ELASTIC ANALYSIS OF CRACKED THIN SHELLS BY THE FINITE ELEMENT METHOD

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ABSTRACT

The stiffness matrix is derived for finite elements with a singular stress distribution, and the procedure is given for its assembly into the global stiffness matrix. In particular, the elastic behaviour of thin shells with linear cracks is analyzed using, in the neighbourhood of the tip of the crack, elements with the asymptotic expansion of the solution as a shape function. Such matrices have been developed and tested for plane stress, plate bending and for cylindrical shells problems.

1. INTRODUCTION

The finite element method, as a procedure for the solution of multivariate boundary value problems, is characterized by approximating the unknown solution function with a piecewise polynomial function. In the stiffness method, for instance, the displacements are approximated by a piecewise polynomial, and so are the stresses, obtained by deriving the displacement distribution; both functions are smooth.

In problems of linear elasticity, however, singular stress distributions may occur because of discontinuities in the loads, or in the geometry (cracks, holes, corners, etc.). In such cases, a straight application of the stiffness method with a locally refined grid of conventional elements provides often poor results at high computational cost. The fact can be explained from well known results of numerical analysis: the errors in a discretized function are magnified in the discretized derivative function, and are dramatized in those fields where the derivative presents sharp variations. Thus, the error in the displacements, due to approximation and round off, blows up the error in the stress distribution where such function presents points of singularity.

Several authors (see for example CHAN et als [2]), starting from an asymptotic expansion of the singularity dependent on parameters, have computed the values of the parameters from the displacements of nodes around the point of singularity; the displacements were obtained with conventional stiffness analysis. A substantial improvement in the results, however, was achieved by Fix [4] in the computation of eigenvalues of elliptic boundary value problems in rectangular polygons. Local asymptotic expansions developed by Lehman represent the eigenfunctions around the corners, and are connected up to the necessary order of derivatives with the polynomial functions representing the eigenfunctions inside the polygon. Fix's approach, in its clear mathematical presentation, offers the groundwork for several extensions.

The present work is specifically oriented to finite element implementation and to thin shell problems. The asymptotic expansions of the displacements are imbedded into special (singular) elements as shape functions; the corresponding stiffness matrices are derived and the connection between the singular and the piecewise polynomial stress fields is performed through a particular procedure of assembly into the global stiffness matrix.

Notice that, quite independently, RAO et als [8] have produced a general formulation for "superelements" that include singularities and discontinuities; the implementation is restricted to plane stress problems.

2. SINGULAR FINITE ELEMENTS

Let P be a point where a stress concentration occurs, and S be a finite element that contains P (see fig.1). The element S is called here "singular" if the displacement and stress distributions across the element coincide with the asymptotic expansion of the displacement and of the stress functions around the singular point P. For analytic development of singular finite elements the singularity in the stress distribution must be of the removable type, and the stress functions must be square summable across the element.

2.1 THE ELEMENT STIFFNESS MATRIX

Let the dispacement vector $\underline{u}$ be represented within the singular element as

$$\underline{u} = F\, \underline{c} \qquad (1)$$

where F is a matrix of functions that contains the asymptotic expansions and the rigid body displacements of the element, and $\underline{c}$ is a vector of constants.

The number of nodes in the singular element is determined by the connection with the adjoining elements and by the necessity of computing the constants $\underline{c}$. The degrees of freedom, $\underline{r}_i$, that define $\underline{c}$ are called independent degrees of freedom; the relationship is given as

$$\underline{c} = C\,\underline{r}_i \qquad (2)$$

From eqs. (1) and (2)

$$\underline{u} = F\,C\,\underline{r}_i = U\,\underline{r}_i \qquad (3)$$

The strain vector $\underline{e}$ is

$$\underline{e} = E\,\underline{c} = EC\underline{r}_i = B\underline{r}_i \qquad (4)$$

Let $\underline{s} = S\,\underline{e}$ represent the stress-strain relationship, and

$$K = \int_V B^T SB \;dV \qquad (5)$$

be the usual definition (Zienkiewicz, [12]) of the stiffness matrix for an element of volume V; by replacing eq.(4) into eq.(5) one defines the stiffness matrix of the singular element as

$$K = C^T \;(\int_V E^T SE\;dV)\;C \qquad (6)$$

The singular functions in the matrix E must be such that the volume integral in eq.(6) is finite. That implies that the strain energy in the singular element must be finite.

The nodal loads $\underline{P}_i$ equivalent to a distributed load $\underline{p}$ are given by

$$\underline{P}_i = \int_V U^T \underline{p}\;dV = C^T \int_V F\;\underline{p}\;dV \qquad (7)$$

The volume integral in (7) must also be finite.

2.2 ASSEMBLY OF THE GLOBAL STIFFNESS MATRIX

Assembling the stiffness matrix of the singular element, given by equation (6), into the global stiffness matrix of the structure may require some special procedures.

Let $n_i = |\underline{r}_i|$ be the total number of independent degrees of freedom and n_a the total number of degrees of freedom of the nodes adjoining the singular element. Notice that n_a is determined by the number and by the type of conventional elements surrounding the singular element. In fig.1, for instance, assuming two degrees of freedom per node: $n_i = 4$ and $n_a = 16$.

When $n_a = n_i$ the stiffness matrix of the singular element is assembled as the matrix of a conventional element.

When $n_a < n_i$, the redundant degrees of freedom of the singular element must be associated with internal nodes.

