Global Solvability for Damped Abstract Nonlinear Hyperbolic Systems *

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Abstract

We consider abstract nonlinear second order in time systems with damping. The nonlinearity is assumed to satisfy a monotonicity condition as well as certain smoothness conditions. Well posedness of solutions is established and several examples of interest are discussed. A nonlinear variation-of-parameters representation for solutions in terms of an associated linear semigroup is also given.

1 Introduction and Motivating Example

In this paper we present new well-posedness results for a class of nonlinear distributed parameter models that arise in a number of applications. Our efforts are a continuation of our earlier endeavors [7, 3] on systems arising in so-called "smart" materials. Indeed, our efforts are basic to our eventual goal of development of computational methodologies for the identification and control of smart material composites undergoing large deformations and/or deformations that fall within the regime of nonlinear stress-strain laws. It is well known [4] in engineering applications that large deformations can occur even when strain levels remain relatively small. More important to certain emerging applications involving composites and certain types of elastomers is that one encounters a nonlinear stress-strain relationship even in the case of small deformations [4, 15, 5]. We describe one such application as a motivating example for the theoretical well-posedness discussions to follow in subsequent sections.

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A problem of fundamental interest and great importance in modern material sciences is the development of both passive and active ("smart") vibration devices constructed from polymer (long molecular chains of covalently bonded atoms often having cross-linking chains) composites such as elastomers filled with carbon black and/or silica or with active elements (i.e., piezoelectric, electrostrictive and magnetic or conductive particles). These rubber based products (even without active elements) involve very complex viscoelastic materials that are not at all like metals (where large deformations lead to permanent material changes) and do not satisfy the usual, well-developed linear theory of (infinitesimal) elasticity for deformable bodies. In considering macroscopic elastic behavior, one finds that the usual constitutive relationships (e.g., Hooke's law) or rheological equations of state for pure elastics are not applicable. Indeed, one observes nonlinearities in both material and geometric behavior - in general, there is a nonlinear relationship between stress and strain even for small strains. Moreover, deformations in the range of practical interest are large and infinitesimal based theories break down.

In spite of these difficulties, there is a substantial literature on modeling of rubber-like elastomers (see [12, 22, 25] for basic texts), predominantly based on one of the two rather distinct approaches: (i) molecular (polymer chain) statistical thermodynamic formulations (ii) phenomenological (usually continuum) formulations involving stored energy or strain energy functions (SEF) and/or finite strain (FS) theories. In the phenomenological approach (which will be the basis of our motivating example here) most investigators begin with an isotropic material under homogeneous strain.

Strain energy function theories typically embody only elastic properties of elastomers or rubbers and hence are mostly used in static (equilibrium) finite element analysis (e.g. see [10]) of materials (e.g. natural gum rubbers) that exhibit little or no hysteretic behavior. SEF material models, such as those of Mooney-Rivlin, Ogden, Treloar and numerous others, are based on strain invariants I_i, where $I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$, $I_2 = \lambda_2^2\lambda_3^2 + \lambda_1^2\lambda_3^2 + \lambda_1^2\lambda_2^2$ and $I_3 = \lambda_1^2\lambda_2^2\lambda_3^2$ and the λ_i are the principal extension ratios (deformed length of unit vectors along directions parallel to the principal axes i.e. the axes of zero shear strain).

The finite strain elastic theory of Rivlin [20, 25] is developed with a generalized Hooke's law in an analogy to infinitesimal strain elasticity but makes no "small deformation" assumption and includes higher order exact terms in its formulation. Moreover, finite stresses are defined relative to the <u>deformed</u> body and hence are the "true stresses" as opposed to the "nominal" or "engineering" stresses (relative to the <u>undeformed</u> body) one usually encounters in the infinitesimal linear elasticity used with metals. This Eulerian measure of strain (relative to a coordinate system convected with the deformations) - as opposed to the usual Lagrangian measure (relative to a fixed coordinate system for the undeformed body) - is an important feature of any development of models for use in analytical/computation/experimental investigations of rubber-like material bodies.

Whether one begins with a choice of the SEF or with Rivlin's finite strain formulation, one can use these along with standard material independent force and moment balance derivations (the Timoshenko theory [24, 9]) as the basis of dynamical models. To illustrate this we take the simplest example: an isotropic, incompressible ($\lambda_1\lambda_2\lambda_3 = 1$) rubber-like rod under simple elongation with a finite applied stress in the principal axis direction x_1. The finite stress theory leads to a true stress $\sigma = \frac{E}{3}(\lambda_1^2 - \frac{1}{\lambda_1})$ for $|\lambda_1| < 1$ or an engineering or nominal stress for what are termed <u>neo-Hookean</u> materials

$$\sigma\text{eng} = \frac{\sigma}{\lambda_1} = \frac{E}{3}\{\lambda_1 - \frac{1}{\lambda_1^2}\} \tag{1.1}$$

where in terms of deformation w in the $x_1 = x$ direction we have (since deformations in the y and z directions are negligible)

$$\lambda_1^2 \quad = \quad \left(1 + \frac{\partial w}{\partial x}\right)^2 \tag{1.2}$$

Here E is a generalized modulus of elasticity.

This can be used in the Timoshenko theory for longitudinal vibrations of a rubber bar to obtain (ρ is the mass density, F is an applied external force)

$$\rho A \frac{\partial^2 w}{\partial t^2} - \frac{\partial S}{\partial x} = F \tag{1.3}$$

where S, the internal (engineering) stress resultant, is given by

$$S = \frac{AE}{3}\{\lambda_1 - \frac{1}{\lambda_1^2}\} = \frac{AE}{3} s\left(\frac{\partial w}{\partial x}\right) \tag{1.4}$$

with $s(\xi) = 1 - \xi - (1+\xi)^{-2}$ for $|\xi| < 1$ and A is the cross sectional area. This leads to the nonlinear partial differential equation

$$\rho A \frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial x}\left(\frac{EA}{3} s\left(\frac{\partial w}{\partial x}\right)\right) = F \tag{1.5}$$

for dynamic longitudinal displacements of a neo-Hookean material rod in extension. Since a series expansion of s yields $s(\xi) = 3\xi - 3\xi^2 + 4\xi^3 - \ldots$, this is readily seen, in the case of small displacements, to reduce to the usual longitudinal deformation equation for Hookean materials. For our subsequent discussions, it is convenient to write (1.5) in the form

$$\rho A \frac{\partial^2 w}{\partial t^2} - \frac{\partial}{\partial x}\left(\frac{EA}{3}\frac{\partial w}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{EA}{3}\tilde{g}\left(\frac{\partial w}{\partial x}\right)\right) = F \tag{1.6}$$

where $\tilde{g}(\xi) = 1 - \frac{1}{(1+\xi)^2}$ for $-1 < \xi < 1$. This can be written in a generalized or variational form for a given set of boundary conditions. To be specific, suppose we have a slender rod of length ℓ that satisfies $w(t,0) = w(t,\ell) = 0$. Then defining $\mathcal{V} = H_0^1(0,\ell)$ and $\mathcal{H} = L^2(0,\ell)$ we obtain the usual Gelfand triple $\mathcal{V} \hookrightarrow \mathcal{H} \approx \mathcal{H}^* \hookrightarrow \mathcal{V}^*$ where $\mathcal{V}^* = H^{-1}(0,\ell)$. Then equation (1.6) along with the specified boundary conditions can be written in variational form

$$\rho A w_{tt} + \mathcal{A}_1 w + D^*\tilde{g}(Dw) = F \quad \text{in } \mathcal{V}^* \tag{1.7}$$

where $\mathcal{A}_1 \in \mathcal{L}(\mathcal{V},\mathcal{V}^*)$ is given by

$$\langle \mathcal{A}_1\varphi, \psi\rangle_{\mathcal{V}^*,\mathcal{V}} = \left\langle \frac{EA}{3} D\varphi, D\psi \right\rangle_{\mathcal{H}}$$

and $D = \frac{\partial}{\partial x} \in \mathcal{L}(\mathcal{V},\mathcal{H})$ is the spatial differentiation operator. This model is unrealistic in that it does not include material damping which is known to be present in typical elastomers. If one assumes an internal damping of the form $\mathcal{A}_2 w_t$ (the exact form of the internal dynamic damping mechanisms in elastomers is a subject of current research - almost nothing is found in the research literature on this even though it is a very important material property that is critical to design of "smart" elastomers), then the model in variational form for the neo-Hookean elastomer rod is given by

$$\rho A w_{tt} + \mathcal{A}_1 w + \mathcal{A}_2 w_t + D^*\tilde{g}(Dw) = F \quad \text{in } \mathcal{V}^* \tag{1.8}$$

2 Formulation of Problem

The remainder of this paper is concerned with establishing global existence of weak solutions for a class of abstract nonlinear damped hyperbolic systems evolving in a complex separable Hilbert space $\mathcal{H}$ (actually holding in the sense of $\mathcal{V}^*$ as explained below):

$$w_{tt} + \mathcal{A}_1 w + \mathcal{A}_2 w_t + \mathcal{N}^* g(\mathcal{N} w) = f(t) \tag{2.1}$$

$$w(0) = \varphi_0 \tag{2.2}$$

$$w_t(0) = \varphi_1 \tag{2.3}$$

Throughout the paper we assume there is a sequence of separable Hilbert spaces $\mathcal{V}$, $\mathcal{V}_2$, $\mathcal{H}$, $\mathcal{V}^*$, $\mathcal{V}_2^*$ forming a Gelfand quintuple [7, 26] satisfying

$$\mathcal{V} \hookrightarrow \mathcal{V}_2 \hookrightarrow \mathcal{H} \hookrightarrow \mathcal{V}_2^* \hookrightarrow \mathcal{V}^*$$

where we assume that the embedding $\mathcal{V} \hookrightarrow \mathcal{V}_2$ is dense and continuous with $\|\varphi\|_{\mathcal{V}_2} \leq c\|\varphi\|_{\mathcal{V}}$ for $\varphi \in \mathcal{V}$ and $\mathcal{V}_2 \hookrightarrow \mathcal{H}$ is a dense compact embedding. We denote by $\langle\ ,\ \rangle_{\mathcal{V}^*,\mathcal{V}}$, etc., the usual duality products [26]. These duality products are the extensions by continuity of the inner product in $\mathcal{H}$, denoted by $\langle\ ,\ \rangle$ throughout. The norm in $\mathcal{H}$ will be denoted by $\|\cdot\|$ while those in $\mathcal{V}$, $\mathcal{V}_2$ etc. will carry an appropriate subscript. The operators $\mathcal{A}_1$ and $\mathcal{A}_2$ are defined (under the assumptions below) as usual in terms of their sesquilinear forms $\sigma_1 : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ and $\sigma_2 : \mathcal{V}_2 \times \mathcal{V}_2 \to \mathbb{C}$. That is, $\mathcal{A}_1 \in \mathcal{L}(\mathcal{V}, \mathcal{V}^*), \mathcal{A}_2 \in \mathcal{L}(\mathcal{V}_2, \mathcal{V}_2^*)$ and $\langle \mathcal{A}_1\varphi, \psi\rangle_{\mathcal{V}^*,\mathcal{V}} = \sigma_1(\varphi, \psi), \langle \mathcal{A}_2\varphi, \psi\rangle_{\mathcal{V}_2^*,\mathcal{V}_2} = \sigma_2(\varphi, \psi)$.

