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Crack Propagation of Cracked Plates

Yung-Tze Chen, Ph.D., P.E.
Sinotech Engineering Consultants, Inc.,Taipei, Taiwan

ABSTRACT

This is a study of crack propagation of the linear elastic cracked plates. An analytical method for crack propagation of the cracked plates subjected to a uniform static loading with simply supported boundary conditions is developed by means of the Galerkin method coupled with integral transform method. Through 3-D plot, crack speed ratios for a typical cracked plate will be exclusively presented. Results for this analysis are used to draw conclusion regarding the ability of relating crack speed ratios to aspect-ratios, to crack-length ratios , and to stress intensity factor.

KEY WORDS: crack propagation, crack speed, aspect-ratio, crack-length ratio, stress intensity factor ratio, full-in-depth crack, cracked plate.

INTRODUCTION

The crack propagation of plates with internal crack has been treated extensively in [1,2]. Up-to-date there is little study on the crack propagation of simply supported plate with internal full-in-depth crack.

In this paper, the Galerkin method coupled with integral transform method [3,4,5,6] are applied for determining the crack propagation of full-in-depth cracked rectangular plates simply supported on four edges. Selected typical 3-D plots tied to the above methods are exclusively expressed as well.

Through the proposed methodology, it makes the related physical phenomena of full-in-depth cracked plate crack propagation more clearly understandable than ever .

ANALYSIS AND FORMULATION

A thin, elastic, isotropic, homogeneous simply supported rectangular plate of uniform thickness h with sides a and b containing an internal full-in-depth crack located at (x_k, y_k) is shown in Figure 1.

Based on the Kirchhoff-Love hypothesis, the governing equation of the subject plate is derived by utilizing the following Hamilton's principle:

$$\delta I = \delta \int_{t_1}^{t_2} (T - U - V)dt = 0 \tag{1}$$

where T = the kinetic energy; U = the strain energy; V = the potential energy produced by external loads; and δ = the first variational operator.

The strain energy U is given by

$$U = \frac{1}{2}\int_o^a \int_o^b (M_x K_x + M_y K_y + 2M_{xy} K_{xy})dxdy \tag{2}$$

where K_x, K_y and K_{xy} = the curvatures and twist of the middle surface, respectively, as defined by

$$K_x = - w_{,xx} \tag{3}$$

$$K_y = - w_{,yy} \tag{4}$$

$$K_{xy} = - w_{,xy} \tag{5}$$

where $w(x, y, t)$ = the deflection.

similarly, M_x, M_y and M_{xy} = the bending and twisting moments per unit width, respectively; they are written as

$$M_x = -Dd(x,y)(w_{,xx} + \upsilon w_{,yy}) \tag{6}$$

$$M = -Dd(x,y)\,(w_{,yy} + \upsilon w_{,xx}) \tag{7}$$

$$M_{xy} = -(1-\upsilon)Dd(x,y)\cdot w_{,xy} \tag{8}$$

in which D = the flexural rigidity of a reference plate neglecting full-in-depth cracks, as given by $D = Eh^3/[12(1-\upsilon^2)]$ (in which h = plate thickness; E = Young's modulus; and υ = Poisson's ratio).
Hence, the strain energy U becomes

$$U = \frac{1}{2}\int_0^a\int_0^b Dd(x,y)\cdot\left[(k_x)^2 + (k_y)^2 + 2\upsilon k_x K_y + 2(1-\upsilon)(k_{xy})^2\right]dxdy \tag{9}$$

The first variation of the potential energy δV produced by external lateral loads p can be expressed as [6]

$$\delta V = -\int_0^a\int_0^b (p - c\dot{w})\delta w dxdy + \int_0^b (m_x\delta w_{,x} - m_{xy}\delta w_{,y} + q_x\delta w)dy \Big|_0^b + \int_0^a (m_y\delta w_{,y} - m_{yx}\delta w_{,x} + q_y\delta w)dx \Big|_0^a \tag{10}$$

in which c = damping coefficient; and q_x, q_y, m_x, m_y, m_{xy}, and m_{yx} = external transverse forces and external moments prescribed on boundary edges, respectively.