When $n_a > n_i$, the most frequent case, the redundant degrees freedom of the adjoining nodes are subjected to the behaviour of the indipendent degrees of freedom; $n_d = n_a - n_i$ is the total number of dependent degrees of freedom.

The dependent degrees of freedom are "condensed" according to the following procedure.

Let K'_{ij} be submatrices of the global stiffness matrix prior to the assembly of the stiffness matrix of the singular element, and let $\underline{r}_j$ be the corresponding components of the unknown displacement vector. Considering $\underline{P}'_i$ as the part of the global nodal loads $\underline{P}_i$ that is supported by the conventional elements, one writes:

$$\ldots + K'_{ee}\,\underline{r}_e + K'_{ei}\,\underline{r}_i + K'_{ed}\,\underline{r}_d = \underline{P}'_e \tag{8}$$

$$K'_{ie}\,\underline{r}_e + K'_{ii}\,\underline{r}_i + K'_{id}\,\underline{r}_d = \underline{P}'_i \tag{9}$$

$$K'_{de}\,\underline{r}_e + K'_{di}\,\underline{r}_i + K'_{dd}\,\underline{r}_d = \underline{P}'_d \tag{10}$$

From eq.(3) one obtains

$$\underline{r}_d = R_d\,\underline{r}_i \tag{11}$$

The dual relationship applies for the loads, namely

$$\underline{P}''_{id} = R_d^T\,\underline{P}''_d \tag{12}$$

where $\underline{P}''_d$ represents the portion of the global nodal loads $\underline{P}_d$ supported by the singular element, and $\underline{P}''_{id}$ are the loads induced by $\underline{P}''_d$ onto the indipendent degrees of freedom.

Let K''_{ii} be the stiffness matrix of the singular element; and let $\underline{P}''_i$ be the part of the global nodal loads $\underline{P}_i$ that is supported by the singular element; one writes.

$$K''_{ii}\,\underline{r}_i = \underline{P}''_i + \underline{P}''_{id} = \underline{P}''_i + R_d^T\,\underline{P}''_d \tag{13}$$

Because of equilibrium

$$\underline{P}_i = \underline{P}'_i + \underline{P}''_i \tag{14}$$

$$\underline{P}_d = \underline{P}'_d + \underline{P}''_d \tag{15}$$

Adding to eq.(9), eq.(13), and eq.(10) premultiplied by R_d^T, and taking into account eqs.(11),(14) and (15), one obtains the condensed version of eqs.(9) and (13)

$$K^o_{ie}\,\underline{r}_e + K^o_{ii}\,\underline{r}_i = \underline{P}^o_i \tag{9'}$$

where:

$$K^{o}_{ie} = K'_{ie} + R^{T}_{d} \; K'_{de}$$

$$K^{o}_{ii} = K'_{ii} + K''_{ii} + R^{T}_{d} \; K'_{id} + R^{T}_{d} \; K'_{dd} R_{d} + K'_{id} \; R_{d}$$

$$\underline{p}^{o}_{i} = \underline{p}_{i} + R^{T}_{d} \; \underline{p}_{d}$$

Similary, introducing eq.(11) into eq.(8), one obtains the condensed version of eq.(8).

$$....+ K'_{ee} \; \underline{r}_{e} + K^{o}_{ei} \; \underline{r}_{i} = \underline{p}'_{e} \qquad (8')$$

Notice that $K^{o}_{ei} = K^{oT}_{ie}$ because of the symmetry of the global stiffness matrix.

3. SINGULAR ELEMENTS FOR LINEAR CRACKS.

The procedure indicated in section 2 is quite general. As an application, singular elements are developed here for the analysis of stress concentrations occurring in thin shells at thetip of a crack. Namely, plane stress, plate bending and cylindrical shell problems are taken into consideration, under the hypotheses that the radius of the crack at the tip is zero (linear crack),and of linearly elastic behaviour.

It can be shown(see for instance [10])that,regardless of the geometry and of the boundary conditions, the stress in the neighbourhood of the crack grows with factor $1/\sqrt{r}$, where r is the distance from the tip of the crack.

3.1 SINGULAR PLANE STRESS ELEMENT

For plane stress problems,Irwin [7] gives the following stress distribution around the tip of a linear crack.

$$\sigma_{x} = \frac{k_{I}}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \left(1-\sin\frac{\theta}{2}\sin 3\frac{\theta}{2}\right) - \frac{k_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2} \left(2+\cos\frac{\theta}{2}\cos 3\frac{\theta}{2}\right) \qquad (16.1)$$

$$\sigma_{y} = \frac{k_{I}}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \left(1+\sin\frac{\theta}{2}\sin 3\frac{\theta}{2}\right) + \frac{k_{II}}{\sqrt{2\pi r}} \sin\frac{\theta}{2} \cos\frac{\theta}{2} \cos 3\frac{\theta}{2} \qquad (16.2)$$

$$\tau_{xy} = \frac{k_{I}}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \sin\frac{\theta}{2} \cos 3\frac{\theta}{2} + \frac{k_{II}}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \left(1 - \sin\frac{\theta}{2}\sin 3\frac{\theta}{2}\right) \qquad (16.3)$$

where k_I is the "stress intensity factor" associated with the symmetric mode of expansion of the crack, and k_{II} the stress intensity factor associated with the antisymmetric mode. The displacements u,v, corresponding to the stress field of eq.(16) are

$$u = \frac{k_I}{2G} \frac{\sqrt{r}}{\sqrt{2\pi}} \cos\frac{\theta}{2}\,(g-1+2\sin^2(\frac{\theta}{2})) + \frac{k_{II}}{2G}\frac{\sqrt{r}}{\sqrt{2\pi}} \sin\frac{\theta}{2}\,(g+1+2\cos^2(\frac{\theta}{2})) \qquad (17.1)$$

$$v = \frac{k_I}{2G} \frac{\sqrt{r}}{\sqrt{2\pi}} \sin\frac{\theta}{2}\,(g+1-2\cos^2(\frac{\theta}{2})) - \frac{k_{II}}{2G}\frac{\sqrt{r}}{\sqrt{2\pi}} \cos\frac{\theta}{2}\,(g-1-2\sin^2(\frac{\theta}{2})) \qquad (17.2)$$

where $g=(3-4\nu)$ for plane strain and $g=(3-\nu)/(1+\nu)$ for plane stress.

