A MODEL FOR RATE-DEPENDENT BUT TIME-INDEPENDENT MATERIAL BEHAVIOR IN CYCLIC PLASTICITY

Y. F. DAFALIAS, M. R. RAMEY, I. SHEIKH

Department of Civil Engineering, University of California, Davis, California 95616, U.S.A.

SUMMARY

It is the purpose of this paper to present a model for rate-dependent but time independent material behavior under cyclic loading in the plastic range. What is referred to as time independent behavior here, is the absence of creep and relaxation phenomena from the behavior of the model. The notion of plastic internal variables (piv) is introduced, as properly invariant scalars or second order tensors, whose constitutive relations are rate-type equations not necessarily homogeneous of order one in the rates, as it would be required for independent plasticity. The concept of a yield surface in the strain space and a loading function in terms of the total strain rate is introduced, where the sign of the loading function defines zero or non-zero value of the rate of piv. Thus rate dependence is achieved without time dependent behavior (no creep or relaxation). In addition, discrete memory parameters associated with the most recent event of unloading-reloading in different directions enter the constitutive relations for the piv.

A particular form of the constitutive relations is assumed, where the rate of piv is a linear combination of the strain rate components, with coefficients depending on the second invariant of the strain rate tensor, which can be viewed as a scalar measure of the rate of deformation in the multiaxial case and a direct generalization of the uniaxial strain rate. This leads to a particularly simple form of the constitutive relations resembling the ones for rate independent plasticity. The uniaxial counterpart would be a relation between the plastic strain rate (as one of the piv) and the total strain rate through a plastic modulus which depends on the strain rate, the piv, and the discrete memory parameters.

Within the above general framework, a generalization of a model proposed by Dafalias and Popov, J. Appl. Mech. 43 Ser. E, No. 4, Dec. 1976, for cyclic rate-independent plasticity in order to incorporate rate-dependence is suggested. The characteristic feature of this model, following from uniaxial experimental observations, was the concept of a "bounding surface" in stress space enclosing the yield surface and where the proximity of these two surfaces in the course of their translation and deformation during plastic loading determines the appropriate piv and discrete memory parameters on which the plastic modulus depends. Similar concepts can be introduced in strain space for the generalized model, where the difference now is that the size of the bounding surface depends also on the second invariant of the strain rate tensor, thus the plastic modulus (different for strain space formulation) depends indirectly on the strain rate. This is a direct generalization of the uniaxial experimental observation that the stress-strain curves converge with bounds in stress-strain space, whose position depends on the strain rate. Besides this difference the mechanism of the new model is identical to the model for rate-independent behavior, and the application in the uniaxial case is illustrated by simple examples, which show good agreement between theory and experiment.

1. Introduction

Rate-dependence of inelastic deformations has been introduced under the form of viscoplasticity theory. The basic feature of the corresponding constitutive equations is that the inelastic strain rate depends on the stress state if this state satisfies certain conditions, namely the corresponding point in stress space must lie outside the so-called static yield surface. In such a case, assuming that the total strain rate equals the sum of the elastic and inelastic strain rates, creep and relaxation phenomena naturally arise if stress or strain are kept constant respectively. These phenomena render the material behavior time-dependent and are inherently built into the viscoplastic constitutive relations.

There is no reason however not to assume an inelastic material response which is rate dependent but time independent in the sense mentioned before, i.e., no creep or relaxation. This is done in the following. Rate dependence is introduced in the form of the constitutive equations for the plastic internal variables (PIV), which are properly invariant scalars or second order tensors; these rate-type constitutive equations are not necessarily homogeneous of order one in the rates, as it would be required for rate-independent plasticity. The concept of the yield surface in the strain space and a loading function in terms of the total strain rate render the behavior time independent. Indeed the sign of the loading function defines zero or non-zero value for the rate of piv, thus a change of stress requires always a change on strain and vice-versa, precluding creep and relaxation. In addition, discrete memory parameters associated with the most recent event of unloading-reloading in different directions enter the constitutive relations for the piv.

