A NOTE ON IMPERFECTION SENSITIVITY OF EXTERNALLY PRESSURIZED SHELLS

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One of the conditions to be judged in the design analysis of primary containment vessels for nuclear power plants is the ability of the vessel to withstand loads that result in compressive stresses. Design information as regards to the elastic stability of shells with rather large radius to thickness ratios (800-1200) is rather limited. An examination of the ASME Boiler & Pressure Vessel Code reveals that it is also woefully meager in this area.

For many classes of vessel geometry, it has been observed that the predicted classical elastic bifurcation pressure is considerably different than the experimentally derived values. In addition, considerable scatter of the experimental results has been shown to exist. In the early 30's Flügge and Donnell suggested that initial imperfections of the model test geometry from a perfect shell could account for this deviation. This concept stirred up a controversy for years. In 1945 Koiter, in his doctorate dissertation, produced the theoretical basis to support the view that deviations from perfect shell geometry could cause significant differences between the critical buckling loads predicted by linear theory and observed in the laboratory. The English translation of this work was not made available until 1963. Budiansky and Hutchinson pursued an independent research effort which resulted in an approach that was similar to Koiter's. Later, Hutchinson examined the imperfection sensitivity of spherical shells. This study was limited to the occurrence of an imperfection shape being present in either an axisymmetric or the minimum energy nonaxisymmetric mode.

The present paper extends the Hutchinson approach to consider the effect of the axisymmetric and nonaxisymmetric imperfections acting simultaneously. The solution is obtained using two interactive modes and the first variation of the potential energy. From this, two equilibrium equations result which contain a nonlinear relationship between the participating modes of deformation. Relationships for the critical pressure under the two mode participation are evaluated. These results are compared to specific numerical results developed from a computer program, NBALL, which is based upon Sanders' equations for the geometrically nonlinear analysis of arbitrary loaded shells of revolution. This comparison reveals the region of validity of Hutchinson's results and the extension of his method presented herein.

1. Introduction

Considerable effort has been expended on buckling in general during the previous half century. Nevertheless, there is a great need for improvement in the quality and quantity of available design information relative to elastic buckling of spherical shells. Spherically shaped primary containment vessels for nuclear power plants or container vessels for a liquid natural gas freighter are examples of spherical shells. An examination of the ASME Boiler & Pressure Vessel Code reveals that it also needs considerable upgrading in this area. Designers of actual shell structures, for the most part, continue to use an impirical knock-down factor applied to the buckling load as predicted by classical, small-deflection theory. The use of a knockdown factor, which represents a lower bound of all observed applicable experimental data, could result in a substantially uneconomical use of material for a large containment vessel having a diameter in excess of 60 meters.

It has been only within the last two decades that initial geometric imperfections were generally accepted as the main cause for the wide experimental scatter observed in buckling tests and for the poor correlation between the predictions of linearized small-deflection theory and experimental results. However, the incorporation of imperfection sensitivity into the design of shells has not found its way into engineering practice (as is implied in the preceding paragraph). This reticence might be due partly to the mathematical complexity that surrounds theoretical buckling studies which consider imperfections in the shell geometry, and partly to the fact that the correlation between the buckling load of a particular shell and its imperfection geometry has yet to be established. In addition, the actual structural configuration and loading may not correspond with those on which buckling studies have been conducted.

It is interesting to note that Flügge [1] as early as 1932, and Donnell [2] in 1934 postulated that initial imperfections in thin walled cylindrical shells might explain the discrepancy between theory and experiment as regards buckling strength. However, this explanation was not generally accepted until some years later when large deflection and postbuckling behavior of shells began to be investigated. Koiter [3] made a significant contribution to the theory of postbuckling behavior of elastic structures when he published his doctoral dissertation in 1945; however, his work was not generally known to the American engineering community until he published a condensed study in English in 1963. Budiansky and Hutchinson [4] in 1964 pursued a similar but less general approach than Koiter in their initial post-buckling analysis of shells.

When the initial postbuckling behavior of thin shells became reasonable well defined, it was observed that alternate equilibrium states could exist for certain shells and loading conditions. The point at which a loaded shell could proceed from the fundamental equilibrium state to an alternate state is called a bifurcation point; and it was found for some cases that the ability of a shell to carry load decreased along an alternative equilibrium path. For these cases imperfections (initial deviations from an assumed perfect structure) will have a deleterious effect upon the critical load of the structure.