The kinetic energy T can be written as

$$T = \frac{1}{2}\int_0^a\int_0^b \rho h(\dot{w})^2\,dx\cdot dy \tag{11}$$

in which ρ = the mass density of the liner plate. Substituting Eq. 9. ~ Eq. 11. into Eq. 1., the differential equation of motion can be obtained as

$$\frac{\rho h\ddot{w}}{D} + \frac{c\dot{w}}{D} - \frac{p}{D} + (dw_{,xx})_{,xx} + (dw_{,yy})_{,yy} + \nu(dw_{,xx})_{,yy} + \nu(dw_{,yy})_{,xx} + 2(1-\nu)(dw_{,xy})_{,xy} = 0 \tag{12}$$

associated with the following boundary conditions

either $w = 0$ or $(Ddw_{,xx})_{,x} + \nu(Ddw_{,yy})_{,x} + 2(1-\nu)(Ddw_{,xy})_{,y} - V_x = 0$

along the edges $x = 0$ and $x = a$ (13)

either $w_{,x} = 0$ *or* $Dd(w_{,xx} + \nu w_{,yy}) - m_x = 0$ along the edges $x=0$ and $x=a$ (14)

either $w = 0$ *or* $2(1-\nu)Ddw_{,xy} = m_x$ at each corner (15)

in which V_x = the Kirchhoff's supplementary force. The similar boundary conditions can be written along the edges $y = 0$ and $y = b$.

CRACK SPEEDS OF PLATE WITH FULL-IN-DEPTH CRACK

Let $\zeta = x - x_k - c(t)$ • $\eta = y_k - y$ (16)

Applying Eq. 16. to Eq. 12 for crack speeds (i.e., $p=0$ and $c=0$), the following equation for $\overline{w}(\xi,\eta)$ is obtained.

$$(d\overline{w}_{,xx})_{,xx} + (d\overline{w}_{,yy})_{,yy} + \nu(d\overline{w}_{,xx})_{,yy} + \nu(d\overline{w}_{,yy})_{,xx} + 2(1-\nu)(d\overline{w}_{,xy})_{,xy} + c^2\cdot\frac{\rho h}{D}\overline{w} = 0 \tag{17}$$

The crack speeds of a rectangular plate with full-in-depth crack are presented by means of the Galerkin method. Now $\overline{w}(\zeta,\eta)$ (x,y) is expressed as

$$\overline{w}(\zeta,\eta)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{w}_{mn}\,sin\left(\frac{m\pi(\zeta+x_k+c)}{a}\right)\cdot sin\left(\frac{n\pi(y_k-\eta)}{b}\right)\cdot\frac{1}{\sqrt{2}}\cdot\{[\eta^2+\zeta^2]^{1/2}+\zeta\}^{1/2} \tag{18}$$

or $$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\overline{w}_{mn}\,sin[\frac{m\pi}{a}(r\cos\theta+x_k+c)]\cdot sin[\frac{n\pi}{b}(y_k-r\sin\theta)]\cdot r^{1/2}\cdot cos(\frac{\theta}{2})$$

where $\overline{w}_{mn}$ = the unknown coefficient; and the rest of functions satisfying the simply supported boundary conditions for the subject plate. After lengthy calculation, the Galerkin equation of Eq. 17. can be simplified as

$$\dot{c}(t)=\frac{a}{m\pi}\sqrt{\frac{D}{\rho h}}\{[\frac{\pi^2}{a^2}(m^2+\frac{a^2}{b^2}n^2)]^2+\frac{m\pi}{a}[(-1)^{m+1}c_8+c_9]/c_1+\frac{n\pi}{b}[(-1)^{n+1}c_{10}+c_{11}]/c_1\}^{1/2} \tag{19}$$

where c_1, c_8, c_9, c_{10} & c_{11} refer to Eq.23 to Eq.27

STRESS INTENSITY FACTORS OF PLATE WITH FULL-IN-DEPTH CRACK

Applying Eq. 16 to Eq. 12 for stress intensity factors, firstly the $\overline{w}_{mn}$ is

$$\overline{w}_{mn}=\frac{4pab}{mn\pi^2 D}/\{(\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2})^2c_1+\frac{m\pi}{a}[(-1)^{m+1}c_8+c_9)]+\frac{n\pi}{b}[(-1)^{n+1}c_{10}+c_{11}]\} \tag{20}$$

m, n = 1,3,5,7,……..