A particular form of the constitutive relations is assumed, where the rate of piv is a linear combination of the strain rate components, with coefficients depending on the second invariant of the strain rate tensor, which can be viewed as a scalar measure of the rate of deformation in the multiaxial case and a direct generalization of the uniaxial strain rate. This leads to a particularly simple form of the constitutive relations resembling the ones for rate independent plasticity. The uniaxial counterpart would be a relation between the plastic strain rate (as one of the piv) and the total strain rate through a plastic modulus which depends on the strain rate, the piv, and the discrete memory parameters.

Within the above general framework, a generalization of a model proposed by Dafalias and Popov [1] for cyclic rate-independent plasticity in order to incorporate rate-dependence is suggested. The characteristic feature of this model, following from uniaxial experimental observations, was the concept of a "bounding surface" in stress space enclosing the yield surface and where the proximity of these two surfaces in the course of their translation and deformation during plastic loading determines the appropriate piv and discrete memory parameters on which the plastic modulus depends. Similar concepts can be introduced in strain space for the generalized model, where the difference now is that the size of the bounding surface depends also on the second invariant of the strain rate tensor, thus the plastic modulus (different for strain space formulation) depends indirectly on the strain rate. This is a direct generalization of the uniaxial experimental observation that the stress-strain curves converge with bounds in stress-strain space, whose position depends on the strain rate. Although the mechanism of the new model in general is the same as the model for rate-independent behavior, a new point of attention is when the stress rate is suddenly increased or decreased. This changes suddenly the position of the bound on the uniaxial stress-strain space (and the size of the bounding surface in the multiaxial case) and a new process begins. The notion of stable and unstable zones is introduced depending on whether or not the point representing the current state in stress-strain space is below or above the bound corresponding to the new strain rate. This makes necessary to differentiate between the two processes following a change of strain rate, where the one begins with the current state at a stable zone and the other at an unstable zone.

Application of the model to uniaxial monotonic loading conditions with change in strain rates show good agreements between theory and experiment.

2. Rate-Dependent Constitutive Relations for Plastic Internal Variables

The internal variables formalism in continuum mechanics is based on the concept that the state of a body is described at each material point by the values of "observed" or "external" variables such as deformation, stress and temperature, and of "hidden" or "internal" variables q_n which are usually taken to be scalars or components of properly invariant second rank tensors (e.g., inelastic strain). The constitutive relations for the q_n are of the general form

$$\dot{q}_n = \hat{q}_n \text{ (ext. var., first-order rate of ext. var., } q_n) \tag{1}$$

where the rates of the external variables in q_n were systematically introduced by Lubliner [2], who also gives a brief review of relevant works.

Proceeding to formulate the constitutive relation for q_n, the notion of a yield surface in strain space is introduced analytically by

$$F(\varepsilon_{ij}, q_n) = 0 \tag{2}$$

with the strain ε_{ij} being the external variables. A set ε_{ij}, q_n satisfying (2) is a plastic state, and a set rendering $F < 0$ is an elastic state. The loading function L is defined at a plastic state by

$$L = \frac{\partial F}{\partial \varepsilon_{ij}} \dot{\varepsilon}_{ij} \tag{3}$$

Loading is said to occur at a plastic state when $L > 0$, unloading when $L < 0$ and neutral loading when $L = 0$.

The common property characterizing the PIV is that their rate $\dot{q}_n$ is different than zero at a given plastic state if $L > 0$; at an elastic state or at plastic state with $L < 0$ or $L = 0$, $\dot{q}_n = 0$. Analytically, the foregoing is expressed, within the general form (1) by

$$\dot{q}_n = \hat{q}_n(\varepsilon_{ij}, \dot{\varepsilon}_{ij}, q_\alpha, \xi_\alpha)H(L) \quad \text{when} \quad F = 0$$
$$\dot{q}_n = 0 \quad \text{when} \quad F < 0 \tag{4}$$

where $H(L)$ is the Heaviside step function, with $H(0) = 0$.