Results of imperfection sensitivity calculations are presently available for a variety of shapes and loadings -- toroids (Hutchinson [5] in 1967); spherical caps (Fitch and Budiansky [6] in 1970); spheroids (Danielson [7] in 1969); cylinders (Budiansky and Amazigo [8] in 1968, and Hutchinson [9] in 1960); and complete spherical shells (Hutchinson [10] in

1967). The limitations are that shallow shell theory has been used in most instances and that the results hold only for small imperfections ($\bar{\xi}/t \leq 1$) in the initial postbuckling regime. The shallowness assumption can be justified if buckle pattern is reproduced many times on the shell surface; therefore, only a small portion of the shell (i.e., a shallow shell) need be considered.

Since Hutchinson had addressed very well the possibility of nonsymmetric buckling patterns in shallow spherical shells, it was felt that his approach might lead to realistic knockdown factors for spherical shells with large R/t ratios and large diameters. Hutchinson found that the two interactive mode cases gave the greatest knockdown factors vs. imperfection values. However, Hutchinson considered each imperfection mode occurring <u>separately</u> in determining the buckling behavior of shells. This paper considers an extension of Hutchinson's approach by assuming that both imperfection modes are present simultaneously. A comparison is also made between Hutchinson's results and those obtained from a computer program, NBALL [11], which is based upon Sanders' equation for the geometrically nonlinear analysis of arbitrary shells of revolution. This comparison reveals the region of validity of Hutchinson's results.

2. Technical Discussion

The classical buckling pressure for a complete spherical shell is given by

$$p_{cr} = \frac{4\, E\, h}{R\, q_o^2}$$

where

$$q_o = \sqrt[4]{12(1-\nu^2)} \quad \sqrt{\frac{R}{h}}$$

When a static nonlinear shell analysis is performed with no imperfection being considered, the classical value of buckling pressure will be obtained. As the imperfection is introduced, a noticeable drop or knockdown is realized. The magnitude and shape of this imperfection will dictate the critical buckling pressure. Kalnins and Biricikoglu [12] found for axisymmetric imperfections that greater knockdown factors could be realized when the magnitude of the imperfection is not limited. A comparison of their results to those obtained by Koiter in some of his work is shown in Fig. 1.

Since Koiter's approach is limited by the size of the imperfection, this has implications in Hutchinson's work [10] wherein he states that his results are limited to those cases where the neglected terms in the potential energy expression are small compared to the terms remaining. As the imperfection magnitude increases, these neglected terms become more significant.

2.1 Equilibrium Equations

Hutchinson assumes the two interactive modes are given in terms of normal displacement, W, as follows:

$$W = W_1 + W_2$$

$$W = \xi_1 h \cos\left(q_o \frac{x}{R}\right) + \xi_2 h \sin\left(q_o \frac{x}{2R}\right) \sin\left(q_o \frac{\sqrt{3}y}{2R}\right) \quad (1)$$

where x and y are the coordinate directions, and R is the radius of the shell. Imperfections, which correspond to the mode shapes, are given by

$$\overline{W} = \bar{\xi}_1 h \cos\left(q_o \frac{x}{R}\right) + \bar{\xi}_2 h \sin\left(q_o \frac{x}{2R}\right) \sin\left(q_o \frac{\sqrt{3}y}{2R}\right) \tag{2}$$

With the above solutions and imperfections substituted into the potential energy expression and the first variation taken, the following equilibrium equations result:

$$\left(1 - \frac{p}{p_c}\right) \xi_1 - \frac{9C}{32} \xi_2^2 = \frac{p}{p_c} \bar{\xi}_1 \tag{3}$$

$$\left(1 - \frac{p}{p_c}\right) \xi_2 - \frac{9C}{8} \xi_1 \xi_2 = \frac{p}{p_c} \bar{\xi}_2 \tag{4}$$

where

$$C = \sqrt{3(1-\nu^2)}$$

It is important to note that the deflections in the modes ξ_1 and ξ_2, as well as the imperfection magnitudes $\bar{\xi}_1$ and $\bar{\xi}_2$, are measured relative to the shell thickness, h.

For comparison the results of Hutchinson are reproduced for the perfect shell ($\bar{\xi}_1=\bar{\xi}_2=0$) and two cases for the imperfect shell in which each imperfection mode occurs separately.