Let S be a singular element as in fig.2; the tip of the crack is in O. Notice that the most suitable shape for S would be a circular one, because of the axisymmetry of the displacement function of eq.(17). The radius of the element should not exceed 1/20 of the crack, in order to preserve the range of validity of eqs.(16) and (17).

Adding rigid body displacements c_3 and c_4 to eqs.(17), one obtains the displacement shape function for the element, in the following compact form

$$u = F_{11}c_1 + F_{12}\,c_2 + c_3 \qquad (18.1)$$

$$v = F_{21}c_1 + F_{22}\,c_2 + c_4 \qquad (18.2)$$

where $c_1 = k_I/\sqrt{\pi}$, $c_2 = k_{II}/\sqrt{\pi}$, while F_{ij} are functions easy to derive from eqs.(17).

The matrix F in eq.(1) assumes here the form

$$F = \begin{vmatrix} F_{11} & F_{12} & 1 & 0 \\ F_{21} & F_{22} & 0 & 1 \end{vmatrix} \qquad (19)$$

In order to compute the four constants $c_1,\ldots,c_4$, two nodes are selected to provide the independent degrees of freedom; for computational convenience such nodes are placed as in position 1 and 2 of fig.2. Calling $(F_{ij})_1$ and $(F_{ij})_2$ the values that function F_{ij} assumes on nodes 1 and 2, and observing that

$$(F_{11})_1 = (F_{11})_2 = (F_{22})_1 = (F_{22})_2 = 0$$

$$(F_{12})_2 = -\,(F_{12})_1$$

$$(F_{21})_2 = -\,(F_{21})_1$$

one obtains

$$c_1 = \frac{r_2 - r_4}{2\,(F_{21})_1} \qquad c_3 = \frac{r_1 + r_3}{2} \qquad (20.1)$$

$$c_2 = \frac{r_1 - r_3}{2\,(F_{12})_1} \qquad c_4 = \frac{r_2 + r_4}{2} \qquad (20.2)$$

and finally, the matrix C is

$$C = \begin{vmatrix} 0 & 1/2(F_{21})_1 & 0 & -1/2(F_{21})_1 \\ 1/2(F_{12})_1 & 0 & -1/2(F_{12})_1 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{vmatrix} \qquad (21)$$

From matrix C and from the geometry of the singular element the stiffness matrix K can be computed through eq.(10). A few simplifications, however, may be introduced.

The rigid body motions c_3 and c_4 do not contribute to the strain vector; thus in eq.(4) $E_{13} = E_{14} = 0$. The remaining terms in matrix E can be computed from eq.(16) applying Hooke's law. Notice that E_{j1} and E_{i2} are functions respectively symmetric and antisymmetric in y. The integral of the product E_{j1} E_{j2} is zero if the singular element has y =o as plane of symmetry.

If $A = \{a_{ij}\}$ is the matrix representing the volume integral in eq.(6), then for the previous argument $a_{ij} = 0$ if $i \neq j$. The final expression for K is

$$K = \begin{vmatrix} \frac{a_{22}}{4(F_{12})_1^2} & 0 & -\frac{a_{22}}{4(F_{12})_1^2} & 0 \\ 0 & \frac{a_{11}}{4(F_{21})_1^2} & 0 & -\frac{a_{11}}{4(F_{21})_1^2} \\ -\frac{a_{22}}{4(F_{12})_i^2} & 0 & \frac{a_{22}}{4(F_{12})_1^2} & 0 \\ 0 & -\frac{a_{11}}{4(F_{21})_1^2} & 0 & \frac{a_{11}}{4(F_{21})_1^2} \end{vmatrix} \qquad (22)$$

3.2. PARTICULAR CASES

When the plane y = 0 is a plane of symmetry or antisymmetry for the structure, only half of the element and only one independent node are taken into account. In the case of symmetry, the constants c_2 and c_4 multiplying antisymmetric functions are zero. Equations (20) an (21) then take the form

$$c_1 = \frac{r_2}{(F_{21})_1} \qquad (23.1)$$

$$c_3 = r_1 \qquad (23.2)$$

$$C = \begin{vmatrix} 0 & 1/(F_{21})_1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{vmatrix} \tag{24}$$

and finally the matrix K

$$K = \begin{vmatrix} 0 & 0 \\ 0 & a_{11}/(F_{21})_1^2 \end{vmatrix} \tag{25}$$

In the case of antisimmetry, starting from $c_1 = c_3 = 0$, one obtains with an analogous procedure

$$K = \begin{vmatrix} a_{22}/(F_{12})_1^2 & 0 \\ 0 & 0 \end{vmatrix} \tag{26}$$

Notice that the volume integral in the computation of a_{11} and a_{22} extends now to half element and affects only the value of a_{11} and a_{22}.