Then the stress intensity factors K_I (mode I) can be calculated as

$$K_I=\frac{-Ez}{1-\nu^2}\sqrt{2\pi}\overline{w}_{mn}\{\frac{\nu m\pi}{a}cos(\frac{\theta}{2})\cdot cos(\frac{m\pi(x_k+c)}{a})\cdot sin(\frac{n\pi y_k}{b})$$

$$+\frac{n\pi}{b}sin(\frac{\theta}{2})\cdot sin(\frac{m\pi(x_k+c)}{a})\cdot cos(\frac{n\pi y_k}{b})\} \tag{21}$$

Furthermore the relationship between crack speed and stress intensity factor can be expressed as

$$\dot{c}=\frac{a}{m\pi}\sqrt{\frac{D}{\rho h}}\sqrt{\frac{Ez\sqrt{2\pi}\cdot 4pab\{\frac{\nu m\pi}{a}\cdot cos(\frac{\theta}{2})\cdot cos(\frac{m\pi(x_k+c)}{a}sin(\frac{n\pi y_k}{b})+\frac{n\pi}{b}sin(\frac{\theta}{2})\cdot sin(\frac{m\pi(x_k+c)}{a})cos(\frac{n\pi y_k}{b})}{(1-\nu^2)K_I\cdot mn\pi^2 Dc_1}} \tag{22}$$

where $$c_1=\int_0^a\int_0^b w(x,y)\,sin(\frac{m\pi x}{a})\,sin(\frac{n\pi y}{b})dxdy \tag{23}$$

$$c_8=\int_0^b w_{xx}(a,y)\,sin(\frac{n\pi y}{b})dy \tag{24}$$

$$c_9=\int_0^b w_{xx}(0,y)\,sin(\frac{n\pi y}{b})dy \tag{25}$$

$$c_{10} = \int_0^a w_{yy}(x,b)\sin(\frac{m\pi x}{a})dx \tag{26}$$

;and

$$c_{11} = \int_0^a w_{yy}(x,0)\sin(\frac{m\pi x}{a})dx \tag{27}$$

NUMERICAL RESULTS

The crack propagation of a plate subjected to a uniformly static loading with full-in-depth crack has been presented by means of the Galerkin method coupled with integral transform method.

Figure 2 gives the stress-intensity-factor ratio $(K_I / p\sqrt{2\pi a})$ vs. crack-length ratio $(c/a = 0 \sim 0.45)$ vs. aspect ratio $(b/a = 0.5 \sim 2.0)$. Figure 3 shows the stress intensity factor ratio $(K_I / p\sqrt{2\pi a})$ vs. crack-length ratio $(c/a = 0 \sim 0.45)$ vs. crack-speed ratio $(\dot{c}(t)/cd$ (i.e., crack speed / dilatation wave speed) $=0.05 \sim 1.0)$ with ν =0.25 and (m, n)=(1, 1). Based on Figure 2, the stress intensity factor ratios for cracked plates increase in sinusoid fashion as aspect ratios are increased and increase as crack-length ratios are decreased. According to Figure 3, among the corresponding crack speed ratios the stress intensity factor ratios for cracked plate decrease in sinusoid fashion as crack-length ratios increase.

CONCLUSION

The rational approach for crack propagation of an elastic, isotropic, homogeneous rectangular plate with full-in-depth crack has been proposed with success by means of the Galerkin method coupled with integral transform method.

Conclusion can be drawn in regards to that the stress intensity factor ratios decrease in sinusoid fashion with increasing crack-length ratios and inversely decrease as crack-speed ratios are increased.

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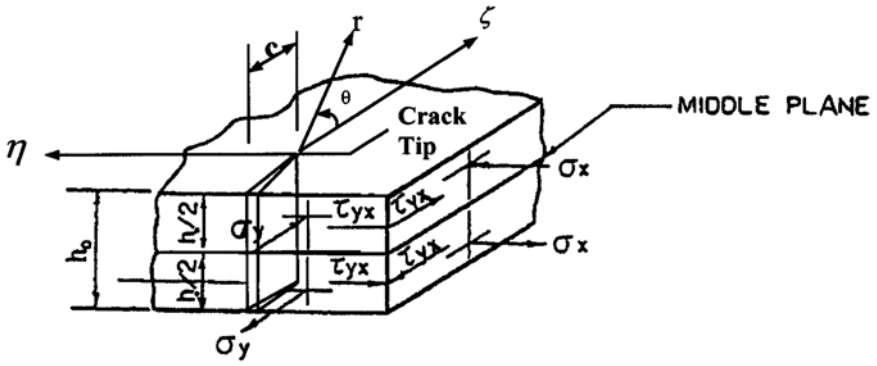
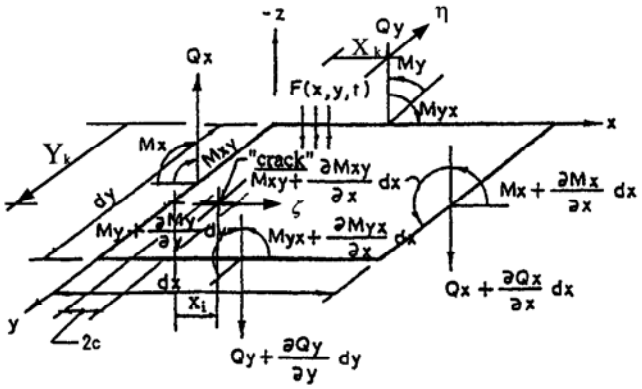