2.2 Perfect Shell ($\bar{\xi}_1 = \bar{\xi}_2 = 0$)

For the perfect shell, eqs. (3) and (4) admit only the trivial solution when the prebuckling pressure is less than the critical value p_c. When p attains p_c, bifurcation from the membrane state of stress occurs, and the equilibrium equations are easily solved for ξ_1 and ξ_2 as follows:

$$\xi_1 = \frac{8}{9C}\left(1 - \frac{p}{p_c}\right)$$

$$\xi_2 = \pm \frac{16}{9C}\left(1 - \frac{p}{p_c}\right)$$

This behavior is sketched in Fig. 2. The equilibrium pressure in the postbuckling regime is greatly reduced even where the buckling deflections are only a small fraction of the shell thickness; i.e., ξ_1 and $\xi_2 \ll 1$.

2.3 Imperfect Shell ($\bar{\xi}_1 > 0$, $\bar{\xi}_2 = 0$)

An imperfect shell suffers deflections in the buckling modes with the first application of external pressure. The behavior for this case is also depicted in Fig. 2. Prior to buckling, the load increased with deflection in the ξ_1 mode with

$$\xi_1 = \frac{\bar{\xi}_1 \, p/p_c}{1 - p/p_c} \tag{5}$$

until the coefficient of ξ_2 in eq. (4) vanished. At this point, bifurcation occurs. Following bifurcation, the equilibrium pressure falls with deflections occurring in both modes; thus the maximum buckling pressure, denoted by p*, is the bifurcation pressure which satisfies

$$\left(1 - \frac{p^*}{p_c}\right)^2 = \frac{9C}{8} \bar{\xi}_1 \frac{p^*}{p_c} \tag{6}$$

Small imperfections (relative to the shell thickness) result in large reductions of the buckling pressure as shown in the plot of eq. (6) in Fig. 3.

2.4 Imperfect Shell ($\bar{\xi}_1 = 0$, $\bar{\xi}_2 \neq 0$)

If $\bar{\xi}_1 = 0$ but $\bar{\xi}_2 \neq 0$, the maximum value of p is obtained by substituting for ξ_1 in terms of ξ_2 from eq. (3) into eq. (4) and determining then the value of p such that $dp/d\xi_2 = 0$. One finds

$$\left(1 - \frac{p^*}{p_c}\right)^2 = \frac{27\sqrt{3}}{32} \left|\bar{\xi}_2\right| \frac{p^*}{p_c} \tag{7}$$

This formula is also plotted in Fig. 3, and it is seen that an imperfection in the form of the ξ_2 mode causes a greater reduction than an equal imperfection in the ξ_1 mode.

2.5 Imperfect Shell ($\bar{\xi}_1 > 0$, $\bar{\xi}_2 \neq 0$)

It should be remembered the information contained in Fig. 3 is based on imperfection magnitudes $\bar{\xi}_1$ and $\bar{\xi}_2$ occurring separately. The question needs to be answered about what effect imperfections $\bar{\xi}_1$ and $\bar{\xi}_2$ will have on the buckling behavior of the shell if they are present simultaneously. To do this we must take eqs. (3) and (4) and solve them for either ξ_1 or ξ_2. Starting with eq. (3) we have (let $r_p = p/p_c$)

$$\xi_1 = \frac{r_p \bar{\xi}_1 + \frac{9C}{32} \xi_2^2}{1 - r_p}$$

Substituting this into eq. (4) results in

$$\xi_2^3 - \frac{[(1-r_p)^2 - \frac{9C}{8} r_p \bar{\xi}_1]}{\frac{81C^2}{256}} \xi_2 + \frac{(1-r_p)\, r_p\, \bar{\xi}_2}{\frac{81C^2}{256}} = 0 \tag{8}$$

Implicit differentiation of eq. (8) will give

$$3\xi_2^2 d\xi_2 - \frac{[(1-r_p)^2 - \frac{9C}{8} r_p \bar{\xi}_1]}{\frac{81C^2}{256}} d\xi_2 - \frac{[-2(1-r_p) - \frac{9C}{8}\bar{\xi}_1]}{\frac{81C^2}{256}} \xi_2 dr_p + \frac{(1-2r_p)\,\bar{\xi}_2}{\frac{81C^2}{256}} dr_p = 0$$

which will give

$$\frac{dr_p}{d\xi_2} = \frac{\frac{243}{256} C^2 \xi_2^2 - (1-r_p)^2 + \frac{9C}{8} r_p \bar{\xi}_1}{(1-2r_p)\,\bar{\xi}_2 + [2(1-r_p) + \frac{9C}{8}\bar{\xi}_1]\,\xi_2}$$