For a full rectangular element of uniform thickness t (fig.3a), the computation of a_{11}, a_{22} gives

$$a_{11} = \frac{a\ t}{E_t} [(3-\nu) \log \frac{\cos \alpha}{1-\sin\alpha} - (1+\nu) \sin \alpha - - 2(1-\nu)\beta \log tg \frac{\alpha}{2} + (1+\nu)\beta \cos \alpha] \tag{27.1}$$

$$a_{22} = \frac{a\ t}{E_t} [(3-\nu) \log \frac{\cos \alpha}{1-\sin\alpha} + 3(1+\nu) \sin \alpha - - 2(3+\nu) \beta \log tg \frac{\alpha}{2} - 3(1+\nu)\beta \cos \alpha] \tag{27.2}$$

where E_t is Young's modulus of the material.

For a full circular element (fig.3b) of uniform thickness t

$$a_{11} = \frac{a\ t}{4\ E_t} (5-3\nu) \tag{28.1}$$

$$a_{22} = \frac{a\ t}{4\ E_t} (9+\nu) \tag{28.2}$$

3.3 SINGULAR PLATE BENDING ELEMENT

For plate bending problems Williams [11] gives the following stress distribution around the tip of the crack.

$$\sigma_x = -\frac{1-\nu}{2(3+\nu)}\frac{k_1 z}{t\sqrt{2r}}\left[3\cos\frac{\theta}{2}+\cos 5\frac{\theta}{2}\right]+ \qquad (29.1)$$

$$+\frac{1-\nu}{2(3+\nu)}\frac{k_2 z}{t\sqrt{2r}}\left[\sin 5\frac{\theta}{2}-\left(\frac{9+7\nu}{1-\nu}\right)\sin\frac{\theta}{2}\right]$$

$$\sigma_y = \frac{1-\nu}{2(3+\nu)}\frac{k_1 z}{t\sqrt{2r}}\left[\cos 5\frac{\theta}{2}+\frac{11+5\nu}{1-\nu}\cos\frac{\theta}{2}\right]+ \qquad (29.2)$$

$$+\frac{1-\nu}{2(3+\nu)}\frac{k_2 z}{t\sqrt{2r}}\left[\sin\frac{\theta}{2}-\sin 5\frac{\theta}{2}\right]$$

$$\tau_{xy} = \frac{1-\nu}{2(3+\nu)}\frac{k_1 z}{t\sqrt{2r}}\left[\sin 5\frac{\theta}{2}+\frac{7+\nu}{1-\nu}\sin\frac{\theta}{2}\right]+ \qquad (29.3)$$

$$+\frac{1-\nu}{2(3+\nu)}\frac{k_2 z}{t\sqrt{2r}}\left[\frac{5+3\nu}{1-\nu}\cos\frac{\theta}{2}-\cos 5\frac{\theta}{2}\right]$$

where k_1 and k_2 are stress intensity factors analogous to those defined for plane stress. Notice that eqs.(29) follow the notation of Sih [9] , while Williams notation is in polar coordinates.

The displacement w, ortogonal to the middle plane of the plate, corresponding to the stress field of eq.(29) is

$$w = \frac{7+\nu}{3+\nu}\frac{r^{3/2}}{3\sqrt{2}}\frac{k_1}{Gt}\left[\cos 3\frac{\theta}{2}-3\frac{1-\nu}{7+\nu}\cos\frac{\theta}{2}\right]- \qquad (30.1)$$

$$-\frac{5+3\nu}{3+\nu}\frac{r^{3/2}}{3\sqrt{2}}\frac{k_2}{Gt}\left[\sin 3\frac{\theta}{2}-3\frac{1-\nu}{5+3\nu}\sin\frac{\theta}{2}\right]$$

From (30.1):

$$\phi = \frac{\partial w}{\partial y} = -\frac{1-\nu}{3+\nu}\frac{\sqrt{r}}{4}\frac{k_1}{Gt}\left[\sin 3\frac{\theta}{2}+\frac{9-\nu}{1-\nu}\sin\frac{\theta}{2}\right]- \qquad (30.2)$$

$$-\frac{1-\nu}{3+\nu}\frac{\sqrt{r}}{4}\frac{k_2}{Gt}\left[\cos 3\frac{\theta}{2}+\frac{3+5\nu}{1-\nu}\cos\frac{\theta}{2}\right]$$

$$\psi = -\frac{\partial w}{\partial x} = \frac{1-\nu}{3+\nu}\frac{\sqrt{r}}{4}\frac{k_1}{Gt}\left[\cos 3\frac{\theta}{2}-\frac{5+3\nu}{1-\nu}\cos\frac{\theta}{2}\right]+ \qquad (30.3)$$

$$+\frac{1-\nu}{3+\nu}\frac{\sqrt{r}}{4}\frac{k_2}{Gt}\left[-\sin 3\frac{\theta}{2}+\frac{7+\nu}{1-\nu}\sin\frac{\theta}{2}\right]$$

The shape functions for the singular plate element are thus given in compact form as

$$w = F_{12}c_1 + F_{12}c_2 - y\,c_3 - x\,c_4 + c_5 \qquad (31.1)$$

$$\phi = F_{21}c_1 + F_{22}c_2 + c_3 \qquad (31.2)$$

$$\psi = F_{31}c_1 + F_{32}c_2 + c_4 \qquad (31.3)$$

where $c_1 = k_1$, $c_2 = k_2$, and c_3, c_4, c_5 are constants that multiply a rigid body displacement. The functions F_{ij} can be easily derived from eqs.(30). The matrix F in (1) assumes here the form

$$F = \begin{vmatrix} F_{11} & F_{12} & y & -x & 1 \\ F_{21} & F_{22} & 1 & 0 & 0 \\ F_{31} & F_{32} & 0 & 1 & 0 \end{vmatrix} \qquad (32)$$

