NON-LINEAR EFFECTS OF HIGH TEMPERATURE ON THE VIBRATION OF BEAMS WITH TIME DEPENDENT BOUNDARY CONDITIONS

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SUMMARY

In dealing with vibration problems of continuous systems, one generally encounters situations where one or more boundaries are constrained to undergo displacements which are varying with time. For example, a rod subjected at its end to varying external pressure or a plate whose edge is supported by another structure which is oscillating. In such cases the boundary conditions are not "stationary" and for this reason the solutions cannot be obtained by classical methods. Further, if the system is operating in a high temperature environment as in the case of pressure vessels and nuclear reactors, the material behaviour is non-Hookian. The material starts softening at high temperatures and the linear relation between stress-strain is valid only for small strains. For a realistic analysis it is necessary to consider the material behaviour to be non-linear. Mindlin *et al.* analyzed the response of a beam with time dependent boundary conditions. However, the analysis is confined to linear elastic material. Berry *et al.* gave a general solution for the response of a structure with non-homogen boundary conditions.

The present work deals with the influence of non-linear material properties on the dynamic response of beams whose boundaries are constrained to undergo displacements which are varying with time.

The stress-strain relation for the material has been assumed to be of the type,

$$\sigma = E\varepsilon - E^{*}\varepsilon^{m} \tag{1}$$

which has been verified on many materials for the elastic as well as creep strains. The governing equation for the transverse vibration of a uniform slender beam undergoing small deformation can be written as

$$EI\frac{\partial^4 w}{\partial x^4} - E^{*}J\left\{m(m-1)\left(\frac{\partial^3 w}{\partial x^3}\right)^2\left(\frac{\partial^2 w}{\partial x^2}\right)^{m-2} + m\frac{\partial^4 w}{\partial x^4}\left(\frac{\partial^2 w}{\partial x^2}\right)^{m-1}\right\} + \rho\mu\frac{\partial^2 w}{\partial t^2} = 0. \tag{2}$$

The problem has been attempted for four sets of boundary conditions. All other boundary conditions are homogeneous except on displacement, which is given by $(w/h)\cos\omega t$. Solutions have been obtained by perturbation method in conjunction with a finite transform technique and Galerkin's method associated with a transformation suggested by Mindlin and Berry. Response curves have been obtained for different boundary conditions for various values of boundary displacements.

In the later part of the paper the study of the stability of the periodic solution obtained above is made. The governing equation reduces to the well known Mathieu equation. The solution of this equation shows that the transition curve separating the regions of stability and instability is dependent on the magnitude of the forcing function.

1. Introduction

In dealing with vibration problems of continuous system, one quite often encounters situations where one or more boundaries are constrainted in terms of displacements or end moments, which are varying with time. Such is the case in a complex structure where one element interacts with the other. For example, a plate whose edge is supported by another structure which is oscillating or a rod subjected at its end to varying external pressure. In such cases the boundary conditions are not "stationary" and for this reason the solutions cannot be obtained by techniques generally used when the boundary conditions are stationary. Further, if the structure or the system is in a high temperature environment as in the case of pressure vessels, nuclear reactors, the material behaviour is essentially non hookean. The material starts softening with a gradual fall in the elastic modulus with increase in temperature. It was shown by Iyengar and Murthy [1] in a study on beams, that material non-linearity gives rise to softening behaviour unlike structural non-linearity. Further, creep phenomenon becomes important at these high temperatures. For a realistic analysis it is, therefore, necessary to consider the material behaviour to be non-linear to obtain in the response of the system. Mindlin and Goodman [2] obtained the response of a beam with time dependent boundary conditions. However, the analysis is confined to linear elastic material. Berry and Naghdi [3] gave a general solution for the response of a structure with nonhomogeneous boundary conditions. The investigation described here deals with the influence material non-linearities on the dynamic response of slender uniform beams whose boundaries are constrained to undergo displacements which are varying with time. The analysis can be looked upon as a first step towards a complete solution including the effects of hysteresis and creep. In the later part of the paper a study of the stability of the solution is made. Results of the investigation are presented in a graphical form for different boundary displacements.