In addition, we make the following assumptions (note that except for (2.9) assumptions A1) – A6) are the same as in [7]).

A1) The form σ_1 is a Hermitian sesquilinear form: for $\varphi, \psi \in \mathcal{V}$

$$\sigma_1(\varphi, \psi) = \overline{\sigma_1(\psi, \varphi)}. \tag{2.4}$$

A2) The form σ_1 is $\mathcal{V}$ bounded: for $\varphi, \psi \in \mathcal{V}$

$$|\sigma_1(\varphi, \psi)| \leq c_1\|\varphi\|_{\mathcal{V}}\|\psi\|_{\mathcal{V}}. \tag{2.5}$$

A3) The form σ_1 is strictly coercive on $\mathcal{V}$: for $\varphi \in \mathcal{V}$

$$\operatorname{Re}\sigma_1(\varphi, \varphi) = \sigma_1(\varphi, \varphi) \geq k_1\|\varphi\|_{\mathcal{V}}^2, \quad k_1 > 0. \tag{2.6}$$

A4) The form σ_2 is bounded on $\mathcal{V}_2$: for $\varphi, \psi \in \mathcal{V}_2$

$$|\sigma_2(\varphi, \psi)| \leq c_2\|\varphi\|_{\mathcal{V}_2}\|\psi\|_{\mathcal{V}_2}. \tag{2.7}$$

A5) The real part of σ_2 is coercive and is symmetric on $\mathcal{V}_2$:

$$\operatorname{Re}\sigma_2(\varphi, \varphi) + \lambda_0\|\varphi\|^2 \geq k_2\|\varphi\|_{\mathcal{V}_2}^2 \quad k_2 > 0, \lambda_0 \geq 0 \tag{2.8}$$

$$\operatorname{Re}\sigma_2(\varphi, \psi) = \operatorname{Re}\sigma_2(\psi, \varphi), \text{ for any } \varphi, \psi \in \mathcal{V}_2. \tag{2.9}$$

We note that the condition in (2.9) is weaker than requiring that σ_2 be Hermitian.

A6) The forcing term f satisfies

$$f \in L^2([0,T], \mathcal{V}_2^*). \tag{2.10}$$

A7) The operator $\mathcal{N}$ in the nonlinear term satisfies

$$\mathcal{N} \in \mathcal{L}(\mathcal{V}, \mathcal{H}) \text{ with } \|\mathcal{N}\varphi\| \le \sqrt{k}\, \|\varphi\|_{\mathcal{V}}. \tag{2.11}$$

To prove that weak solutions are unique we need to replace A7) by the strengthened condition:

A7a) The operator $\mathcal{N}$ satisfies

$$\mathcal{N} \in \mathcal{L}(\mathcal{V}_2, \mathcal{H}) \text{ with } \|\mathcal{N}\varphi\| \le \sqrt{\tilde{k}}\, \|\varphi\|_{\mathcal{V}_2} \tag{2.12}$$

and the range of $\mathcal{N}$ on $\mathcal{V}$ is dense in $\mathcal{H}$.

Note that (2.12) implies (2.11) with $k = c^2\tilde{k}$.

A8) The nonlinear function $g : \mathcal{H} \to \mathcal{H}$ is a continuous nonlinear mapping of real gradient (or potential) type. This means that there exists a continuous Frechet-differentiable nonlinear functional $G : \mathcal{H} \to R^1$, whose Frechet derivative $G'(\varphi) \in \mathcal{L}(\mathcal{H}, R^1)$ at any $\varphi \in \mathcal{H}$ can be represented in the form

$$G'(\varphi)\psi = \mathrm{Re}\langle g(\varphi), \psi\rangle \quad \text{for any } \psi \in \mathcal{H}. \tag{2.13}$$

We also require that there are constants C_1, C_2, C_3 and $\varepsilon > 0$ such that

$$-\frac{1}{2}k^{-1}(k_1 - \varepsilon)\|\varphi\|^2 - C_1 \le G(\varphi) \le C_2\|\varphi\|^2 + C_3, \tag{2.14}$$

where k is from (2.11) and k_1 from (2.6).

In the case $\mathcal{V} = \mathcal{V}_2$ it is possible to take $\varepsilon = 0$.

A9) The nonlinear function g also satisfies

$$\|g(\varphi)\| \le \tilde{C}_1\|\varphi\| + \tilde{C}_2, \quad \varphi \in \mathcal{H}, \tag{2.15}$$

for some constants $\tilde{C}_1$, $\tilde{C}_2$.

An additional condition is necessary for uniqueness of solutions as well as for the integral equation semigroup formulation of the problem discussed in Section 7 below.

A10) For any $\varphi \in \mathcal{H}$ the Frechet derivative of g exists and satisfies

$$g'(\varphi) \in \mathcal{L}(\mathcal{H}, \mathcal{H}) \text{ with } \|g'(\varphi)\|_{\mathcal{L}(\mathcal{H},\mathcal{H})} \leq \widetilde{C}_3. \tag{2.16}$$

Let $\mathcal{L}_T$ denote the space of functions $w : [0, T] \to \mathcal{H}$ such that

$$w \in C_W([0,T], \mathcal{V}_2) \cap L^\infty([0,T], \mathcal{V})$$

(W means weak continuity), and

$$w_t \in C_W([0,T], \mathcal{H}) \bigcap L^2([0,T], \mathcal{V}_2),$$

where the time derivative w_t is understood in the sense of distributions with values in a Hilbert Space (see, e.g., [16]). The space $\mathcal{L}_T$ is equipped with the norm

$$\|w\|_{\mathcal{L}_T} = \operatorname*{ess\,sup}_{t\in[0,T]} \left(\|w_t(t)\| + \|w(t)\|_{\mathcal{V}}\right) + \left(\int_0^T \|w_t(t)\|^2_{\mathcal{V}_2} dt\right)^{1/2}. \tag{2.17}$$

A11) We assume that for any $u, v \in \mathcal{L}_T$, the following inequality is satisfied for any $t \in [0,T]$:

$$\begin{aligned}&\int_0^t \Big\{ \operatorname{Re}\langle g(\mathcal{N}u(\tau)) - g(\mathcal{N}v(\tau)), \mathcal{N}u(\tau) - \mathcal{N}v(\tau)\rangle \\ &+k_1 k^{-1} \|\mathcal{N}u(\tau) - \mathcal{N}v(\tau)\|^2 \Big\}\, dt \\ &+a\left(\left(\int_0^t \|u(\tau) - v(\tau)\|^2\, dt\right)^{1/2}\right) \geq 0,\end{aligned} \tag{2.18}$$

where $a(\xi) \geq 0$ is a continuous function in $\xi \geq 0$ such that

i) $a(0) = 0$,

ii) there exists a first derivative such that $a'(0) = 0$.

Note that (2.18) is satisfied if, for example,

$$\operatorname{Re}\langle g(\varphi) - g(\psi), \varphi - \psi\rangle + k_1 k^{-1} \|\varphi - \psi\|^2 \geq 0 \tag{2.19}$$

for any $\varphi, \psi \in \mathcal{H}$, where k and k_1 are the constants in (2.11) and (2.6).

We say that $w \in \mathcal{L}_T$ is a *weak solution* of the problem (2.1) – (2.3) with $\varphi_0 \in \mathcal{V}$ and $\varphi_1 \in \mathcal{H}$ if it satisfies the equation:

$$\int_0^t \Big[- \langle w_\tau(\tau), \eta_\tau(\tau) \rangle + \sigma_1 (w(\tau), \eta(\tau)) + \sigma_2 (w_\tau(\tau), \eta(\tau)) +$$

$$+ \langle g(\mathcal{N} w(\tau)), \mathcal{N}\eta(\tau) \rangle \Big] d\tau + \langle w_t(t), \eta(t) \rangle =$$

$$= \langle \varphi_1, \eta(0) \rangle + \int_0^t \langle f(\tau), \eta(\tau) \rangle_{\mathcal{V}_2^*, \mathcal{V}_2} d\tau, \tag{2.20}$$

for any $t \in [0, T]$ and any $\eta \in \mathcal{L}_T$, as well as the initial condition

$$w(0) = \varphi_0. \tag{2.21}$$

We note that this notion of weak solution of (2.1)-(2.3) agrees with the usual one [16] in that it yields $w_{tt} \in L^2([0,T], \mathcal{V}^*) = L^2([0,T], \mathcal{V})^*$ with (2.1) holding in the sense of $L^2([0,T], \mathcal{V}^*)$. Also, the class of test functions η used in this definition are somewhat smoother than necessary (e.g., see the remarks following (6.20) in the existence theorem below).