The independent nodes are placed as in fig.2; notice that for the plate element the following relationship exist among the F_{ij}

$$(F_{11})_1 = (F_{11})_2 = (F_{22})_1 = (F_{22})_2 = (F_{31})_1 = (F_{31})_2 = 0$$

$$(F_{12})_2 = - (F_{12})_1$$

$$(F_{21})_2 = - (F_{21})_1$$

$$(F_{32})_2 = - (F_{32})_1$$

With computations similar to the plane stress case one finally obtains for the singular plate element the matrices C and K

$$C = \begin{vmatrix} 0 & 1/2(F_{21})_1 & 0 & -1/2(F_{21})_1 & 0 \\ 0 & 0 & 1/2(F_{32})_1 & 0 & -1/2(F_{32})_1 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 1 & 0 & -\frac{1}{2}(\frac{(F_{12})_1}{(F_{32})_1}+a) & 0 & +\frac{1}{2}(\frac{(F_{12})_1}{(F_{32})_1}+a) \end{vmatrix} \qquad (33)$$

$$K = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & a_{11}/4(F_{21})_1^2 & 0 & -a_{11}/4(F_{21})_1^2 & 0 \\ 0 & 0 & a_{22}/4(F_{32})_1^2 & 0 & -a_{22}/4(F_{32})_1^2 \\ 0 & -a_{11}/4(F_{21})_1^2 & 0 & a_{11}/4(F_{21})_1^2 & 0 \\ 0 & 0 & -a_{22}/4(F_{32})_1^2 & 0 & a_{22}/4(F_{32})_1^2 \end{vmatrix} \qquad (34)$$

Notice that such element provides no stiffness to the z.- displacement.

3.4 PARTICULAR CASES.

When the structure has the plane y=0 as plane of symmetry,the costants c_2 and c_3 that multiply antisymmetric functions are zero. The stiffness matrix K is

$$K = \begin{vmatrix} 0 & 0 & 0 \\ 0 & a_{11}/(F_{21})_1^2 & 0 \\ 0 & 0 & 0 \end{vmatrix} \qquad (35)$$

In the case of antisymmetry, $c_1 = c_4 = 0$, and, from the usual computations

$$K = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & a_{22}/(F_{32})_1^2 \end{vmatrix} \qquad (36)$$

For a full rectangular element, of uniform thickness t, as in fig.3a, the values of a_{11} and a_{22} are

$$a_{11} = \frac{t\ a}{12\ E_t}\left(\frac{1-\nu}{3+\nu}\right)^2 \{\log \frac{\cos\alpha}{1-\sin\alpha}\ [1+\left(\frac{5+3\nu}{1+\nu}\right)^2 + \frac{2(1+\nu)(3+\nu)^2}{(1-\nu)^2} + \frac{2\nu(5+3\nu)}{(1-\nu)}] +$$
$$+ \sin\alpha \frac{4(4+\nu)(1+\nu)}{(1-\nu)} - \beta \log \operatorname{tg}\frac{\alpha}{2}\ 4\ [1+\left(\frac{3+\nu}{1-\nu}\right)^2 + \frac{8(1+\nu)}{(1-\nu)^2} + \frac{2\nu(3+\nu)}{(1-\nu)}] -$$
$$- \beta \cos\alpha \frac{4(7+\nu)(1+\nu)}{(1-\nu)}\ \} \qquad (37.1)$$

$$a_{22} = \frac{t\ a}{12\ E_t}\left(\frac{1-\nu}{3+\nu}\right)^2 \{\log \frac{\cos\alpha}{1-\sin\alpha}\ [1+\left(\frac{5+3\nu}{1-\nu}\right)^2 + \frac{2(1+\nu)(3+\nu)^2}{(1-\nu)^2} + \frac{2\nu(5+3\nu)}{(1-\nu)}] -$$
$$- \sin\alpha \frac{4(5+3\nu)(1+\nu)}{(1-\nu)} - \beta \log \operatorname{tg}\frac{\alpha}{2} \frac{8(3+\nu)(1+\nu)^2}{(1-\nu)^2} + \qquad (37.2)$$
$$+ \beta\cos\alpha \frac{4(5+3\nu)(1+\nu)}{(1-\nu)}\ \}$$

and for a full circular element of uniform thickness t

$$a_{11} = \frac{\pi\ a\ t}{96\ E_t}\left(\frac{1-\nu}{3+\nu}\right)^2 \{11+\left(\frac{11+5\nu}{1-\nu}\right)^2 - 2\nu - \frac{6\nu(11+5\nu)}{(1-\nu)} + 2(1+\nu)$$
$$[1+\left(\frac{7+\nu}{1-\nu}\right)^2]\} \qquad (38.1)$$

$$a_{22} = \frac{\pi a\ t}{96\ E_t}\left(\frac{1-\nu}{3+\nu}\right)^2 \{3+\left(\frac{9+7\nu}{1-\nu}\right)^2 + \frac{4\nu(5+3\nu)}{(1-\nu)} + 2(1+\nu) \qquad (38.2)$$
$$[1+\left(\frac{5+3\nu}{1-\nu}\right)^2]\}$$

3.5 SINGULAR ELEMENT FOR CYLINDRICAL SHELLS

The problem here is to analyze the stress distribution in a cylindrical shell of finite lenght and radius R, with a linear crack of length 2d in the direction of the axis (see fig.5a).