The ξ_α are internal variables (not PIV) associated with abrupt changes of the plastic loading processes, which in the present context means drastic change of loading direction usually but not always obtained by unloading-reloading. Such changes are likely to affect subsequently the material behavior, hence the motivation to introduce ξ_α in (4). The ξ_α are given by relations of the form

$$\xi_\alpha = \hat{\xi}_\alpha(\varepsilon_{ij}{}^0, q_n{}^0) \tag{5}$$

in terms of the values $\varepsilon_{ij}{}^0$, $q_n{}^0$ of the state variables when such a drastic change occurs. Note that ξ_α remain constant during the loading process, thus they are not PIV. They are however internal variables, since a discontinuous relation of the general form (1) can be written for $\dot{\xi}_\alpha$, but it is easier to use (5) instead. Thus it is convenient to think of ξ_α as discrete memory parameters whose role is to store in the material memory events associated with drastic changes of loading direction, with or without passing through a phase of unloading-reloading.

If the constitutive relation (4) is to be continuous throughout stress and temperature space, the following relation must be satisfied:

$$\hat{q}_n = 0 \quad \text{whenever} \quad L = 0 \tag{6}$$

The rates of the state variables are not independent when $L > 0$, as they must satisfy the consistency condition which insures that the point representing the current state remains on the yield surface during loading. Using (2) and (3) this is expressed analytically by

$$\dot{F} = L + \frac{\partial F}{\partial q_n} \dot{q}_n = 0 \tag{7}$$

It follows from (4) that the inelastic response is rate-dependent in general, unless $\hat{q}_n$ is homogeneous of degree one in $\dot{\varepsilon}_{ij}$. In addition the response is time indepdendent. Indeed, with $\dot{\varepsilon}_{ij} = 0$ we have $L = 0$ and consequently the plastic strain rate $\dot{\varepsilon}_{ij}{}^p = 0$ since $\varepsilon_{ij}{}^p$ is one of the piv. Thus, $\dot{\varepsilon}_{ij}{}^e = 0$ and so is $\dot{\sigma}_{ij}$. Therefore, any change in stress requires a change in strain with strain rate dependence for loading, and vice-versa. A particularly useful rate dependent form of (4) can be obtained if we assume

$$\hat{q}_n = A_{nk\ell}\,[\varepsilon_{ij}, q_\alpha, \xi_\alpha, (\dot{\varepsilon}_{ij}\,\dot{\varepsilon}_{ij})^{1/2}]\,\dot{\varepsilon}_{k\ell} \tag{8}$$

From eqs. (3), (6) and (8) we obtain

$$(A_{nk\ell} - \lambda_n \frac{\partial F}{\partial \varepsilon_{k\ell}})\,\dot{\varepsilon}_{k\ell} = 0 \tag{9}$$

for all $\dot{\varepsilon}_{k\ell}$ satisfying $L = 0$, where λ_n plays the role of a Lagrange multiplier. This last condition means that $\dot{\varepsilon}_{k\ell}$ can be represented by a vector in strain space which is perpendicular to the normal vector $\partial F/\partial\varepsilon_{k\ell}$ on the yield surface, i.e., $\dot{\varepsilon}_{k\ell}$ lies on the tangent plane at the point representing the current state on the yield surface. Consider now the vector $\dot{\varepsilon}_{k\ell}$ rotating around $\partial F/\partial\varepsilon_{k\ell}$, remaining in the tangent plane so that $L = 0$, and with $(\dot{\varepsilon}_{ij}\,\dot{\varepsilon}_{ij})^{1/2}=$ constant. This of course implies that the individual components of $\dot{\varepsilon}_{ij}$ may change, but the important point is that $\dot{\varepsilon}_{ij}$ can assume any direction on the tangent plane. The quantity in parenthesis in (9) depends on $\dot{\varepsilon}_{ij}$ only through $(\dot{\varepsilon}_{ij}\,\dot{\varepsilon}_{ij})^{1/2}$ which remains constant, thus the only way that (9) can be satisfied for all directions of $\dot{\varepsilon}_{ij}$ is for the vector represented by this quantity to be colinear with $\partial F/\partial\varepsilon_{k\ell}$, which yields

$$A_{nk\ell} = r_n \frac{\partial F}{\partial \varepsilon_{k\ell}} \tag{10}$$

Finally, (4), (8) and (10) yield

$$\dot{q}_n = r_n[\varepsilon_{ij}, q_\alpha, \xi_\alpha, (\dot{\varepsilon}_{ij}\, \dot{\varepsilon}_{ij})^{1/2}]\langle L\rangle \text{ when } F = 0 \tag{11}$$

where $\langle\ \rangle$ is the Macauley bracket defining the operation $\langle L\rangle = L\,H(L)$.