3 The Main *a priori* Estimate

Under our standing assumption A1)-A9) and the formulations of the previous section, Equation (2.1) or equivalently (2.20) can be written

$$\langle w_{tt}, \eta \rangle_{\mathcal{V}^*, \mathcal{V}} + \sigma_1(w, \eta) + \sigma_2(w_t, \eta) + \langle g(\mathcal{N} w), \mathcal{N}\eta \rangle = \langle f, \eta \rangle_{\mathcal{V}_2^*, \mathcal{V}_2} \tag{3.1}$$

for all $\eta \in \mathcal{L}_T$ and for almost all $t \in [0, T]$.

Treating this equation formally for the present, choosing $\eta = w_t$ and taking the real part we find that if a solution exists, it must satisfy:

$$\frac{d}{dt} \left\{ \frac{1}{2} \|w_t\|^2 + \frac{1}{2} \sigma_1(w, w) + G(\mathcal{N} w) \right\} + \operatorname{Re} \sigma_2(w_t, w_t) = \operatorname{Re} \langle f, w_t \rangle_{\mathcal{V}_2^*, \mathcal{V}_2} \tag{3.2}$$

Here we have used the fact that due to (2.13) we have

$$\frac{d}{dt} G(\mathcal{N} w) = \operatorname{Re} \langle g(\mathcal{N} w), \mathcal{N} w_t \rangle.$$

Using the conditions (2.4) – (2.8), (2.10), (2.11), (2.14) we obtain from (3.2):

$$\|w_t\|^2 + \varepsilon \|w\|_{\mathcal{V}}^2 + k_2 \int_0^t \|w_\tau(\tau)\|_{\mathcal{V}_2}^2 d\tau \le \|\varphi_1\|^2 + \tilde{c}_1 \|\varphi_0\|_{\mathcal{V}}^2 +$$

$$+ \frac{1}{k_2} \int_0^t \|f(\tau)\|_{\mathcal{V}_2^*}^2 d\tau + 2\lambda_0 \int_0^t \|w_\tau(\tau)\|^2 d\tau + 2C_1 + 2C_3, \tag{3.3}$$

where $\tilde{c}_1 = c_1 + 2kC_2$.

To obtain (3.3) we first use (2.8) in (3.2) to obtain

$$\frac{d}{dt}\left\{\frac{1}{2}\|w_t\|^2 + \frac{1}{2}\sigma_1(w,w) + G(\mathcal{N}w)\right\} + k_2\|w_t\|^2_{\mathcal{V}_2} \leq$$

$$\delta\|w_t\|^2_{\mathcal{V}_2} + \frac{1}{4\delta}\|f\|^2_{\mathcal{V}_2^*} + \lambda_0\|w_t\|^2 \tag{3.4}$$

for any $\delta > 0$. We next choose $\delta = k_2/2$, integrate the terms in (3.4) from 0 to t and use

$$\sigma_1(w,w) + 2G(\mathcal{N}w) \geq k_1\|w\|^2_{\mathcal{V}} - k^{-1}(k_1 - \varepsilon)\|\mathcal{N}w\|^2 - 2C_1$$

$$\geq k_1\|w\|^2_{\mathcal{V}} - (k_1 - \varepsilon)\|w\|^2_{\mathcal{V}} - 2C_1 = \varepsilon\|w\|^2_{\mathcal{V}} - 2C_1,$$

$$G(\mathcal{N}\varphi_0) \leq kC_2\|\varphi_0\|^2_{\mathcal{V}} + C_3$$

and

$$|\sigma_1(\varphi_0,\varphi_0)| \leq c_1\|\varphi_0\|^2_{\mathcal{V}}.$$

Having established (3.3), by ignoring the 2nd and 3rd terms on the left in (3.3) and applying Gronwall's lemma, we obtain

$$\|w_t(t)\|^2 \leq \left(\|\varphi_1\|^2 + c_1\|\varphi_0\|^2_{\mathcal{V}} + \frac{1}{k_2}\int_0^T \|f(\tau)\|^2_{\mathcal{V}_2^*}d\tau + 2C_1 + 2C_3\right)e^{2\lambda_0 t}. \tag{3.5}$$

Substituting (3.5) back into (3.3) we have

$$\|w_t\|^2 + \varepsilon\|w\|^2_{\mathcal{V}} + k_2\int_0^t \|w_\tau(\tau)\|^2_{\mathcal{V}_2}d\tau \leq C, \tag{3.6}$$

where the constant $C = C(\|\varphi_1\|, \|\varphi_0\|_{\mathcal{V}}, \|f\|_{L^2([0,T],\mathcal{V}_2^*)})$ is easily computable.

4 Galerkin Approximations

Let $\{\psi_k\}_{k=1}^{\infty} \subset \mathcal{V}$ be any total linearly independent system in $\mathcal{V}$. We assume without loss of generality that the elements ψ_j have been normalized in $\mathcal{V}$ and hence are uniformly bounded in $\mathcal{H}$ and $\mathcal{V}$.

We define the "Galerkin" approximations for (2.1) by

$$w^N(t) = \sum_{k=1}^{N} c_k^N(t)\psi_k, \tag{4.1}$$

where the $\{c_k^N(t)\}_{k=1}^N$ are chosen so that $w^N(t)$ is the unique solution of

$$\frac{d^2}{dt^2}\langle w^N(t), \psi_j\rangle + \sigma_1(w^N(t), \psi_j) + \frac{d}{dt}\sigma_2(w^N(t), \psi_j) +$$
$$\langle g(\mathcal{N} w^N(t)), \mathcal{N}\psi_j\rangle = \langle f, \psi_j\rangle_{\mathcal{V}_2^*, \mathcal{V}_2} \tag{4.2}$$

for $j = 1, \ldots, N,$ with initial conditions

$$c_k^N(0) = c_{0k}^N, \quad \frac{d}{dt}c_k^N(0) = c_{1k}^N,$$

where $\{c_{0k}^N\}, \{c_{1k}^N\}$ are chosen so that $\varphi_0 = \lim_{N\to\infty}\sum_1^N c_{0k}^N\psi_k, \varphi_1 = \lim_{N\to\infty}\sum_1^N c_{1k}^N\psi_k$ where the limits are in the $\mathcal{V}$ and $\mathcal{H}$ sense, respectively.

Multiplying (4.2) by $\frac{d}{dt}\overline{c_j^N(t)}$ and summing over $j = 1, \ldots, N$, we obtain (3.2) with w replaced by w^N. Repeating the above arguments, we then obtain

$$\|w_t^N\|^2 + \varepsilon\|w^N\|_{\mathcal{V}}^2 + k_2\int_0^t \|w_\tau^N(\tau)\|_{\mathcal{V}_2}^2 d\tau \le \widetilde{C}, \tag{4.3}$$

where the constant $\tilde{C}$ is independent of N, depending only on φ_0, φ_1 and f as in the constant C of (3.6). (We note that the convergences $\varphi_0^N \to \varphi_0$ in $\mathcal{V}$, $\varphi_1^N \to \varphi_1$ in $\mathcal{H}$ guarantee uniform boundedness of $\|\varphi_0^N\|_{\mathcal{V}}$ and $\|\varphi_1^N\|$.)

5 Convergence of the Galerkin Approximations

To establish existence of solutions to (2.1)-(2.3), we shall use the bounds of (4.3) to extract successive subsequences of the Galerkin approximations and argue that the final subsequence converges to a solution for the problem. In these arguments, we shall not distinguish subsequences but shall denote by the same symbol $\{w^N\}$ the subsequences of $\{w^N\}_{N=1}^\infty$ selected at each step.

It follows from (4.3) that the set $\{w^N\}$ is bounded in $C([0,T],\mathcal{V}) \subset L^2([0,T],\mathcal{V})$ and $\{w_t^N\}$ is bounded in $C([0,T],\mathcal{H})$ and in $L^2([0,T],\mathcal{V}_2)$. This allows us to conclude that there exist a subsequence such that

$$w^N \longrightarrow w \text{ weakly in } L^2([0,T],\mathcal{V}) \tag{5.1}$$

$$w_t^N \longrightarrow \widehat{w} \text{ weakly in } L^2([0,T]),\mathcal{V}_2). \tag{5.2}$$

Reasoning as in [7], we can readily show that $w_t(t)$ exists in the $\mathcal{V}_2$ sense and $w_t(t) = \widehat{w}(t)$ a.e. in $[0,T]$. These considerations were sufficient to treat the linear problem in [7]. The nonlinear case requires some additional effort.

The main lemma needed to carry out the proof of existence is the following.

Lemma 5.1 *There exists a subsequence $\{w^N\}$ of the original sequence of Galerkin approximations and $w \in \mathcal{L}_T$ such that the following statements hold.*

a)

$$w^N \to w \text{ weakly in } L^2([0,T],\mathcal{V}); \tag{5.3}$$

b) The set $\{w^N\}$ is an equicontinuous and bounded subset of $C([0,T],\mathcal{V}_2)$; moreover,

$$w^N(t) \to w(t) \text{ weakly in } \mathcal{V}_2 \tag{5.4}$$

uniformly in $t \in [0,T]$, i.e., $w^N \to w$ in $C_W([0,T],\mathcal{V}_2)$;

c)

$$w_t^N \to w_t \text{ weakly in } L^2([0,T],\mathcal{V}_2); \tag{5.5}$$

d) The set $\{w_t^N\}$ is bounded in $C([0,T],\mathcal{H})$ and equicontinuous in $C_W([0,T],\mathcal{H})$; moreover

$$w_t^N(t) \to w_t(t) \text{ weakly in } \mathcal{H} \tag{5.6}$$

uniformly in $t \in [0,T]$;

e)

$$w_t^N \to w_t \text{ strongly in } L^2([0,T],\mathcal{H}); \tag{5.7}$$

f) There exists $h \in L^2([0,T],\mathcal{H})$ such that

$$g(\mathcal{N}w^N) \to h \text{ weakly in } L^2([0,T],\mathcal{H}). \tag{5.8}$$

Remark 5.1 i) The statements (5.3) and (5.5) are just repetitions of (5.1) and (5.2). Statement f) follows immediately from (4.3), A7) and A9).

ii) We shall make use of the following version of the Arzela-Ascoli theorem [19, Thm 3.17.24]: If Y is a complete metric space and $\mathcal{F} \subset C([0,T],Y)$, then $\mathcal{F}$ is relatively compact if and only if $\mathcal{F}$ is equicontinuous and $\{f(t) : f \in \mathcal{F}\}$ is relatively compact in Y for each $t \in [0,T]$.

iii) Note that b) along with the compactness of the embedding $\mathcal{V}_2 \subset \mathcal{H}$ implies that

$$w^N \to w \text{ strongly in } C([0,T],\mathcal{H}). \tag{5.9}$$

iv) Statement (5.5) does not imply (5.7), since the embedding $L^2([0,T],\mathcal{V}_2) \subset L^2([0,T],\mathcal{H})$ is not compact even though $\mathcal{V}_2$ embeds compactly in $\mathcal{H}$.