2. Material Stress-Strain Law

Experiments have shown that for most materials that stress-strain relation is nonlinear except for very small strains. This nonlinearity increases with increase in temperature. Following Ramberg-Osgood [4] , one can write the uniaxial stress-strain relation as

$$\varepsilon = p\sigma + q\,\sigma^{R} \tag{1}$$

where p, q and R are constants depending on the material and temperature. In Ref. [5] extension of this law to include creep strain is described. The constants then become functions of time in addition to temperature. In the study described in this paper, this law has been modified and used in the following form :

$$\sigma = E(T)\,\varepsilon - E^{*}(T)\,\varepsilon^{m} \tag{2}$$

It should be noted that neither relations (1) nor (2) include the hysteresis and creep effects. E and E^{*} are material properties.

3. Governing Equation of Motion

In order to derive the equations of motion governing the transverse oscillations of a slender uniform beam, the following assumptions are made.

(i) The amplitudes of oscillations are small (of the order of half beam thickness).

(ii) Axial movements of the supports are allowed.

(iii) Shear deformation and rotatory inertia effects are neglected.

(iv) Damping and hysteresis effects are neglected.

The equation of motion for transverse oscillation can be written as [6]

$$EI \frac{\partial^4 W}{\partial x^4} - E^*J\{m \frac{\partial^4 W}{\partial x^4} (\partial^2 W/\partial x^2)^{m-1} + m(m-1)\partial^3 W/\partial x^3 \; (\partial^2 W/\partial x^2)^{m-2} + \rho A \frac{\partial^2 W}{\partial t^2} = 0 \qquad (3)$$

Eq. (3) is a non-linear partial differential equation. It is rather difficult if not impossible to obtain the solution of eq. (3) for any general value of 'm' and, therefore, it is necessary to consider diserete values of 'm'. Here the value of 'm' is chosen as 3. The approach however, is quite general for any odd integral value of 'm'.

Introducing non-dimensional parameters,

$$V = W/h \;,\; \xi = x/L \text{ and } \theta = \omega t \;, \qquad (4)$$

the equation governing the transverse motion of the beam reduces to

$$a^2(\partial^4 V/\partial \xi^4) + \omega^2 \; \partial^2 V/\partial \theta^2 - \varepsilon[3(\partial^2 V/\partial \xi^2)^2 \; \partial^4 V/\partial \xi^4 + 6(\partial^3 V/\partial \xi^3)^2 \; \partial^2 V/\partial \xi^2] = 0 \qquad (5)$$

where $\varepsilon = \dfrac{E^* h^6}{80 \, \rho \, L^8}$ and $a^2 = \dfrac{Eh^2}{12 \, \rho \, L^4}$

The parameter ε is of order 10^{-3}, which makes eq. (5) 'weakly non-linear'.

4. Boundary and Initial conditions.

The problem has been attempted for four sets of boundary conditions, namely (i) Simply supported-simply supported (ii) Clamped-clamped (iii) clamped-free and (iv) clamped-simply supported. The equation and methods of obtaining solutions are valid for any other type of boundary conditions. All the conditions on the boundary are homogeneous except on displacement, which is given by $(W/h) \cos \theta$.

To reduce the complexity of the problem, the initial conditions are chosen as follows.

$$V(\xi,0) = f(\xi) \qquad \text{and} \qquad \partial V/\partial t(\xi,0) = 0 \qquad (6)$$

If the boundary conditions are homogencous, then $f(\xi)$ can be chosen as the initial deflected shape. In the present case, however, it has to satisfy the imposed boundary conditions on displacement. For simplicity $f(\xi)$ is chosen as

$$f(\xi) = A_n X_n(\xi) + \varepsilon \frac{W_o}{h} G(\xi) \;,\; W = \varepsilon W_o \qquad (7)$$

where $X_n(\xi)$ is the mode shape and $G(\xi)$ takes care of the boundary conditions.

5. Method of Analysis

Eq. (5) is the governing equation for free vibration of the beam. However, in view of the boundary conditions the problem can be looked upon as a forced vibration problem. The results, therefore, are confined to the steady state response of the system.

Eq. (5) is a non-linear partial differential equation and it is futile to seek for closed form solutions. Hence approximate methods are used to study the behaviour of the system in the neighbourhood of linear system. In this paper perturbation and Mindlin Galerkin techniques have been used to solve the above equations. These two methods are used to serve as a mutual check in view of the lack of any experimental data.