3. The Model

The most important PIV is the plastic rate. The basic relation

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}_{ij}^{\,e} + \dot{\varepsilon}_{ij}^{\,p} \tag{12}$$

is assumed, where $\dot{\varepsilon}_{ij}$, $\dot{\varepsilon}_{ij}^{\,e}$, $\dot{\varepsilon}_{ij}^{\,p}$ are the rates of the total, elastic and plastic strain tensors, respectively. In accordance with (11) the constitutive relation for $\dot{\varepsilon}_{ij}^{\,p}$ can be written as

$$\dot{\varepsilon}_{ij}^{\,p} = \frac{1}{K}\langle L\rangle \rho_{ij} \tag{13}$$

where ρ_{ij} is a directional unit vector in strain space ($\rho_{ij}\,\rho_{ij} = 1$) and K is a modulus depending on ε_{ij}, q_α, ξ_α and $(\dot{\varepsilon}_{ij}\,\dot{\varepsilon}_{ij})^{1/2}$. The objective now is to find an appropriate functional dependence of K on the above mentioned quantities.

The uniaxial counterpart of (13) is (omitting the brackets)

$$\dot{\varepsilon}^p = (1 + \frac{E^p}{E^e})^{-1}\,\dot{\varepsilon} \tag{14}$$

where E^e is the elastic modulus and E^p the plastic modulus relating stress rate to plastic strain rate, i.e.,

$$\dot{\varepsilon}^p = \frac{\dot{\sigma}}{E^p} \tag{15}$$

The parenthesis in (14) is the uniaxial counterpart of K in (13) and is a function of ε, q_α, ξ_α and $\dot{\varepsilon}$ which is the uniaxial expression for $(\dot{\varepsilon}_{ij}\,\dot{\varepsilon}_{ij})^{1/2}$. Since E^e is assumed constant, that implies that E^p is function of those quantities. Thus, considering (15) for simplicity, the functional dependence of E^p on ε, q_α, ξ_α and $\dot{\varepsilon}$ is sought.

The model developed by Dafalias and Popov [1] for rate independent cyclic plasticity will be used as a basis. The details are given in the above reference and here only a brief summary is presented. Every stress-strain curve converges with bounds which can be represented by lines in stress-strain space. It was found that the material behavior can be described by considering the plastic modulus E^p a function of the distance δ of the stress state from the corresponding bound, and of the initial value δ_{in} of δ,

$$E^p = \hat{E}^p\,(\delta, \delta_{in}) \tag{16}$$

For a smooth transition from elastic to elastic-plastic region, we have $\hat{E}^p(\delta_{in}, \delta_{in}) = \infty$. On the bounds $\delta = 0$, and $\hat{E}^p(0, \delta_{in}) = E_0^p$, the slope of the bound in $\sigma - \varepsilon^p$ space. The δ_{in} changes only upon unloading-reloading and is the discrete memory parameter ξ_α in this case. The δ depends on ε and q_α, thus E^p depends through δ and δ_{in} on ε, q_α and ξ_α. No dependence on $\dot{\varepsilon}$ exists, since we are dealing with rate independent plasticity. Projection on the σ-axis and generalization to multiaxial stress space introduces through the bounds the concept of the "bounding surface" enclosing the yield surface and where the proximity of these two surfaces measured by generalized δ and δ_{in} determines the plastic modulus. A projection on the ε-axis and generalization to the multiaxial strain space introduces similar concepts in strain space.

Once the dependence of E^p on the rate $\dot{\varepsilon}$ has been systematically obtained by the preceeding strain space formulation for rate dependence, we can for simplicity deal in the uniaxial case with the stress formulation and the relation (15). The following observation now leads to the proper generalization of the rate-independent Dafalias-Popov model, to include rate dependence. It has been observed from experiments that the position of the bounds in the stress-strain space depends on the strain rate $\dot{\varepsilon}$. This is illustrated in Fig. 1, where for $\dot{\varepsilon}_2 > \dot{\varepsilon}_1$ the bound XX is parallel and above the bound X'X'. Thus, if the bound position is expressed in terms of $\dot{\varepsilon}$ from experiments, δ and δ_{in} will depend indirectly on $\dot{\varepsilon}$ and so will E^p through them. In the multiaxial case that would mean that the size of the bounding surface depends on $(\dot{\varepsilon}_{ij}\ \dot{\varepsilon}_{ij})^{1/2}$. The rest remains the same and the previous mechanism of the model is applied, both for monotonic and cyclic loading.

