and we get control $g^{\text{opt}} = 0.4806$ in 4 Newton iterations. The resulting flow field is given in Figure 3. We also carried out computations to find the flow field corresponding to the optimal control input computed using the reduced order model which is given in Figure 4. They all are in good agreement with the desired flow field given in Figure 2.

3.2. Control of channel flows. In this section, we consider the problem of control channel flows. We will consider the forward-facing step channel, a schematic of this geometry s given in Figure 5. The aim is to shape the flow to a "desired" configuration by means of controlled movement of boundary along some part of the boundary. In this work we consider the minimization of vorticity in the flow. Thus we consider the following cost functional:

$$\mathcal{J}(\mathbf{u}, g) = \int_\Omega |\nabla \times \mathbf{u}|^2 \, d\Omega,$$

where the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$. The control problem for the channel is posed in the following form:

$$\text{Minimize} \quad \mathcal{J}(\mathbf{u}) = \int_\Omega |\nabla \times \mathbf{u}|^2 \, d\Omega \tag{3.9}$$

subject to

$$\tfrac{1}{Re} a(\mathbf{u}, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}) \quad \text{for all } \mathbf{v} \in \mathbf{H}_0^1(\Omega), \tag{3.10}$$

$$b(\mathbf{u}, q) = 0 \quad \text{for all } q \in L_0^2(\Omega), \tag{3.11}$$

$$\mathbf{u}|_{\Gamma_1} = \mathbf{b} \quad \text{and} \quad \mathbf{u}|_{\Gamma_2} = g\boldsymbol{\chi},$$

where Γ_2 is part of the boundary where boundary surface is moving (control input) and Γ_2 is rest of the boundary. Then $\mathbf{b} \in \mathbf{H}^{\frac{1}{2}}(\Gamma)$ corresponds to the inflow, outflow boundary conditions and zero boundary conditions at the walls. Also, $\boldsymbol{\chi} = (\chi_1, \chi_2)$ is a characteristic function defined on the boundary Γ_2 and g is the magnitude of the boundary surface velocity. In the channel geometry our choice of control portion Γ_2 is not the only one possible. But it is motivated by the fact that if one wants maximum influence in the flow, then the control has to be applied in that vicinity.

We assume that the inflow and outflow are parabolic with $u(y) = u_i = y(1 - y/3)/3$ and $u(y) = u_o = 3(3 - y)(y - 1)/8$, respectively. Figure 6 qualitatively demonstrate the situation for high Reynolds number. Our objective in this case is to remove the recirculation that occurs on the top of the step. The boundary conditions with which the basis vectors computed are tabulated in Table 2. Then we set $\mathbf{u} = \mathbf{u}_0 + g\mathbf{u}_5 + \sum_1^4 \alpha_i \boldsymbol{\phi}_i$, where $\boldsymbol{\phi}_1 = \mathbf{u}_6 - 3\mathbf{u}_2 + 2\mathbf{u}_1$, $\boldsymbol{\phi}_2 = \mathbf{u}_6 - 2\mathbf{u}_3 + \mathbf{u}_1$, $\boldsymbol{\phi}_3 = \mathbf{u}_6 - 1.5\mathbf{u}_4 + .5\mathbf{u}_1$, $\boldsymbol{\phi}_4 = \mathbf{u}_6 - 1.2\mathbf{u}_5 + .2\mathbf{u}_1$ and $\boldsymbol{\phi}_5 = \mathbf{u}_6 - \mathbf{u}_1$.

We take the control region to be the line segment between $x = 1$ and $x = 5$ at $y = 3$ we note that at $y = 3$ is where the channel changes its cross section area. Also, we take $\boldsymbol{\chi} = (1, 0)$, that is the movement of the wall is horizontal and $g \in \mathbb{R}$ completely determines the control input.

Then, for the vorticity cost (**C2**), with the Reynolds number 1000 ($\nu = 1/1000$), we obtain the optimal control $g^{\text{opt}} = 0.3041$ in 17 Newton iterations. The resulting flow is shown in Figure 7. We also simulated the flow corresponding to the optimal control computed from the reduced order model and the result is shown in Figure 8. The results show significant reduction in the circulation on the top of the step.

4. Concluding Remarks. Reduced basis approach to control of fluid flow problems is presented. Numerical results are given for driven cavity and channel flow control problems in steady viscous incompressible flows. An extended version of this work [4] will be published elsewhere. Feedback control methods for fluid dynamics based on reduced basis method are currently underway and will be reported in a forthcoming article.

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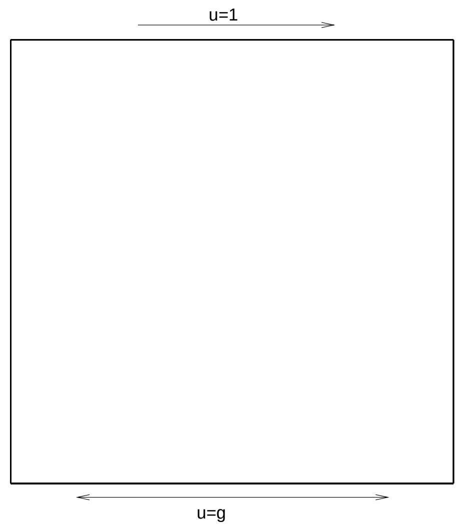