5.1 Perturbation Technique

Perturbation techniques are based on the principle of building up the final solution from a simple basic solution which generally refers to a corresponding linear problem. The displacement $V(\xi,\theta)$ as well as frequency ω are represented by an asymptotic expansion in powers of

$$V(\xi,\theta) = V_o(\xi,\theta) + \varepsilon^2 V_1(\xi,\theta) + \varepsilon^2 V_2(\xi,\theta) + \dots \tag{8a}$$

$$\omega(\varepsilon) = \omega_o + \varepsilon\,\omega_1 + \varepsilon^2\,\omega_2 + \dots \tag{8b}$$

Substituting relations (8a) and (8b) into eq. (5) and the boundary conditions and then grouping the coefficients of different powers of ε yields

$$\varepsilon^{(0)}\ ;\quad a^2 \frac{\partial^4 V_o}{\partial\xi^4} + \omega_o^2 \frac{\partial^2 V_o}{\partial\theta^2} = 0 \tag{9a}$$

$$\varepsilon^{(1)}\ ;\quad a^2 \frac{\partial^4 V_1}{\partial\xi^4} + \omega_o^2 \frac{\partial^2 V_1}{\partial\theta^2} = \left[3 \frac{\partial^4 V_o}{\partial\xi^4}\left(\frac{\partial^2 V_o}{\partial\xi^2}\right)^2 + 6\left(\frac{\partial^3 V_o}{\partial\xi^3}\right)^2 \frac{\partial^2 V_o}{\partial\xi^2}\right] - 2\omega_o\omega_1 \frac{\partial^2 V_o}{\partial\theta^2} \tag{9b}$$

$$\varepsilon^{(2)}\ ;\quad a^2 \frac{\partial^4 V_2}{\partial\xi^4} + \omega_o^2 \frac{\partial^2 V_2}{\partial\theta^2} = 3\left[\frac{\partial^4 V_1}{\partial\xi^4}\left(\frac{\partial^2 V_o}{\partial\xi^2}\right)^2 + 2\,\frac{\partial^4 V_o}{\partial\theta^4}\,\frac{\partial^2 V_1}{\partial\theta^2}\,\frac{\partial^2 V_o}{\partial\theta^2} + 2\,\frac{\partial^3 V_1}{\partial\xi^3}\cdot\frac{\partial^3 V_o}{\partial\xi^2}\,\frac{\partial^2 V_o}{\partial\xi^2} + \frac{\partial^2 V_1}{\partial\xi^2}\cdot\left(\frac{\partial^3 V_o}{\partial\xi^3}\right)^2\right] - 2\omega_o\omega_1 \frac{\partial^2 V_1}{\partial\theta^2} - \omega_1^2 \frac{\partial^2 V_1}{\partial\theta^2} - 2\omega_o\omega_2 \frac{\partial^2 V_o}{\partial\theta^2} \tag{9c}$$

and so on for higher order ε

The boundary conditions for each order of ε are

$$\varepsilon^{(o)}\ ;\quad V_o(0,\theta) = V_o(1,\theta) = 0 \tag{10a}$$

$$\varepsilon^{(1)}\ ;\quad V_1(0,\theta) = V_1(1,\theta) = \frac{W_o}{h}\cos\theta \tag{10b}$$

$$\varepsilon^{(2)}\ ;\quad V_2(0,\theta) = V_2(1,\theta) = 0 \tag{10c}$$

and

$$\frac{\partial^2 V_i}{\partial\xi^2}(0,\theta) = \frac{\partial^2 V_i}{\partial\xi^2}(1,\theta) = \frac{\partial^3 V_i}{\partial\theta^3}(1,\theta) = 0 \qquad \text{for } i = 0,1,2 \tag{10d}$$

The initial conditions are

$$\varepsilon^{(0)}\ ;\ V_o(\xi,0) = \Lambda_n X_n(\xi) \qquad \dot{V}_o(\xi,0) = 0 \tag{11a}$$

$$\varepsilon^{(1)}\ ;\ V_1(\xi,0) = \frac{W_o}{h}G(\xi) \qquad \dot{V}_1(\xi,0) = 0 \tag{11b}$$

$$\varepsilon^{(2)}\ ;\ V_2(\xi,0) = 0 \qquad \dot{V}_2(\xi,0) = 0 \tag{11c}$$

Eqs. (9) are linear partial differential equations with constant coefficients and can be solved as a sequence of linear differential equations with appropriate boundary and initial conditions. It is observed that the linear operator on V_i's same as on the generating solution and equations corresponding to higher order of ε are inhomogeneous form of the basic equation and if are they are subjected to same type of boundary conditions, then the higher order equations will have a nontrivial periodic solution if and only if the inhomogeneous part of them are orthogonal to all the solutions of the adjoint homogeneous equation [7] . This condition is used to determine ω_i's . However, the eq. (9b) is associated with a nonhomogeneous boundary condition. Since the operator is self adjoint a finite transform technique may be employed to eliminate the nonhomogenity in the boundary condition [8] . The transformation is defined as