4. Change of Strain-Rate

Assuming that the position of the bound has been defined as a function of $\dot{\varepsilon}$, a constant strain-rate loading is automatically given by the model as shown by the curves converging with X'X' and XX in Fig. 1 for $\dot{\varepsilon}_1$ and $\dot{\varepsilon}_2$ respectively. However, when a change of strain rate occurs during a loading process, special attention is required.

An increase of strain rate for $\dot{\varepsilon}_1$ to $\dot{\varepsilon}_2$ occurs at point A, the bound changes from X'X' to XX and the stress-strain curve from OAX' to OAX. At A the jump of the curve means that the deformation is purely elastic at this point. Thus E^p becomes infinite, and this is compatible with choosing at A a new δ_{in} measured from A to the corresponding point $\bar{A}$ on the new bound, with $\delta = \delta_{in}$ at A. All points below the corresponding bound are characterized by positive values of the tangent modulus E^t (and of E^p), thus for a given $\dot{\varepsilon}$ and a corresponding bound these points constitute a stable zone.

A sudden decrease of the strain rate may however bring the new bound below the point, as illustrated in Fig. 2. Then the curve OAX changes to OAX', and the new bound XX' is below the point A. The stress-strain curve at A becomes unstable, i.e., has negative E^t and E^p, until eventually both pass from zero become again positive and finally E^p in the limit becomes $E_o{}^p$ as the curve converges with X'X'. All points above a bound for a given strain rate, constitute an unstable zone. At any point $\dot{\varepsilon} = \dot{\sigma}(E^p + E^e)/E^pE^e$. At point A $\dot{\varepsilon} > 0$ but $\dot{\sigma} < 0$. Thus $(E^p + E^e)/E^pE^e$ which was positive, must become negative, thus it assumes at A the value 0, therefore $E^p = -E^e$ instantaneously.

5. Numerical Examples

The functional form for E^p used in [1] was

$$E^p = E_o{}^p + h\left[\frac{\delta}{\delta_{in}-\delta}\right]$$

with h a shape parameter determined by a stress-strain curve as shown in [1]. In consistency with the previous section, we have:

$$\text{Stable zone :}\quad E_o{}^p \leq E^p \leq \infty ,\quad h > 0$$

$$\text{Unstable zone:}\quad -E^e \leq E^p \leq E_o{}^p ,\quad h < 0$$

Experimental data from [3, 4] are compared against the present model. In [3] a titanium specimen is strained in two different strain rates, $\dot{\varepsilon}_1 = 1.6 \times 10^{-5}\ \sec^{-1}$ and $\dot{\varepsilon}_1 = 3.2 \times 10^{-3}\ \sec^{-1}$ with an increase from $\dot{\varepsilon}_1$ to $\dot{\varepsilon}_2$ and a decrease from $\dot{\varepsilon}_2$ to $\dot{\varepsilon}_1$. From the curves it was measured $E^e = 12500$ kg/mm^2, $E_0^p = 22.76$ kg/mm^2 and yield stress = 20 kg/mm. From one of the curves and with the method described in [1], the shape parameter h was calculated to be $h = 587$. The strain history was given in small increaments and the model predicted the stress history. Comparison between theory and experiment is shown in Figs. 3 and 4, where the position of the bounds is indicated by dashed lines.

In [4] an aluminum 1100-0 specimen was subjected to two different strain rates, in torsion, $\dot{\gamma}_1 = 5 \times 10^{-5}\ \sec^{-1}$ and $\dot{\gamma}_2 = 850\ \sec^{-1}$. The corresponding torsion stress-strain curve was converted to a tensile curve by $\sigma = \sqrt{3}\,\tau$ and $\gamma = \sqrt{3}\,\varepsilon$ The measured data were $E^e = 10000$ ksi, $E_0^p = 34.4$ and yield stress = 3.03 ksi. The shape parameter $h = 105$. Theory versus experiment is shown in Fig. 5.