For this lemma, we thus only need to prove the statements b), d) and e). We consider b) first.

From the main *a priori* estimate (4.3) we see that $\{w^N\}$ is bounded in $C([0,T],\mathcal{V})$:

$$\max_{t\in[0,T]} \|w^N(t)\|_{\mathcal{V}}^2 \le \varepsilon^{-1}\tilde{C}. \tag{5.10}$$

Since $\mathcal{V} \hookrightarrow \mathcal{V}_2$, the set is also bounded in $C([0,T],\mathcal{V}_2)$.

Further, we have

$$\begin{aligned} \|w^N(t+\triangle t) - w^N(t)\|_{\mathcal{V}_2}^2 &= \left\| \int_t^{t+\Delta t} w_\tau^N(\tau)d\tau \right\|_{\mathcal{V}_2}^2 \\ \le \left(\int_t^{t+\Delta t} \|w_\tau^N(\tau)\|_{\mathcal{V}_2} d\tau \right)^2 &\le \triangle t \int_t^{t+\Delta t} \|w_\tau^N(\tau)\|_{\mathcal{V}_2}^2 d\tau \\ &\le k_2^{-1}\tilde{C} \triangle t, \end{aligned} \tag{5.11}$$

and the desired equicontinuity follows. The convergence statement (5.4) then results from use of the Arzela-Ascoli theorem (see ii) of Remark 5.1) with Y chosen as the appropriate closed bounded subset in $\mathcal{V}_2$ taken with the weak topology. (Recall that Y is then a compact metric space - see [11, p. 424,426,434], and the equicontinuity in the sense of (5.11) implies the equicontinuity in the sense of the metric of Y.)

Next, we turn to e) and use a lemma of Aubin [1] (see also [17, p.57-], [23, p.271], [8, Lemma 8.4]) which can be stated succinctly as follows: Let $X_0 \hookrightarrow X \hookrightarrow X_1$ where the embedding $X_0 \hookrightarrow X$ is compact. Define the space $Y = \{y \in L^2([0,T],X_0) : y_t \in L^2([0,T],X_1)\}$ with norm

$$\|y\|_Y = \|y\|_{L^2([0,T],X_0)} + \|y_t\|_{L^2([0,T],X_1)}.$$

Then the embedding $Y \hookrightarrow L^2([0,T],X)$ is compact.

It is actually a corollary of the lemma which we use; it can be stated as follows for our configuration of spaces: Suppose $\{w_t^N\}$ is bounded in $L^2([0,T],\mathcal{V}_2)$ and $\{w_{tt}^N\}$ is bounded in $L^2([0,T],\mathcal{V}^*)$ then $\{w_t^N\}$ is relatively compact in $L^2([0,T],\mathcal{H})$. To obtain this corollary, choose $X_0 = \mathcal{V}_2$, $X = \mathcal{H}$ and $X_1 = \mathcal{V}^*$ in Aubin's lemma.

Since we have $\{w_t^N\}$ bounded in $L^2([0,T],\mathcal{V}_2)$ by the *a priori* estimate (4.3), it suffices for e) of Lemma 5.1 to argue that $\{w_{tt}^N\}$ is bounded in $L^2([0,T],\mathcal{V}^*)$. Since $L^2([0,T],\mathcal{V}^*) =$

$L^2([0,T],\mathcal{V})^*$, it is suffices to show that

$$\begin{aligned} |w_{tt}^N(\Phi)| &= |\int_0^T \langle w_{tt}^N(\tau), \Phi(\tau)\rangle_{\mathcal{V}^*,\mathcal{V}}\, d\tau| \\ &\le K\|\Phi\|_{L^2([0,T],\mathcal{V})} \quad \text{for any } \Phi \in L^2([0,T],\mathcal{V}). \end{aligned}$$

For fixed M, let Φ_M be of the form $\Phi_M(t) = \sum_{k=1}^M a_k(t)\psi_k$ where $a_k \in C^1[0,T]$. From the equation for w^N we find that for $N \ge M$ we must have

$$\begin{aligned} |w_{tt}^N(\Phi_M)| &= |\int_0^T \langle w_{tt}^N(\tau), \Phi_M(\tau)\rangle_{\mathcal{V}^*,\mathcal{V}}\, d\tau| \\ &\le c_1 \int_0^T \|w^N(\tau)\|_{\mathcal{V}} \|\Phi_M(\tau)\|_{\mathcal{V}}\, d\tau + c_2 \int_0^T \|w_t^N(\tau)\|_{\mathcal{V}_2} \|\Phi_M(\tau)\|_{\mathcal{V}_2}\, d\tau \\ &+ \sqrt{k} \int_0^T \|g(\mathcal{N}w^N(\tau))\|\, \|\Phi_M(\tau)\|_{\mathcal{V}}\, d\tau + \int_0^T \|f(\tau)\|_{\mathcal{V}_2^*} \|\Phi_M(\tau)\|_{\mathcal{V}_2}\, d\tau. \end{aligned}$$

From the *a priori* bounds, A9), A6) and standard inequalities, we find that this estimate leads to

$$|w_{tt}^N(\Phi_M)| \le K\|\Phi_M\|_{L^2([0,T],\mathcal{V})} \tag{5.12}$$

where the constant K depends on $c_1, c_2, k, \tilde{C}, \tilde{C}_1, \tilde{C}_2$ but not N or Φ_M. Since elements of the form $\{\Phi_M\}_{M=1}^\infty$ form a dense subset of $L^2([0,T],\mathcal{V})$, the desired boundedness is readily inferred and thus e) is established.

Finally, we consider d). The boundedness statement follows from (4.3) and the convergence statement will once again follow from an application of Arzela-Ascoli in $C_W([0,T],\mathcal{H})$ once we establish the equicontinuity. To do this we first note that

$$|\langle w_t^N(t+\Delta t) - w_t^N(t), v\rangle| \;\le\; \hat{k}\|v\|_{\mathcal{V}}\sqrt{|\Delta t|} \tag{5.13}$$

for $v \in \mathcal{V}$ which is obtained using arguments similar to those employed in obtaining (5.11) and (5.12). Assume now that $\varphi \in \mathcal{H}$ and fix $\epsilon > 0$. For $v \in \mathcal{V}$ (and $t, t+\Delta t \in [0,T]$) we have

$$\begin{aligned} &|\langle w_t^N(t+\Delta t) - w_t^N(t), \varphi\rangle| \\ &\le |\langle w_t^N(t+\Delta t) - w_t^N(t), v\rangle| + |\langle w_t^N(t+\Delta t) - w_t^N(t), \varphi - v\rangle| \\ &\le \hat{k}\|v\|_{\mathcal{V}}\sqrt{|\Delta t|} + 2\tilde{C}\|\varphi - v\|, \end{aligned} \tag{5.14}$$

where $\tilde{C}$ is the constant from (4.3). Selecting v so that $2\tilde{C}\|\varphi - v\| \le \epsilon/2$ we can conclude that the right hand side of (5.14) does not exceed ϵ for any N if $|\Delta t| \le \delta = (\epsilon/2\hat{k}\|v\|_{\mathcal{V}})^2$, and the desired equicontinuity follows.

Remark 5.2 We note that the statement b) of Lemma 5.1 can be strengthened. Namely, the set $\{w^N\}$ is equicontinuous in $C_W([0,T],\mathcal{V})$. Since, due to (4.3), it is also bounded in $C([0,T],\mathcal{V})$ we can conclude that

$$w^N \to w \text{ in } C_W([0,T],\mathcal{V}). \tag{5.15}$$

The convergence (5.15) will not be used in the next section, where we show that w satisfies (2.20) and (2.21). However, (5.15) is important to verify that $w \in \mathcal{L}_T$. Indeed, from Lemma 5.1 we can only conclude that $w \in C_W([0,T],\mathcal{V}_2) \cap L^2([0,T],\mathcal{V})$ and $w_t \in C_W([0,T],\mathcal{H}) \cap L^2([0,T],\mathcal{V}_2)$, which is sufficient for (2.20) to make sense, but we cannot conclude that $w \in L^\infty([0,T],\mathcal{V})$.