$$V^*(\alpha_n,\theta) = \int_0^1 V(\xi,\theta)\, X_n(\xi)\, d\xi \tag{12}$$

where $X_n(\xi)$ is defined in the domain of the space variable and satisfies the homogeneous boundary conditions. Detailed steps of analysis for a simply supported beam is given in Ref. [6] .

5.2 Mindlin-Galerkin Technique

The Galerkin technique in its usual form cannot be applied directly as the space variable has to satisfy time dependent boundary conditions. However, the boundary conditions are linear and hence the transformation of the type

$$V(\xi,\theta) = V(\zeta,\theta) + \frac{W}{h}G(\xi)\cos\theta \tag{13}$$

may be used to homogenize the nonhomogeneous boundary conditions. Choice of a function $X(\xi)$, which satisfies both geometric and natural boundary conditions and application of Galerkin's technique leads to an ordinary nonlinear differential equation in terms of $T(\theta)$.

$$\omega^2 \frac{d^2T}{d\theta^2} + \alpha_n^4 a^2 T - \varepsilon \sum_{n=0}^{3} \lambda_n \varepsilon^n \cos^n \theta \, T^{3-n}$$

$$= \varepsilon W^* \cos \theta \qquad (14)$$

where
$$W^* = \frac{W_o}{h} \int_0^1 (a^2 G^{iv} - \omega^2 G) X_n \, d\xi / \int_0^1 X_n^2(\xi) \, d\xi \qquad (15)$$

and
$$\lambda_o = \int_0^1 [3X_n^{iv}(X_n'')^2 + 6(X_n'')^2 X_n'''] X_n(\xi) \, d\xi / \int_0^1 X_n^2(\xi) \, d\xi \qquad (16)$$

$$\lambda_1 = \frac{W_o}{h} \int_0^1 \frac{[6G'' X_n'' X_n^{iv} + 3G^{iv}(X_n'')^2 + 6G''(X_n''')^2 + 12G''' X_n'' X_n'''] X_n(\xi)}{\int_0^1 X_n^2(\xi) \, d\xi} \qquad (17)$$

$$\lambda_2 = \frac{(\frac{W_o}{h})^2 \int_0^1 [3X_n^{iv}(G'')^2 + 6G' G^{iv} X'' + 6(G''')^2 X_n'' + 12G'' G''' X_n'''] X_n(\xi) d\xi}{\int_0^1 X_n^2(\xi) \, d\xi} \qquad (18)$$

$$\lambda_3 = \frac{(\frac{W_o}{h}) \int_0^1 [3G^{iv}(G'')^2 + 6(G''')^2 G''] X_n(\xi) \, d\xi}{\int_0^1 X_n^2(\xi) \, d\xi} \qquad (19)$$

where α_n's are eigenvalues of the functions.

The initial conditions are transformed to

$$T(0) = A_n \int_0^1 X_n^2(\xi) \, d\xi \qquad (20)$$

$$\dot{T}(0) = 0$$

Eq. (14) is solved by Fourier expansion method, wherein the non-linear terms are expanded in terms of a fourier series using the basic linear solution. Replacement of non-linear terms in the form of a series and collection of coefficient of like terms yield a frequency amplitude relation. Ref. [6] gives the detailed calculation for a simply supported beam.

6. Stability of Periodic Solutions

To investigate the stability of the solution, we confine ourselves to the ordinary differential equation derived using Mindlin-Galerkin's approach. A small variation $\tilde{u}$ from the periodic state of equilibrium To is considered. The variational equation for $\tilde{u}$ can be written as

$$\omega^2 \ddot{\tilde{u}} + \omega_0^2 \tilde{u} - \varepsilon [\sum_{m=0}^{2} \lambda_m \varepsilon^m (3-m) A_n^{2-m}] \cos^2 \theta \tilde{u} = 0 \tag{21}$$