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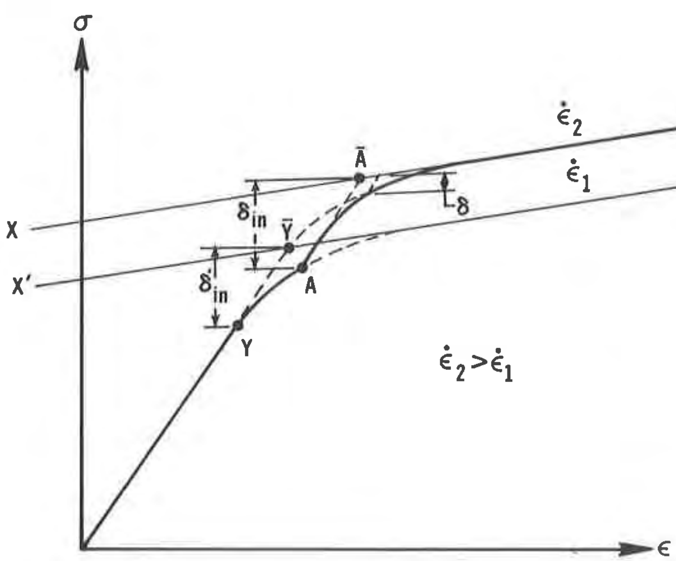

Fig. 1 Schematic illustration of strain rate increase

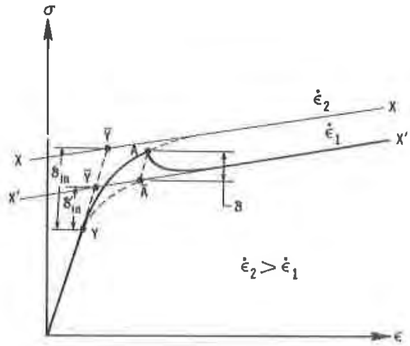

Fig. 2 Schematic illustration of strain rate decrease

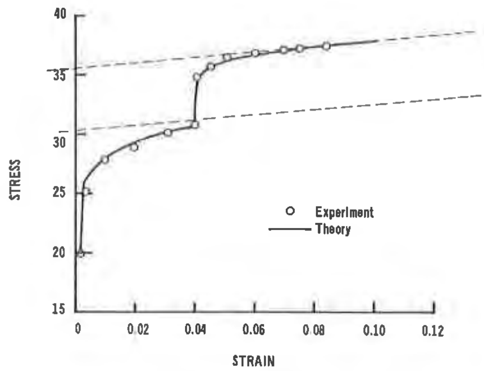

Fig. 3 Strain rate increase for titanium

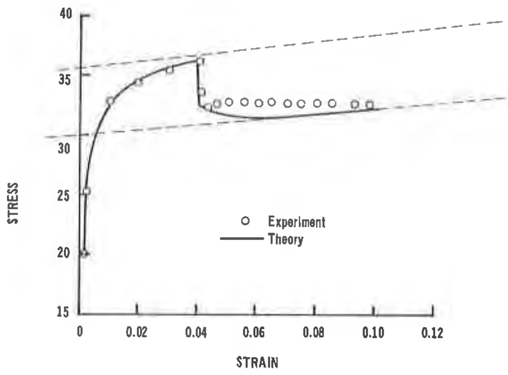

Fig. 4 Strain rate decrease for titanium

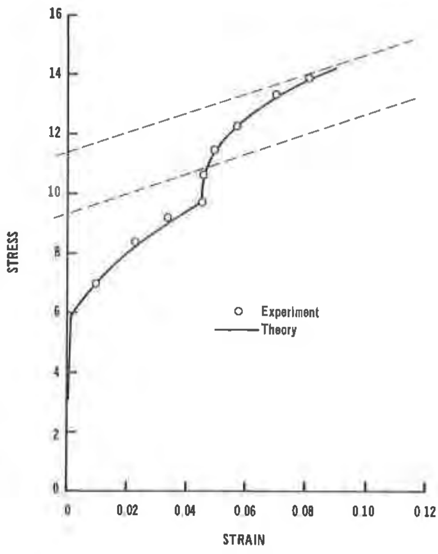

Fig. 5 Strain rate increase for aluminum