We sketch the proof of (5.15) which establishes, in fact, that $w \in C_W([0,T],\mathcal{V})$. Note that, due to our assumptions A1)-A3) on σ_1, the mapping $\mathcal{A}_1 : \mathcal{V} \to \mathcal{V}^*$ is a topological isomorphism (see, e.g., [26], sec. 17). Define $\mathrm{dom}\mathcal{A}_1 = \{v \in \mathcal{V} : \mathcal{A}_1 v \in \mathcal{H}\} = \mathcal{A}_1^{-1}(\mathcal{H})$. Since $\mathcal{H}$ is dense in $\mathcal{V}^*$, we see that $\mathrm{dom}\mathcal{A}_1$ is dense in $\mathcal{V}$. We prove the equicontinuity of $\{w^N\}$ in $C_W([0,T],\mathcal{V})$ assuming that $\mathcal{V}$ is equipped with the inner product $\sigma_1(\cdot,\cdot)$, which is equivalent to the original inner product in $\mathcal{V}$. Let $v \in \mathrm{dom}\mathcal{A}_1$. Then we have

$$\begin{aligned} |\sigma_1(v, w^N(t+\Delta t) - w^N(t))| &= |\langle \mathcal{A}_1 v, w^N(t+\Delta t) - w^N(t)\rangle| \\ &\le \|\mathcal{A}_1 v\| \int_t^{t+\Delta t} \|w_\tau^N(\tau)\| d\tau \le \sqrt{\tilde{C}} \|\mathcal{A}_1 v\| \, |\Delta t|, \end{aligned} \tag{5.16}$$

where $\tilde{C}$ is the constant from (4.3). Since $\mathrm{dom}\mathcal{A}_1$ is dense in $\mathcal{V}$, the desired equicontinuity can be deduced from (5.16) by an argument similar to the one used in deriving d) from (5.13).

6 Existence of Weak Solutions

In this section we verify, based on Lemma 5.1, that it is possible to pass to the limit in an integral identity (see (6.3) below) for the Galerkin approximations. We thus obtain the fundamental existence results.

Theorem 6.1 *Under assumptions A1) - A9) and A11), there exists a weak solution of (2.1)-(2.3) (or equivalently (3.1), (2.2), (2.3)). If in addition, A7a) and A10) hold, then the solution is unique.*

To give the arguments for this theorem, we denote by $\mathcal{P}_M$ ($M = 1, 2, \cdots$) the class of functions $\eta \in \mathcal{L}_T$, which can be represented in the form

$$\eta(t) = \sum_{k=1}^{M} a_k(t)\psi_k, \tag{6.1}$$

where $a_k \in C^1([0,T])$. Let

$$\mathcal{P} = \bigcup_{M=1}^{\infty} \mathcal{P}_M. \tag{6.2}$$

It is obvious that $\mathcal{P}$ is dense in $\mathcal{L}_T$. Recalling the definition of the Galerkin approximation in (4.1), (4.2), we multiply the jth equation in (4.2) by $a_j(t)$, take the sum from 1 to M and integrate over $[0,t]$ to obtain

$$\begin{aligned}&\int_0^t \Big[-\langle w_\tau^N(\tau), \eta_\tau(\tau)\rangle + \sigma_1(w^N(\tau), \eta(\tau)) + \sigma_2(w_\tau^N(\tau), \eta(\tau)) + \\ &\langle g(\mathcal{N} w^N(\tau)), \mathcal{N}\eta(\tau)\rangle\Big]\, d\tau + \langle w_t^N(t), \eta(t)\rangle - \langle w_t^N(0), \eta(0)\rangle \\ &= \int_0^t \langle f(\tau), \eta(\tau)\rangle_{\mathcal{V}_2^*, \mathcal{V}_2}\, d\tau\end{aligned} \tag{6.3}$$

which is satisfied for all $\eta \in \mathcal{P}_M$, for $M \leq N$.

Now, fix $\eta \in \mathcal{P}_M$ with $M \leq N$. Using (5.3)-(5.8), we can pass to the limit $N \to \infty$ in (6.3) and obtain

$$\begin{aligned}&\int_0^t [-\langle w_\tau(\tau), \eta_\tau(\tau)\rangle + \sigma_1(w(\tau), \eta(\tau)) + \sigma_2(w_\tau(\tau), \eta(\tau)) + \\ &\langle h(\tau), \mathcal{N}\eta(\tau)\rangle]\, d\tau + \langle w_t(t), \eta(t)\rangle - \langle \varphi_1, \eta(0)\rangle \\ &= \int_0^t \langle f(\tau), \eta(\tau)\rangle_{\mathcal{V}_2^*, \mathcal{V}_2}\, d\tau.\end{aligned} \tag{6.4}$$

Let us explain the latter statement in a more detailed manner. Note, first of all, that all the statements of Lemma 5.1 are true for any interval $[0,t]$, $t \leq T$. Now, let us examine all four terms under the integral in the left-hand side of (6.3). To pass to the limit in the first term we only need the weak convergence $w_\tau^N \to w_\tau$ in $L^2([0,t], \mathcal{H})$, which follows, e.g., from (5.5) or from (5.7). To treat the second term we note that for a fixed $\eta \in \mathcal{L}_T$ the mapping $u \to \int_0^t \sigma_1(u(\tau), \eta(\tau))d\tau$ is a bounded linear functional on $L^2([0,t], \mathcal{V})$ due to (2.5). Therefore, this functional is weakly continuous, and we can pass to the limit due to (5.3). A similar argument holds for the third term due to (2.7). In the fourth term we can pass to the limit due to (5.8). Finally, in the first term outside the integral in the left side of (6.3) we can pass to the limit due to (5.6) and in the second term due to the fact that $w_t^N(0) \to \varphi_1$ in $\mathcal{H}$ as $N \to \infty$.

Equation (6.4) is satisfied for all $\eta \in \mathcal{P}_M$ for all M, and, therefore for all $\eta \in \mathcal{L}_T$ since $\mathcal{P}$ is dense in $\mathcal{L}_T$. Except for the term involving the limit function h, this is the equation for weak solutions (see (2.20)). The condition (2.21) is clearly satisfied since $w^N(0) \to \varphi_0$ in $\mathcal{V}$ as $N \to \infty$. We argue that the h term is the correct term involving $g(\mathcal{N}w(t))$ to yield that the limit function w is a weak solution. To prove this we use the Minty-Browder monotonicity method [2, 18] whose application to quasilinear parabolic equations can be found in [14].

Lemma 6.1 *For any $\eta \in \mathcal{L}_T$ and for $t \in [0,T]$*

$$\int_0^t \langle g(\mathcal{N}w(\tau)), \mathcal{N}\eta(\tau)\rangle \, d\tau = \int_0^t \langle h(\tau), \mathcal{N}\eta(\tau)\rangle \, d\tau \tag{6.5}$$

Proof: The condition (2.18) plays a crucial role in this proof. Combining (2.18) with (2.11), we obtain

$$\int_0^t \Big[\mathrm{Re}\langle g(\mathcal{N}u(\tau)) - g(\mathcal{N}v(\tau)), \mathcal{N}u(\tau) - \mathcal{N}v(\tau)\rangle + \tag{6.6}$$
$$k_1 \|u(\tau) - v(\tau)\|_{\mathcal{V}}^2 \Big] \, d\tau + a\left(\left(\int_0^t \|u(\tau) - v(\tau)\|^2 \, d\tau \right)^{1/2} \right) \geq 0,$$

for any $u, v \in \mathcal{L}_T$.

Now consider (6.6) with $u = w^N \in \mathcal{L}_T$ and any $v \in \mathcal{P}_M \subset \mathcal{L}_T$ with $M \leq N$. Taking into account (2.6) we obtain

$$\int_0^t \big[\mathrm{Re}\langle g(\mathcal{N}w^N(\tau)) - g(\mathcal{N}v(\tau)), \mathcal{N}w^N(\tau) - \mathcal{N}v(\tau)\rangle + \tag{6.7}$$
$$\sigma_1(w^N(\tau) - v(\tau), w^N(\tau) - v(\tau)) \big] \, d\tau + a\left(\|w^N - v\|_{L^2([0,T],\mathcal{H})} \right) \geq 0.$$

We next return to (6.3) with $\eta = w^N - v$, (notice that this is possible since $w^N \in \mathcal{P}_N$ and $v \in \mathcal{P}_M$ with $M \leq N$). Taking the real parts of both sides, we obtain an expression which can be written in the form

$$\mathrm{Re} \int_0^t \langle g(\mathcal{N}w^N(\tau)), \mathcal{N}w^N(\tau) - \mathcal{N}v(\tau)\rangle \, d\tau = \tag{6.8}$$
$$\mathrm{Re} \int_0^t \big[\langle w_\tau^N(\tau), w_\tau^N(\tau) - v_\tau(\tau)\rangle - \sigma_1(w^N(\tau), w^N(\tau) - v(\tau)) -$$
$$-\sigma_2(w_\tau^N(\tau), w^N(\tau) - v(\tau)) \big] \, d\tau - \mathrm{Re}\langle w_t^N(t), w^N(t) - v(t)\rangle +$$
$$+ \mathrm{Re}\langle w_t^N(0), w^N(0) - v(0)\rangle + \mathrm{Re} \int_0^t \langle f(\tau), w^N(\tau) - v(\tau)\rangle_{\mathcal{V}_2^*, \mathcal{V}_2} \, d\tau$$

Substituting (6.8) into (6.7) we obtain, after a straightforward simplification: for all $v \in \mathcal{P}_M$ $(M \le N)$

$$
\begin{aligned}
&\|w_t^N\|^2_{L^2([0,t],\mathcal{H})} - \frac{1}{2}\operatorname{Re}\sigma_2(w^N(t), w^N(t)) \qquad (6.9)\\
&+\frac{1}{2}\operatorname{Re}\sigma_2(w^N(0), w^N(0)) + \operatorname{Re}\int_0^t \Big[-\langle w_\tau^N(\tau), v_\tau(\tau)\rangle + \\
&+\sigma_2(w_\tau^N(\tau), v(\tau)) + \langle f(\tau), w^N(\tau) - v(\tau)\rangle_{\mathcal{V}_2^*,\mathcal{V}_2} \\
&-\langle g(\mathcal{N}v(\tau)), \mathcal{N}w^N(\tau) - \mathcal{N}v(\tau)\rangle - \sigma_1(v(\tau), w^N(\tau) - v(\tau))\Big]\, d\tau \\
&-\operatorname{Re}\langle w_t^N(t), w^N(t) - v(t)\rangle + \operatorname{Re}\langle w_t^N(0), w^N(0) - v(0)\rangle \\
&+a\left(\|w^N - v\|_{L^2([0,T],\mathcal{H})}\right) \ge 0.
\end{aligned}
$$