Substituting $2\theta = \phi$ the above equation may be reduced to Mathieu's equation as

$$d^2\tilde{u}/d\phi^2 + (\delta + \nu \cos \phi) \tilde{u} = 0 \tag{22}$$

where

$$\delta = \frac{1}{4} [\omega_0^2 - \frac{\varepsilon}{2} \{ \sum_{m=0}^{2} \lambda_m \varepsilon^m (3-m) A_n^{2-m} \}] / \omega^2 \tag{23}$$

and

$$\nu = \frac{1}{4} [- \frac{1}{2} \varepsilon \{ \sum_{m=0}^{2} \lambda_m \varepsilon^m (3-m) A_n^{2-m} \}] / \omega^2 \tag{24}$$

For an equation of this type, the criteria for stability are

$$\delta \leq \frac{1}{4} + \nu/2$$
$$\delta \geq \frac{1}{4} - \nu/2 \tag{25}$$

where in equality sign indicates the transition from stable to unstable zone. Substitution of values for δ and ν in relation (25) and further simplification leads to following expressions

$$\frac{1}{4} \varepsilon \sum_{m=0}^{2} \lambda_m \varepsilon^m (3-m) A_n^{2-m} \leq \omega_0^2 - \omega^2$$
$$\frac{3}{4} \varepsilon \sum_{m=0}^{2} \lambda_m \varepsilon^m (3-m) A_n^{2-m} \geq \omega_0^2 \tag{26}$$

7. Numerical Results and Discussion

Influence of material non-linearity as a consequence of high temperatures on the dynamic response of beams have been studied for various end conditions by perturbation and Mindlin-Galerkin approach. Numerical calculations are done on IBM 7044 Computer. Fig. 1, shows the plot of non-dimensional amplitude versus non-dimensional frequency for various end conditions for the fundamental mode. It is evident from the figure that the influence of material non-linearity on the response is of the softening type. The effect is maximum for the clamped-clamped beam and least for the clamped-free beam.

The above results were obtained by the perturbation technique. Evaluation of higher order terms get more complex. Results have been computed upto $O(\varepsilon^2)$, and it was observed that the convergence was fairly rapid. The ratio of the third term to the second term for the frequency, in the case of simply supported beam was of the order of 0.0062. This ratio, however, increases with the increase in forcing function.

Galerkin's method with one term approximation gave results which are very close to those obtained by perturbation method. Fig. 2 shows the results for a clamped-simply supported beam by both the methods. Both the methods are powerful in the sense that they have the ability to converge to exact solution. It is however, felt that for higher mode calculations, Galerkin's approach may be better in view of the ease of computation.

Fig . 3 shows the plot of response curve for a clamped-free beam along with regions of stability. The solution of the Eq. (21) shows that the transition curve seperating the regions of stability and instability are dependent on the magnitude of the forcing function. It may be of interest to note that in the usual problems of forced vibrations such a phenomenon is seldom observed.

Fig . 4 shows the influence of support excitation on the mode shapes for different end conditions. It may be observed that a support excitation does not result in a rigid body displacement, but changes the amplitude to a considerable extent. This changes has to be considered for the determination of frequency.

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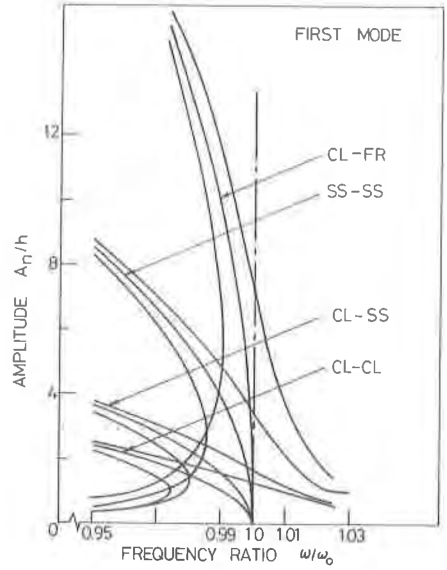

FIG.1. AMPLITUDE FREQUENCY RESPONSE CURVES FOR BEAMS WITH VARIOUS END CONDITIONS.