In plane elements, the plane stress and plate bending stiffness matrices are uncoupled,both for conventional and for singular elements. In conventional curved elements coupling exists between membrane and bending stiffness.

The existence of similar coupling is now investigated for a singular curved element.

In cylindrical shells, neglecting the ratio t/R with respect to 1,the strain vector is given as

$$\varepsilon_x = \frac{\partial u}{\partial x} - z \frac{\partial^2 w}{\partial x^2} \tag{39.1}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} + \frac{w}{R} - z \frac{\partial^2 w}{\partial y^2} \tag{39.2}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - 2z \frac{\partial^2 w}{\partial x \, \partial y} \tag{39.3}$$

Folias [6] shows that the membrane and the bending components of the strain vector in the neighborhood of a linear crack are identical to the strain vector for plane stress and plate bending respectively. The curvature of the element provides only an interaction between the associated stress intensity factors.

In eqs.(39),the strains due to bending are associated to the displacement w as in the case of the plate; one assumes for w, in the curved singular element, the same expression as eq.(30.1).

The membranal strains, in eqs.(39), differ from the plane stress case in the additional term w/R for ε_y. Calling ε_{y_m} the membrane component of ε_y in eq.(39.2), and indicating by k_m and k_b the membrane and bending stress intensity factors, one obtains from (39.2)

$$\frac{\partial v}{\partial y} = \varepsilon_{ym} - \frac{w}{R} = \frac{k_m}{\sqrt{r}} f_m(\theta) - \frac{k_b}{R} r^{3/2} f_b(\theta) \tag{40}$$

Within the range of validity of the asymptotic expansion, the second term in the third member of eq.(40) can be neglected with repect to the first.Then

$$\varepsilon_{ym} = \frac{\partial v}{\partial y}$$

and the same expressions as in eqs.(17) may be assumed for the displacements u and v.

In conclusion, the stiffness matrix of the curved singular element is easily obtained by combining the uncoupled stiffness matrices for the plate bending case. .and for the plane stress case.

4. EXAMPLES

A family of singular elements was implemented into the SELF finite element system (see CELLA et al [1]).

As a first test, three problems were investigated for which the exact solution is known; singular rectangular elements are used.

In fig.4, an infinite plate is indicated subjected to a uniform in-plane stress field of the form $\sigma_y = \sigma_o$.

The numerical computations are confined to a finite rectangular plate with variable side lengths 2A and 2B. Considering the finite element partition shown in fig. 4b., the results are given in table 1; notice that the exact solution for the infinite plate is:

$$c_1 = \frac{k_1}{\sqrt{\pi}} = \sigma \sqrt{l} \qquad (41)$$

In the second case the infinite plate of fig.4a is subjected to uniform moments per unit lenght: $M_y = M_o$ and $M_x = M_o$. A similar computation performed on a finite plate in bending gives the results reported in table 1. The exact solution is now

$$c_1 = k_1 = \frac{6M_o}{t^2} \sqrt{l} \qquad (42)$$

Table 1 contains also the values of c_1 derived from the displacement of the nearest pint to the tip of the crack; such displacements are computed using only conventional finite elements.

Finally, a cylindrical shell is analysed of radius R and infinite lenght, subjected to uniform pressure p, and with a linear crack parallel to the axis (see fig.5a). Because of symmetry, a quarter of a finite cylinder centered around the crack has been examined (see fig.5b), using plane rectangular elements.

The results are given in Table 2, where the computed value of c_1 is compared with the theoretical values given by Folias [6] and Erdogan and Kibler [3]; the subscripts m and b stand for the membrane and the bending component of the stress intensity factor. Notice that in table II the two sets of results are associated with different lengths of the crack.

5. CONCLUSIONS

In the results reported in Table I the constants c_1 are, in all cases, closer to the theoretical value, when computed with the singular element. In problems of plane stress and plate bending, then, it is possible to improve the prediction of the stresses in the neighbourhood of the tip of a linear crack, at the same computational cost, using a singular element. Notice, in the plate case, a slight divergence in the results of the last example for both singular and conventional elements. It is believed that such divergence is the effect of cumulated round-off errors on the solution of the system of linear equations (644 unknowns).

In table II, the membrane stress intensity factor is reasonable well predicted with the singular element, even using a small number of flat elements. With conventional elements, on the other hand, the prediction is quite poor. The results on the bending stress intensity factor are in neither case satisfactory. The fact is attributed, in part to the use of few elements, and in part on the relatively small influence that the bending factor has on the singular stress distribution (see ERDOGAN et al [3]). It is believed that the results would improve, at the same number of partitions, using curved cylindrical shell elements.

The present work does not deal with the convergence properties of the singular elements. An extensive study on the convergence of singular finite elements is under way at Harvard University[5]. From that basis, one may later evaluate, in full perspective, the numerical results so far produced.

AKNOWLEDGMENT

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6. REFERENCES

[1] CELLA A., MORANDI CECCHI M.,"A system for generalized finite elements" (in Italian), 1st AIMETA Congress, Udine,(It.), (June 1971).

[2] CHAN S.K., TUBA I.S., WILSON K.W., "On the finite element method in linear fracture mechanics", Eng-Fracture Mech, 2, (1970), 1-17.

[3] ERDOGAN F., KIBLER J.T., "Cylindrical and spherical shells with cracks", Int. Jour of Fracture Mech., 5, (1969), 229-237.