Here we have used the fact that, due to the symmetry of the real part of σ_2 it follows that:

$$\operatorname{Re}\sigma_2(w_t^N, w^N) = \frac{1}{2}\frac{d}{dt}\operatorname{Re}\sigma_2(w^N, w^N).$$

Now the most important observation is that we can pass to the limit $N \to \infty$ in (6.9) to obtain

$$
\begin{aligned}
&\|w_t\|^2_{L^2([0,T],\mathcal{H})} - \frac{1}{2}\operatorname{Re}\sigma_2(w(t), w(t)) \qquad (6.10)\\
&+\frac{1}{2}\operatorname{Re}\sigma_2(w(0), w(0)) + \operatorname{Re}\int_0^t [-\langle w_\tau(\tau), v_\tau(\tau)\rangle \\
&+\sigma_2(w_\tau(\tau), v(\tau)) + \langle f(\tau), w(\tau) - v(\tau)\rangle_{\mathcal{V}_2^*,\mathcal{V}_2} \\
&-\langle g(\mathcal{N}v(\tau)), \mathcal{N}w(\tau) - \mathcal{N}v(\tau)\rangle - \sigma_1(v(\tau), w(\tau) - v(\tau))]\, d\tau \\
&-\operatorname{Re}\langle w_t(t), w(t) - v(t)\rangle + \operatorname{Re}\langle w_t(0), w(0) - v(0)\rangle \\
&+a\left(\|w - v\|_{L^2([0,T],\mathcal{H})}\right) \ge 0
\end{aligned}
$$

The inequality (6.10) requires some discussion. In the first term in (6.9), we can pass to the limit due to the strong convergence in e) of Lemma 5.1. In the third term we can pass to the limit because $w^N(0) \to w(0) = \varphi_0$ strongly in $\mathcal{V}$ and, therefore, also in $\mathcal{V}_2$. In all the terms under the integral we can pass to the limit due to the weak convergence a), b), c) and d) in Lemma 5.1. This limit can be justified by precisely the same arguments that were used to justify passing to the limit in (6.3). We only note that in the third term under the integral in (6.9) we use the weak convergence

$$w^N \to w \text{ in } L^2([0,T],\mathcal{V}_2),$$

which follows, from (5.3) or (5.4), and to pass to the limit in the fourth term we observe that (5.3) and (2.11) imply the weak convergence

$$\mathcal{N} w^N \to \mathcal{N} w \text{ in } L^2([0,T],\mathcal{H}).$$

In the next two terms outside the integral we can pass to the limit due to (5.6) and (5.9). Finally, in the last term we use the strong convergence $w^N \to w$ in $L^2([0,T],\mathcal{H})$, which follows from (5.9), and the fact that the function a is continuous. It remains to explain why we can pass to the limit in the second term of (6.9). Here we only have the *weak* convergence

$$w^N(t) \to w(t) \text{ in } \mathcal{V}_2 \tag{6.11}$$

for any $t \in [0,T]$ (see (5.4)).

From (2.7), (2.8), (2.9), $\operatorname{Re}\sigma_2(\cdot,\cdot) + \lambda_0\|\cdot\|^2$ is topologically equivalent to the norm inner product on $\mathcal{V}_2$. Since norms are weakly lower semicontinuous in Hilbert spaces, when passing to the limit we have

$$\operatorname{Re}\sigma_2(w(t),w(t)) \leq \underline{\lim}_{N\to\infty} \operatorname{Re}\sigma_2(w^N(t),w^N(t)). \tag{6.12}$$

Taking into account the inequality in (6.9), we can thus obtain the desired inequality (6.10) when passing to the limit.

Note that (6.10) is valid for any $v \in \mathcal{P} = \cup_{M=1}^{\infty}\mathcal{P}_M$ and, therefore, for any $v \in \mathcal{L}_T$.

Now we return to (6.4). Observe that in this relation we can set $\eta = w$, since $w \in \mathcal{L}_T$ and (6.4) is valid for $\eta \in \mathcal{L}_T$. Taking the real parts of both sides, we obtain after a straightforward computation

$$\begin{aligned} &-\|w_t\|^2_{L^2([0,t],\mathcal{H})} + \frac{1}{2}\operatorname{Re}\sigma_2(w(t),w(t)) - \frac{1}{2}\operatorname{Re}\sigma_2(w(0),w(0)) + \\ &+ \int_0^t \left[\sigma_1(w(\tau),w(\tau)) + \operatorname{Re}\langle h(\tau),\mathcal{N}w(\tau)\rangle - \operatorname{Re}\langle f(\tau),w(\tau)\rangle_{\mathcal{V}_2^*,\mathcal{V}_2}\right] d\tau + \\ &+ \operatorname{Re}\langle w_t(t),w(t)\rangle - \operatorname{Re}\langle w_t(0),w(0)\rangle = 0. \end{aligned} \tag{6.13}$$

Let us now consider (6.4) with $\eta = -v$ where v is from (6.10). Taking the real parts of both sides we obtain

$$\begin{aligned}\mathrm{Re}\int_0^t \Big[\langle w_\tau(\tau), v_\tau(\tau)\rangle - \sigma_1(w(\tau), v(\tau)) - \sigma_2(w_\tau(\tau), v(\tau)) - \\ -\langle h(\tau), \mathcal{N}v(\tau)\rangle + \langle f(\tau), v(\tau)\rangle_{\mathcal{V}_2^*,\mathcal{V}_2}\Big]\, d\tau - \mathrm{Re}\langle w_t(t), v(t)\rangle + \\ + \mathrm{Re}\langle w_t(0), v(0)\rangle = 0\end{aligned} \tag{6.14}$$

We next add the inequality (6.10) and the relations (6.13) and (6.14). After considerable cancellation we arrive at

$$\begin{aligned}\int_0^t \Big[\mathrm{Re}\langle h(\tau) - g(\mathcal{N}v(\tau)), \mathcal{N}w(\tau) - \mathcal{N}v(\tau)\rangle + \\ +\sigma_1(w(\tau) - v(\tau), w(\tau) - v(\tau))\Big] + a\left(\|w - v\|_{L^2([0,T],\mathcal{H})}\right) \geq 0.\end{aligned} \tag{6.15}$$

Now take any $\theta > 0$ and let $\zeta \in \mathcal{L}_T$. Select

$$v(t) = w(t) - \theta\zeta(t). \tag{6.16}$$

Substituting (6.16) into (6.15) and dividing by $\theta > 0$, we obtain

$$\begin{aligned}\int_0^t \Big[\mathrm{Re}\langle h(\tau) - g(\mathcal{N}w(\tau) - \theta\mathcal{N}\zeta(\tau)), \mathcal{N}\zeta(\tau)\rangle + \\ +\theta\sigma_1(\zeta(\tau), \zeta(\tau))\Big]\, d\tau + \theta^{-1}a\Big(\theta\|\zeta\|_{L^2([0,t],\mathcal{H})}\Big) \geq 0,\end{aligned} \tag{6.17}$$

for any $\zeta \in \mathcal{L}_T$, $\theta > 0$. In (6.17) we can pass to the limit $\theta \to 0$ and obtain

$$\mathrm{Re}\int_0^t \langle h(\tau) - g(\mathcal{N}w(\tau)), \mathcal{N}\zeta(\tau)\rangle\, d\tau \geq 0. \tag{6.18}$$

Here we have used the fact that $g : \mathcal{H} \to \mathcal{H}$ is a continuous mapping. We have also used the fact that

$$\lim_{\theta\to 0} \frac{a(\theta\rho)}{\theta} = 0, \text{ for any } \rho \geq 0,$$

which follows from condition ii) following (2.18): $a(0) = a'(0) = 0$.

The inequality (6.18) holds for all $\zeta \in \mathcal{L}_T$ only if it holds for equality. Indeed, suppose that for some ζ we have a strict inequality in (6.18), then replacing ζ by $-\zeta$ we obtain a contradiction to (6.18). Thus we have

$$\mathrm{Re}\int_0^t \langle h(\tau) - g(\mathcal{N}w(\tau)), \mathcal{N}\zeta(\tau)\rangle\, d\tau = 0 \tag{6.19}$$

for all $\zeta \in \mathcal{L}_T$.

It remains to observe, that replacing ζ in (6.19) by $i\zeta$ we obtain that the imaginary part of the integral in (6.19) is also equal to zero. Lemma 6.1 is thus established and the proof of existence is complete.

We turn next to the uniqueness statement of Theorem 6.1. We follow arguments in the spirit of those of [7] which are similar to the classic Ladyzhenskaya arguments [14, 13, 16, 26]. Let w and v be two solutions of (3.1) corresponding to the data φ_0, φ_1, f. Then $u \equiv w - v$ satisfies $u(0) = u_t(0) = 0$ and

$$\langle u_{tt}, \eta \rangle_{\mathcal{V}^*, \mathcal{V}} + \sigma_1(u, \eta) + \sigma_2(u_t, \eta) + \langle g(\mathcal{N}w) - g(\mathcal{N}v), \mathcal{N}\eta \rangle = 0 \tag{6.20}$$

for all $\eta \in \mathcal{L}_T$.