Figure 1.Schematic of controlled driven cavity

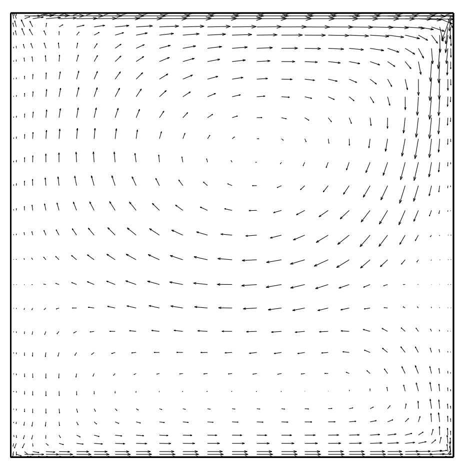

Figure 2.Desired velocity field at Re=500

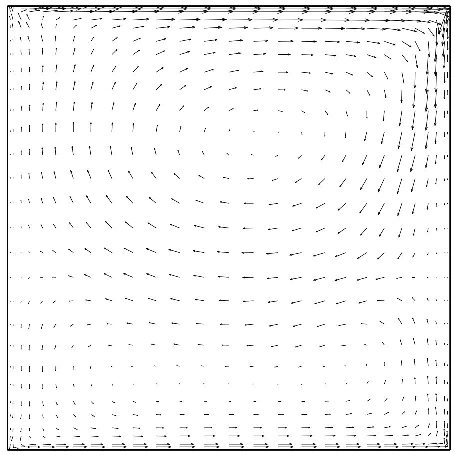

Figure 3.Controlled velocity field at Re=500

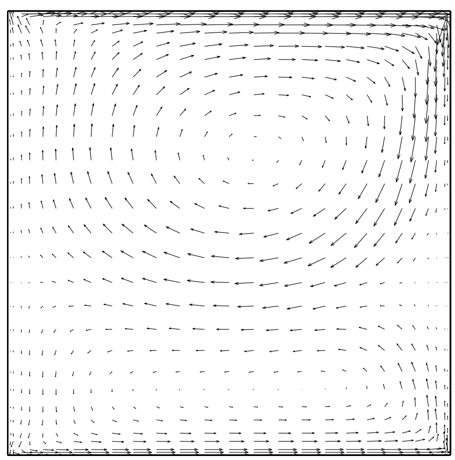

Figure 4.Cavity flow with optimal control input at Re=500

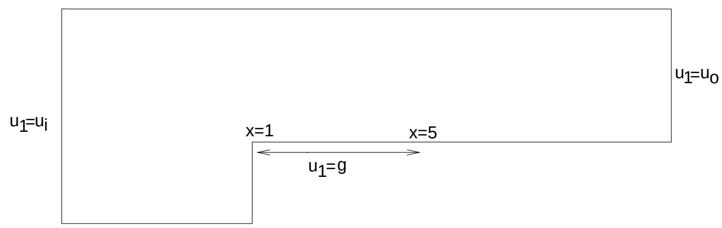

Figure 5.Schematic of controlled forward-facing step channel

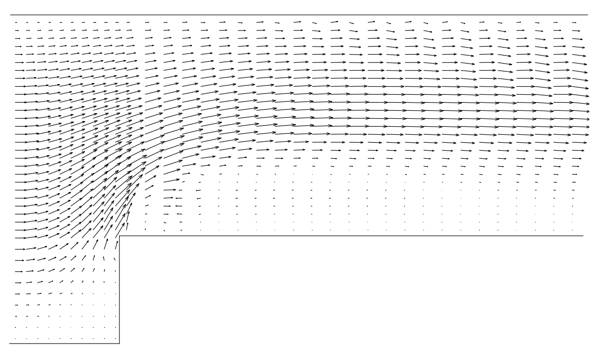

Figure 6.Uncontrolled velocity field at Re=1000

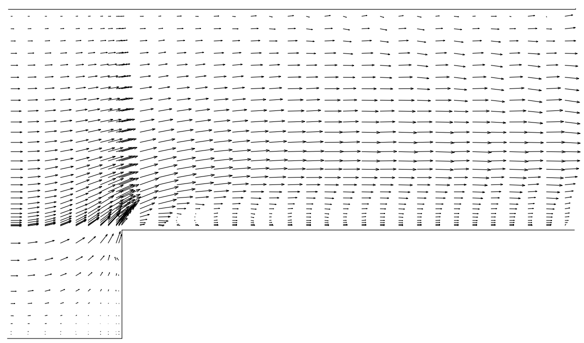

Figure 7.Controlled channel flow at Re=1000

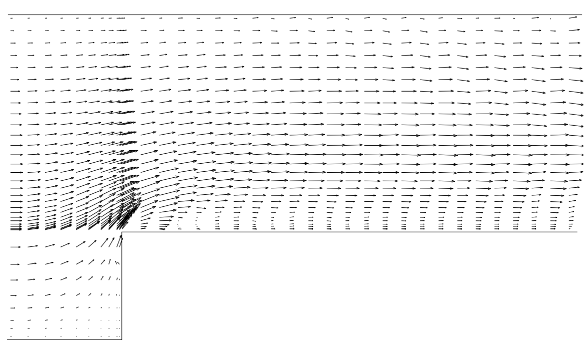

Figure 8. Channel flow with optimal control input at Re=1000

Basis vectors	$\mathbf{u}_0$	$\mathbf{u}_1$	$\mathbf{u}_2$	$\mathbf{u}_3$
Top wall velocity	1	1	1	0
Bottom wall velocity	0	1	-1	1

Table 1. Wall velocities for basis vector generation

Basis vectors	$\mathbf{u}_0$	$\mathbf{u}_1$	$\mathbf{u}_2$	$\mathbf{u}_3$	$\mathbf{u}_4$	$\mathbf{u}_5$
Wall velocity	0	0.1	0.15	0.2	0.25	0.3

Table 2. Wall velocities for basis vector generation