[4] FIX G., "Higher order Rayleigh-Ritz approximations", Jour of Math and Mech, 18, (1969), 645-657.

[5] FIX G., Private comunication, (July 1971).

[6] FOLIAS E.S., "An axial crack in a pressurized cylindrical shell", Int.Jour of Fracture Mech., 1, (1965), 104-113.

[7] IRWIN G.R., "Analysis of stresses and strains near the end of a crack traversing a plate ", Jour of Appl. Mech., 24, (1957), 361-364.

[8] RAO A.K., RAJU I.S., KRISHNAMURTHY A.V., "A powerful hybrid method in finite element analysis", Report no. AE 254 s, Dept. of Aer. Eng., Indian Institute of Science, Bangalore, (1969).

[9] SIH G.H., LIEBOWITZ H., "Mathematical theories of brittle fracture", in Fracture, vol. 2 (ed. H. Liebowitz), Academic Press, New Kork, (1968), 67-190.

[10] SNEDDON I.N., LOWENGRUB M., Crack problems in the classical theory of elasticity, Wiley, New York, (1969).

[11] WILLIAMS M.L., "The bending stress distribution at the base of a stationary crack", Jour. of Appl. Mech., 28, (1961), 78-82.

[12] ZIENKIEWICZ O.C., The finite element method in engineering science, Mc Graw-Hill, New York, (1971).

TAB. I-Cracked plates

PLANE STRESS												
A mm	B mm	l mm	t mm	σ_o Kg/mm^2	c_1 (+) Kg $\frac{\sqrt{mm}}{mm}2$	c_1 (++) Kg $\frac{\sqrt{mm}}{mm}2$	c_1 (+++) Kg $\frac{\sqrt{mm}}{mm}2$	a' mm	b' mm	n_a	n_l	n_b
108	72	36	6	1	8.89	9.55	6	8	4	4	3	5
180	144	36	6	1	7.35	7.84	6	8	4	8	3	9
252	216	36	6	1	7.04	7.52	6	8	4	12	3	13
PLATE BENDING												
A mm	B mm	l mm	t mm	M_o Kg $\frac{mm}{mm}$	c_1 (+) Kg $\frac{\sqrt{mm}}{mm}2$	c_1 (++) Kg $\frac{\sqrt{mm}}{mm}2$	c_1 (+++) Kg $\frac{\sqrt{mm}}{mm}2$	a' mm	b' mm	n_a	n_l	n_b
91	55	36	6	6	6.87	8.48	6	8	4	3	3	4
108	72	36	6	6	6.66	8.25	6	8	4	4	3	5
180	144	36	6	6	6.29	8.00	6	8	4	8	3	9
252	216	36	6	6	6.47	8.31	6	8	4	12	3	13

(+) singular element

(++) conventional elements

(+++) theoretical value for plate of infinite dimension

TAB. II - Cracked cylindrical shell

A mm	R mm	l mm	t mm	p $\frac{Kg}{mm^2}$	$C_{1m}^{(+)}$ $Kg.\frac{\sqrt{mm}}{mm^2}$	$C_{1m}^{(++)}$ $Kg.\frac{\sqrt{mm}}{mm^2}$	$C_{1m}^{(+++)}$ $Kg.\frac{\sqrt{mm}}{mm^2}$	$C_{1b}^{(+)}$ $Kg.\frac{\sqrt{mm}}{mm^2}$	$C_{1b}^{(++)}$ $Kg.\frac{\sqrt{mm}}{mm^2}$	$C_{1b}^{(+++)}$ $Kg.\frac{\sqrt{mm}}{mm^2}$	a' mm	b' mm	n_a	n_l	n_b
124	76	3I	5.5	1	101.8	328.1	151.8	3.4	862.5	27.7	6.2	6.2	4	3	3
124	76	31	5.5	1	145.9	240.5	151.8	0.7	208.5	27.7	6.2	6.2	4	3	5
124	76	31	5.5	1	154.2	260.2	151.8	0.6	206.5	27.7	6.2	6.2	5	4	5
200	76	50	5.5	1	263.1	703.3	273.9	3.3	926.6	21.1	10.0	6.2	4	3	3
200	76	50	5.5	1	295.8	493.1	273.9	0.4	199.7	21.1	10.0	6.2	4	3	5
200	76	50	5.5	1	313.6	540.1	273.9	0.3	197.5	21.1	10.0	6.2	5	4	5

(+) singular element

(++) conventional plane elements

(+++) theoretical value from Erdogan e Kibler [3]