At this point we observe that (6.20), as well as (3.1) and (2.20), will still be satisfied if we extend the class of test functions η. Namely, (6.20) holds for $\eta \in \mathcal{M}_T$, where $\mathcal{M}_T$ denotes the space of functions $\eta : [0, T] \to \mathcal{H}$ such that

$$\begin{aligned} &\eta \in C_W([0, T], \mathcal{V}_2) \cap L^\infty([0, T], \mathcal{V}), \\ &\eta_t \in L^2([0, T], \mathcal{V}_2). \end{aligned}$$

For fixed $s \in (0, T)$, let ψ be defined by

$$\psi(t) = \begin{cases} -\int_t^s u(\theta) d\theta & t < s \\ 0 & t \geq s \end{cases}$$

so that $\psi(T) = 0$, $\psi(s) = 0$ and $\psi(t) \in \mathcal{V}$ for each t. Indeed, $\psi \in \mathcal{M}_T$. (Note that $\psi \notin \mathcal{L}_T$, since $\psi_t \notin C_W([0, T], \mathcal{H})$.) The usual arguments reveal that

$$\int_0^s \{\langle u_{tt}(t), \psi(t) \rangle_{\mathcal{V}^*, \mathcal{V}} + \langle u_t(t), u(t) \rangle\} \; dt = \int_0^s \frac{d}{dt} \langle u_t(t), \psi(t) \rangle \; dt = 0.$$

Hence, choosing $\eta = \psi$ in (6.20) and integrating we obtain

$$\int_0^s \{\langle u_t(t), u(t) \rangle - \sigma_1(u(t), \psi(t)) - \sigma_2(u_t(t), \psi(t)) - \langle \Delta g(t), \mathcal{N}\psi(t) \rangle\} \, dt = 0$$

where $\Delta g(t) \equiv g(\mathcal{N}w(t)) - g(\mathcal{N}v(t))$. Since $\psi_t(t) = u(t)$, we can rewrite this as

$$\begin{aligned} &\int_0^s \frac{d}{dt} \left\{ \|u(t)\|^2 - \sigma_1(\psi(t), \psi(t)) \right\} dt \\ &= \; 2Re \int_0^s \{\sigma_2(u_t(t), \psi(t)) + \langle \Delta g(t), \mathcal{N}\psi(t) \rangle\} \, dt. \end{aligned}$$

But since $u(0) = 0, \psi(s) = 0$, this implies

$$\|u(s)\|^2 + \sigma_1(\psi(0), \psi(0)) = 2Re \int_0^s \{\sigma_2(u_t(t), \psi(t)) + \langle \Delta g(t), \mathcal{N}\psi(t)\rangle\}\, dt.$$

However,

$$\begin{aligned} \int_0^s \sigma_2(u_t(t), \psi(t))dt &= \int_0^s \Big\{\frac{d}{dt}\sigma_2(u(t), \psi(t)) - \sigma_2(u(t), u(t))\Big\}dt \\ &= -\int_0^s \sigma_2(u(t), u(t))\, dt \end{aligned}$$

so that we obtain

$$\|u(s)\|^2 + \sigma_1(\psi(0), \psi(0)) + 2Re \int_0^s \sigma_2(u(t), u(t))dt = 2Re \int_0^s \langle \Delta g(t), \mathcal{N}\psi(t)\rangle\, dt. \tag{6.21}$$

Considering the last term in this equality, using A10) and A7a), we find

$$\begin{aligned} &| \int_0^s \langle \Delta g(t), \mathcal{N}\psi(t)\rangle\, dt| \\ &= | \int_0^s \Big\langle \int_0^1 g'(\theta \mathcal{N}w(t) + (1-\theta)\mathcal{N}v(t))\, [\mathcal{N}w(t) - \mathcal{N}v(t)]\, d\theta, - \int_t^s \mathcal{N}u(\theta)d\theta \Big\rangle\, dt| \\ &\leq \int_0^s \{\tilde{C}_3 \|\mathcal{N}w(t) - \mathcal{N}v(t)\| \int_t^s \|\mathcal{N}u(\theta)\|d\theta\}dt \\ &\leq \int_0^s \tilde{C}_3 \|\mathcal{N}u(t)\|dt \int_0^s \|\mathcal{N}u(\theta)\|d\theta \\ &= \tilde{C}_3 (\int_0^s \|\mathcal{N}u(t)\|dt)^2 \leq \tilde{C}_3 \tilde{k} (\int_0^s \|u(t)\|_{\mathcal{V}_2}dt)^2 \\ &\leq \tilde{C}_3 \tilde{k}\; s \int_0^s \|u(t)\|^2_{\mathcal{V}_2}dt. \end{aligned}$$

Using this along with A5), A3) in (6.21) we obtain

$$\begin{aligned} &\|u(s)\|^2 + k_1 \|\psi(0)\|^2_{\mathcal{V}} + 2k_2 \int_0^s \|u(t)\|^2_{\mathcal{V}_2}dt \\ &\leq 2\tilde{C}_3 \tilde{k}\; s \int_0^s \|u(t)\|^2_{\mathcal{V}_2}dt + 2\lambda_0 \int_0^s \|u(t)\|^2 dt. \end{aligned}$$

This implies

$$\|u(s)\|^2 + (2k_2 - 2\tilde{C}_3\tilde{k}s) \int_0^s \|u(t)\|^2_{\mathcal{V}_2}dt \leq 2\lambda_0 \int_0^s \|u(t)\|^2 dt.$$

Hence for $s < s_0 \equiv k_2/\tilde{C}_3\tilde{k}$ we have

$$\|u(s)\|^2 \leq 2\lambda_0 \int_0^s \|u(t)\|^2 dt.$$

By Gronwall's lemma, we thus find $u(s) \equiv 0$ on $[0, s_0)$ where s_0 is independent of the solutions w, v. It follows that one must have $u \equiv 0$ on $[s_0, 2s_0)$, etc so that $u \equiv 0$ on any finite interval $[0, T]$.

7 Semigroup Formulation

In this section we show that the weak solution of our problem (2.1)-(2.3) satisfies a variation of parameters type integral equation. Our arguments are strongly dependent on the results of Section 3 of [7]. Let us first formally derive this equation. Eq. (2.1) can be formally rewritten as

$$\dot{z}(t) = \mathbf{A}z(t) + F(t) \tag{7.1}$$

where

$$z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} \equiv \begin{pmatrix} w(t) \\ w_t(t) \end{pmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & I \\ -\mathcal{A}_1 & -\mathcal{A}_2 \end{bmatrix}, \tag{7.2}$$

and

$$F(t) = \begin{pmatrix} 0 \\ \Phi(t) \end{pmatrix}, \quad \Phi(t) = f(t) - \mathcal{N}^* g(\mathcal{N} z_1(t)). \tag{7.3}$$

As it was shown in [7], the operator $\mathbf{A}$ generates a C_0-semigroup $S(t)$ on the space $\mathcal{Z} \equiv \mathcal{V} \times \mathcal{H}$, where we can, without loss of generality, use the equivalent σ_1 inner product on $\mathcal{V}$. Moreover, the operator $\mathbf{A}$ can be extended to an operator $\hat{\mathbf{A}} : \mathcal{Z} \to \mathcal{W}$ where $\mathcal{W} = \mathcal{Y}^*, \mathcal{Y} = [dom\mathbf{A}^*]$ with inner product $\langle \Phi, \Psi \rangle_{\mathcal{Y}} = \langle (\lambda - \mathbf{A}^*)\Phi, (\lambda - \mathbf{A}^*)\Psi \rangle_{\mathcal{Z}}$ where $\lambda > \lambda_0$ and $\hat{\mathbf{A}}$ is the infinitesimal generator of a C_0-semigroup $\hat{S}(t)$ on the space $\mathcal{W}$. This semigroup $\hat{S}(t)$ is an extension of $S(t)$ from $\mathcal{Z}$ to $\mathcal{W}$. Moreover, $\mathcal{Z}^* \subset \mathcal{W}$ with $\|\Psi\|_{\mathcal{W}} \le C\|\Psi\|_{\mathcal{Z}^*}$ for $\Psi \in \mathcal{Z}^*$. This semigroup can be used to formally rewrite Eq. (7.1) in the form

$$z(t) = \hat{S}(t)z(0) + \int_0^t \hat{S}(t-\tau)F(\tau)\, d\tau. \tag{7.4}$$

Theorem 7.1 *In addition to the assumptions A1)-A9), A11) used to prove existence of a weak solution in Theorem 6.1, we also assume A7a) and A10). Then the weak solution w satisfies the integral equation (7.4).*

Proof: The proof is based on results in [7] for the corresponding linear problem. First of all we notice that the statement (5.8) of Lemma 5.1 can be strengthened if (2.16) is satisfied. Namely,

$$g(\mathcal{N}w^N) \to h, \text{ in } C_W([0,T], \mathcal{H}) \tag{7.5}$$

or, in other words,

$$g(\mathcal{N}w^N(t)) \to h(t), \text{ weakly in } \mathcal{H} \tag{7.6}$$

uniformly with respect to $t \in [0,T]$. This is certainly correct for a subsequence of $\{w^N\}$ (recall our convention at the beginning of Section 5).

By the Arzela-Ascoli theorem, to prove (7.6) it suffices to show that the set $\left\{g(\mathcal{N}w^N)\right\}_{N=1}^{\infty}$ is uniformly bounded and equicontinuous on $[0,T]$ in the $\mathcal{H}$-norm and recall that bounded sets in $\mathcal{H}$ are sequentially compact in the weak topology. We have

$$\begin{aligned}&\|g(\mathcal{N}w^N(t))\| \le \widetilde{C}_1\|\mathcal{N}w^N(t)\| + \widetilde{C}_2 \\ &\le \widetilde{C}_1\sqrt{k}\|w^N(t)\|_{\mathcal{V}} + \widetilde{C}_2 \le \varepsilon^{-1/2}\sqrt{k}\,\widetilde{C}_1\widetilde{C}^{1/2} + \widetilde{C}_2\end{aligned} \tag{7.7}$$

where we have used (2.15), (2.11) and the main *a priori* estimate (4.3). To show equicontinuity we check that the set $\left\{\frac{d}{dt}g(\mathcal{N}w^N)\right\}_{N=1}^{\infty}$ is bounded in $L^2([0,T],\mathcal{H})$. Here $\frac{d}{dt}$ means the strong $\mathcal{H}$-derivative. We have

$$\begin{aligned}&\|\frac{d}{dt}g(\mathcal{N}w^N(t))\|^2_{L^2([0,T],\mathcal{H})} = \\ &= \int_0^T \|g'(\mathcal{N}w^N(t))\mathcal{N}w_t^N(t)\|^2\,dt \le \\ &\le \widetilde{C}_3^2\int_0^T \|\mathcal{N}w_t^N(t)\|^2\,dt \le \\ &\le \widetilde{k}\widetilde{C}_3^2\int_0^T \|w_t^N(t)\|^2_{\mathcal{V}_2}\,dt \le \widetilde{k}k_2^{-1}k\widetilde{C}_3\widetilde{C},\end{aligned} \tag{7.8}$$

where we have used (2.12), (2.16), and (4.3). Note that in (7.8) we have also used the fact that $(\mathcal{N}w^N)_t(t) = \mathcal{N}w_t^N(t)$ for all $t\in[0,T]$, where on the left the derivative is understood in the sense of distributions with values in $\mathcal{H}$ and on the right with values in $\mathcal{V}_2$. Thus (7.6) is established.