(a)



(b)

Figure 1 (a) Positive State of Stresses for Cracked Plate, (b)Differential Cracked Plate Element (Middle Surface) with Stress Resultants.

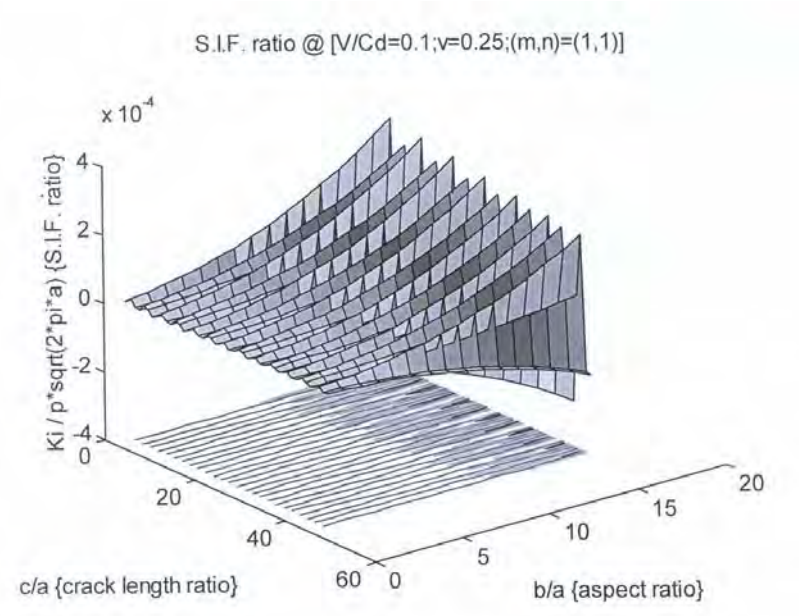


Figure 2 Stress Intensity Factor Ratios due to a Uniform Static Loading.
(V/C_d=0.1; $\nu = 0.25$)

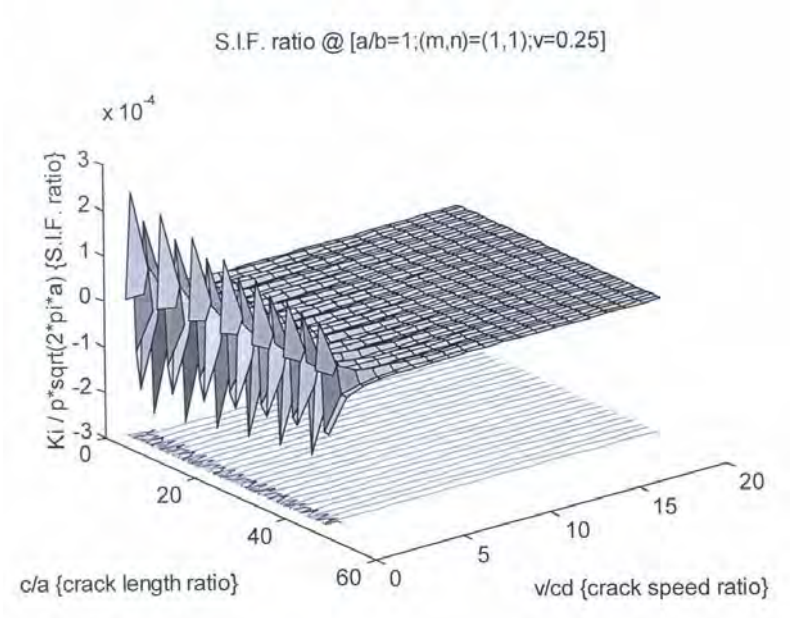


Figure 3 Stress Intensity Factor Ratios due to a Uniform Static Loading.
(a/b=1; $\nu = 0.25$)