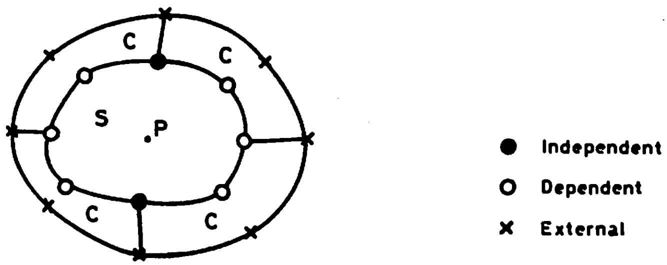


fig. 1 - Connectivity of the singular element

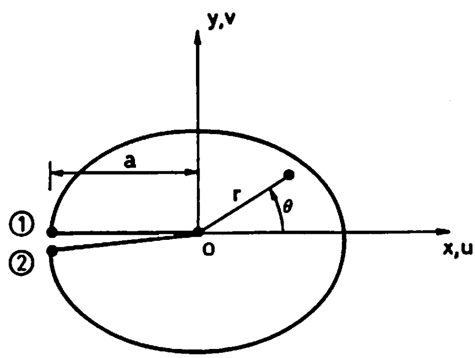
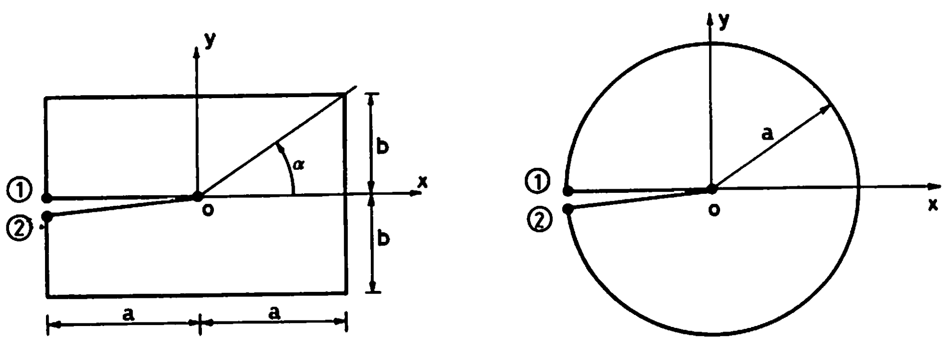
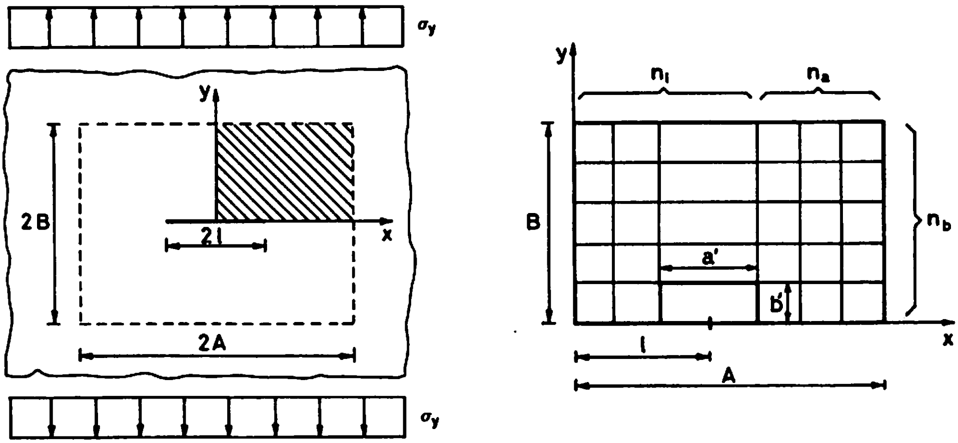


fig. 2 - The element for linear crack



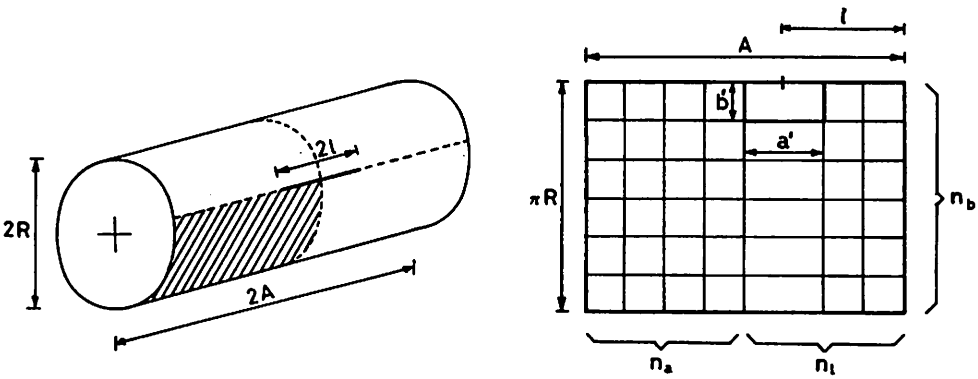
a) b)

fig. 3 - Rectangular and circular elements



a) b)

fig. 4 - The plane stress problem



a) b)

fig. 5 - The cylindrical shell problem

DISCUSSION

Q D. FISCHER, Austria

Is it possible to involve in your computer program real material behaviour near the crack (before a crack is appearing, always material becomes plastical before !) ?

A C. CARMIGNANI, Italy

Until now the plastic behaviour cannot be taken into account in our computer program. But this is not an impossible thing: in the future we shall study this problem.

Q R. H. GALLAGHER, U. S. A.

1. What do you mean by "conventional" elements, in membrane and bending action ?
2. In the shell analysis problem, the "conventional" element results appear to diverge with improved grid and the singularity element solution becomes worse (with reference to the comparison solution) as the grid is refined. How do you explain these results ?

A C. CARMIGNANI, Italy

1. The "conventional" elements used in the numerical calculations are :

a) for the plane stress problem, rectangular elements with 4 nodal points and 8 degrees of freedom.

b) for the plate bending problem, rectangular elements with 4 nodal points and 12 degrees of freedom.

2. The numerical results in the cracked shell problem may be affected by the following:

a) the singularity element must have dimensions which are limited by the validity zone of asymptotic solution. In the partition used in these first numerical results we have one small singularity flat element in conjunction with relatively large conventional flat elements.

b) we used only flat elements. The use of curved elements would improve the numerical results, especially when we want to use a small number of elements.

c) about convergence we think that it can be oscillatory.

All these points will be examined in the near future.