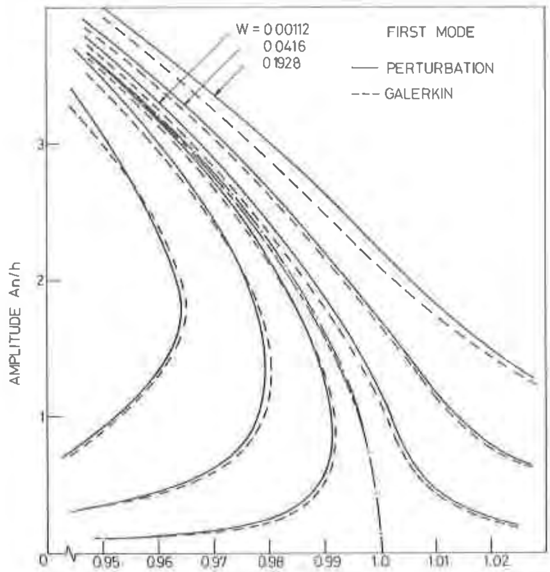

FIG.2. AMPLITUDE FREQUENCY RESPONSE CURVES FOR A CLAMPED-SIMPLY SUPPORTED BEAM-COMPARISON BETWEEN PERTURBATION AND GALERKIN TECHNIQUES.

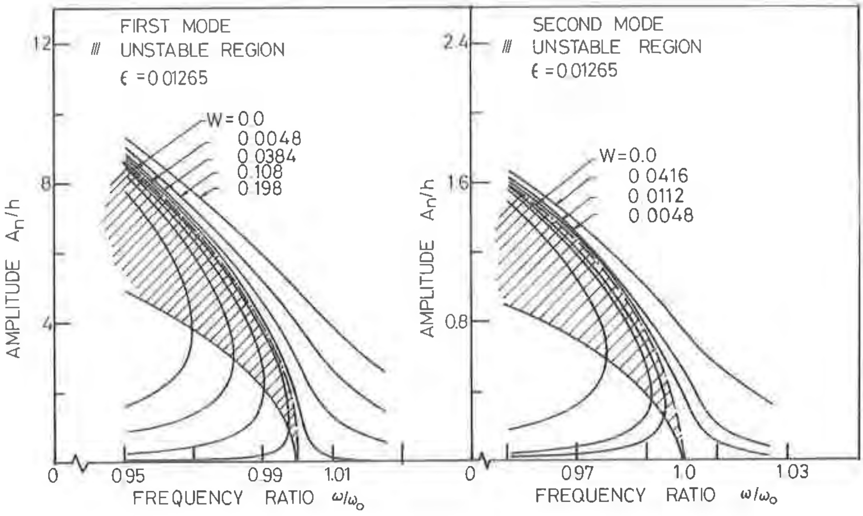

FIG.3. STABILITY CURVES FOR A CLAMPED FREE BEAM.

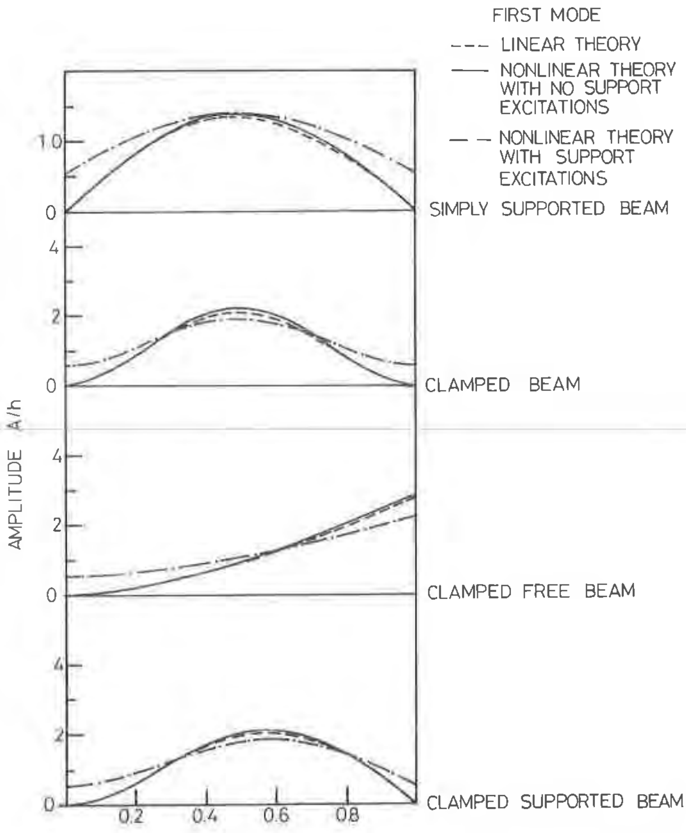

FIG.4. MODE SHAPES FOR BEAMS WITH VARIOUS END CONDITIONS.