Next we use the additional assumption in A7a) that $\mathcal{N}(\mathcal{V})$ is dense in $\mathcal{H}$. From this assumption and the statement (6.5) in Lemma 6.1 we can conclude that

$$g(\mathcal{N}w(t)) = h(t), \quad \text{for a.e. } t\in[0,T]. \tag{7.9}$$

Comparing (7.9) with (7.6) we conclude that

$$g(\mathcal{N}w^N) \rightharpoonup g(\mathcal{N}w) \text{ in } C_W([0,T],\mathcal{H}). \tag{7.10}$$

Thus we can choose our subsequences and limit function w so that we have

$$g(\mathcal{N}w(t)) \in \mathcal{H} \text{ for all } t\in[0,T]. \tag{7.11}$$

Recall that we have imposed the additional restriction (2.12) on $\mathcal{N}$. Hence we have

$$\mathcal{N}^*g(\mathcal{N}w(t)) \in \mathcal{V}_2^* \text{ for all } t\in[0,T] \tag{7.12}$$

and, moreover,

$$\mathcal{N}^* g(\mathcal{N} w) \in C_W([0,T], \mathcal{V}_2^*). \tag{7.13}$$

From (7.13) we conclude that, in particular,

$$\mathcal{N}^* g(\mathcal{N} w) \in L^2([0,T], \mathcal{V}_2^*). \tag{7.14}$$

From this last conclusion we can consider our original equation (2.1) as a linear equation with right side term

$$\Phi = f - \mathcal{N}^* g(\mathcal{N} w) \in L^2([0,T], \mathcal{V}_2^*). \tag{7.15}$$

Then the statement of the theorem follows from Theorem 4.3 in [7].

8 An Explicit Example

In this section we present an example of a system governed by a partial differential equation for which all the assumptions are satisfied. In particular, we consider an m-dimensional, nonlinear damped membrane with fixed boundary.

Let $\Omega \subset R^m$ be a bounded domain with C^1-smooth boundary Γ. We consider the problem

$$w_{tt} + \kappa_1 \Delta^2 w + \kappa_2 \Delta^2 w_t + \Delta g(\Delta w) = f \tag{8.1}$$

$$w|_{\Gamma=0} \tag{8.2}$$

$$\left.\frac{\partial w}{\partial n}\right|_{\Gamma} = 0 \tag{8.3}$$

$$w(x,0) = \varphi_0(x) \in H_0^2(\Omega), \quad w_t(x,0) = \varphi_1(x) \in L^2(\Omega) \tag{8.4}$$

$$x = (x_1, \cdots, x_m) \in \Omega, \ t \in [0,T], (x,t) \in \Omega \times [0,T] \equiv Q_T$$

We assume that $f(\cdot,t) \in H^{-2}(\Omega)$ for almost all $t \in [0,T]$ and

$$\int_0^T \|f(\cdot,t)\|^2_{H^{-2}(\Omega)}\, dt < \infty. \tag{8.5}$$

Assumption 8.1 *We assume that*

$$G(\xi) = \int_0^\xi g(\tau)\, d\tau, \quad g(\xi) = G'(\xi) \tag{8.6}$$

satisfies

1. *There exist positive constants C_j for $j = 1, 2, 3$ such that*

$$-\frac{1}{2}(\kappa_1 + \kappa_2 - \epsilon)|\xi|^2 - C_1 \le G(\xi) \le C_2|\xi|^2 + C_3 \tag{8.7}$$

   for $\epsilon > 0$.

2. *There are positive constants $\widetilde{C}_j$, $j = 1, 2$ such that*

$$|g(\xi)| \le \widetilde{C}_1|\xi| + \widetilde{C}_2. \tag{8.8}$$

3. *We also assume that*

$$g'(\xi) \ge -k_1. \tag{8.9}$$

Notice that in this problem

$$\mathcal{V} = \mathcal{V}_2 = H_0^2(\Omega) = \left\{\psi \in H^2(\Omega) : \psi|_\Gamma = \left.\frac{\partial\psi}{\partial n}\right|_\Gamma = 0\right\}$$

and

$$\mathcal{A}_1 = \mathcal{A}_2 = \Delta^2, \quad \mathcal{N} = \Delta, \quad k = \widetilde{k} = 1.$$

Let us check that (8.9) implies the monotonicity condition (2.19). We have for $\varphi, \psi \in L^2(\Omega)$

$$\begin{aligned}
&(g(\varphi) - g(\psi), \varphi - \psi) = \int_\Omega [g(\varphi) - g(\psi)]\,(\overline{\varphi(x)} - \overline{\psi(x)})\,dx \\
&= \int_\Omega \left[\int_0^1 ds \frac{d}{ds} g(s\varphi(x) + (1-s)\psi(x)\right] (\overline{\varphi(x)} - \overline{\psi(x)})\,dx \\
&= \int_\Omega \left[\int_0^1 ds\ g'(s\varphi(x) + (1-s)\psi(x))\right] |\varphi(x) - \psi(x)|^2\,dx \\
&\ge -k_1\|\varphi - \psi\|^2_{L^2(\Omega)},
\end{aligned}$$

and the result follows. All other conditions A1)-A11) are also satisfied.

In concluding this section, we note that the motivating example on nonlinear elastomers of Section 1 also falls within the class of examples that can be treated with the theory developed in this paper. Of course, the neo-Hookean nonlinearity $\tilde{g}$ of (1.6) (which is only locally defined) must be appropriately extended to a map $\tilde{g} : R^1 \to R^1$. Once this is properly done, the functions $g : \mathcal{H} \to \mathcal{H}, \mathcal{H} = L^2(0,\ell)$, defined by $g(\varphi)(x) = \tilde{g}(\varphi(x))$ and $G(\varphi) = \int_0^1 \tilde{G}(\varphi(x))dx = Re\langle\tilde{G}(\varphi), 1\rangle$ where $\tilde{G}(\xi) = \int_0^\xi \tilde{g}(s)ds$, $\tilde{G}(\varphi)(x) = \tilde{G}(\varphi(x))$ will satisfy the necessary hypotheses for the theory of Sections 2-7.

9 Concluding Remarks

In the previous sections we have presented arguments of existence, uniqueness and regularity for solutions of abstract systems described by (2.1)-(2.3). These arguments are constructive in the sense that they also establish convergence of certain classes of finite element Galerkin approximations that are the foundation of computational methods.

To be specific, suppose we have a family of approximation spaces

$$\mathcal{H}^N \equiv span\{\psi_1^N, \psi_2^N, \ldots, \psi_N^N\},\ N = 1, 2, \ldots,$$

where the basis elements $\{\psi_j^N\}$ satisfy the standard finite element condition:

(C1) For each N, $\mathcal{H}^N \subset \mathcal{V}$ and for each $\psi \in \mathcal{V}$, we have

$$\|\psi - P^N\psi\|_{\mathcal{V}} \to 0 \text{ as } N \to \infty$$

where $P^N : \mathcal{H} \to \mathcal{H}^N$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}^N$.

We can then define the Galerkin approximations in the standard manner: $w^N(t) = \sum_{k=1}^N c_k^N(t)\psi_k^N$ are chosen to satisfy (4.2) for each test function $\psi_j = \psi_j^N,\ j = 1, 2, \ldots, N$, with initial conditions

$$w^N(0) = P^N\varphi_0,\ w_t^N(0) = P^N\varphi_1.$$

We note that (C1) immediately yields that $w^N(0) \to \varphi_0$ in $\mathcal{V}$ and $w_t^N(0) \to \varphi_1$ in $\mathcal{H}$ for φ_0 and φ_1 in $\mathcal{V}$ and $\mathcal{H}$ respectively. Then under the additional condition

(C2) For all N, $\mathcal{H}^N \subset \mathcal{H}^{N+1}$,

we can prove that $w^N \to w$ in $C([0,T],\mathcal{H})$. The arguments follow almost immediately from those of Section 6 above. In both (5.12) and (6.1) we choose test functions $\Phi_M(t) = \eta(t) = \sum_{k=1}^M a_k^M(t)\psi_k^M$ with the a_k^M arbitrary $C^1([0,T])$ functions. We then have $\eta \in \mathcal{P}_M$ for every $M \leq N$ (we use the condition (C2) here only), so that (6.3) again holds for $\eta \in \mathcal{P}_M,\ M \leq N$. Then the remainder of the arguments of Section 6 remain unchanged and we thus conclude that beginning with *any* subsequence of the Galerkin sequence $\{w^N\}$, we can obtain a further subsequence which converges to w, the unique solution of (2.1)-(2.3). Hence the original Galerkin sequence itself must converge in $C([0,T],\mathcal{H})$ to w.

The condition (C1) is standard in finite element and spectral family approximation schemes. The condition (C2) is also readily satisfied in certain finite element and spectral approximations. For example, consider from Section 1 the one-dimensional elastomer rod with strong damping (so that $\mathcal{V}_2$ embeds compactly into $\mathcal{H} = L^2(0,\ell)$) and let ψ_j^K be the usual piecewise linear elements corresponding to the discretizations of $0 < x < \ell$ with $\Delta x = \ell/K$. Define $V^K = span\{\psi_1^K, \ldots, \psi_K^K\}$ and then choose $\mathcal{H}^N = V^{2^N}$ in the Galerkin scheme described above. It is readily seen that (C1) and (C2) hold where $\mathcal{V} = H_0^1(0,\ell)$ as in Section 1.

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