To find r_p^*, the maximum pressure ratio, we need to set $\frac{dr_p}{d\xi_2} = 0$. This will give

$$\frac{243}{256} C^2 \xi_2^2 - (1-r_p)^2 + \frac{9C}{8} r_p \bar{\xi}_1 = 0 \tag{9}$$

That ξ_2^* at which r_p^* will occur can be obtained by solving eq. (9)

$$\xi_2^* = \frac{1}{D\sqrt{3}} \sqrt{(1-r_p^*)^2 - 2Dr_p^* \bar{\xi}_1} \tag{10}$$

where

$$D = \frac{9C}{16}$$

Rewriting eq. (8) in the following form, we have

$$D^2\xi_2^3 - [(1-r_p)^2 - 2Dr_p\,\overline{\xi}_1]\,\xi_2 + r_p(1-r_p)\,\overline{\xi}_2 = 0 \tag{11}$$

Substituting eq. (10) for ξ_2 in eq. (11) results in

$$\frac{[(1-r_p^*)^2 - 2Dr_p^*\,\overline{\xi}_1]^{\frac{3}{2}}}{3D\sqrt{3}} - \frac{[(1-r_p^*)^2 - 2Dr^*_p\,\overline{\xi}_1]^{\frac{3}{2}}}{D\sqrt{3}} + r_p^*(1-r_p^*)\,\overline{\xi}_2 = 0$$

Simplifying, we obtain

$$[(1-r_p^*)^2 - 2Dr_p^*\,\overline{\xi}_1]^{\frac{3}{2}} = \frac{3D\sqrt{3}}{2}\,r_p^*(1-r_p^*)\,\overline{\xi}_2 \tag{12}$$

Substituting back for "D" in the above equation and writing it in a form similar to Hutchinson's will yield

$$(1-r_p^*)^2 = [\frac{27\sqrt{3}\,C}{32}\,r_p^*\,(1-r_p^*)\,\overline{\xi}_2]^{\frac{2}{3}} + \frac{9C}{8}\,r_p^*\,\overline{\xi}_1 \tag{13}$$

One may check this equation and determine for either $\overline{\xi}_1 = 0$ or $\overline{\xi}_2 = 0$ that eq. (13) reduces to eq. (7) or eq. (6) respectively.

Figure 4 illustrates the critical pressure ratio, r_p^* as a function of $\overline{\xi}_1$ and $\overline{\xi}_2$. A study of Fig. 4 reveals that buckling pressures are <u>lower</u> when $\overline{\xi}_1$ and $\overline{\xi}_2$ imperfection magnitudes are present simultaneously.

3. Numerical Results

The use of the assumed imperfection of the form of eq. (1) requires that the analysis be directed towards the shallow part of the shell and away from any boundaries. Thus, eq. (1) will not maintain the correct buckle pattern (or imperfection geometry) for a complete spherical shell.

The only zone over which the Hutchinson equations are directly applicable is at the equator of the sphere. If the same rectangular coordinate system is used adjacent to the pole, the resulting imperfection severely stiffens the shell. Therefore, in order to make a comparison to the computer program NBALL [11], which is based upon a shell of revolution coordinate system, the imperfection geometry needs to be referenced to a polar coordinate system which may be accomplished by a coordinate transformation. In order to generate the proper function for the pole, we must use the following relationships involving Bessel functions of the first kind:

$$\begin{aligned}
\cos\,(z\cos\theta) &= J_0(z) + 2\sum_{n=1}^{\infty}(-1)^n\,J_{2n}(z)\cos 2n\theta \\
\sin\,(z\cos\theta) &= 2\sum_{n=0}^{\infty}(-1)^n\,J_{2n+1}(z)\cos\,(2n+1)\,\theta \\
\sin\,(z\sin\theta) &= 2\sum_{n=0}^{\infty}J_{2n+1}(z)\sin\,(2n+1)\,\theta
\end{aligned} \tag{14}$$

The imperfection geometry takes on the form:

$$\overline{W} = \overline{\xi}_1 h \{J_o(\frac{q_o r}{R}) + 2 \sum_{k=1}^{\infty} (-1)^k J_{2k}(\frac{q_o r}{R}) \cos 2k\theta\} + \overline{\xi}_2 h \{\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} (-1)^k$$
$$\times J_{2k+1}(\frac{q_o r}{2R}) J_{2n+1}(\frac{\sqrt{3}q_o r}{2R}) \sin(2k+2n+2)\theta - \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} (-1)^k \qquad (15)$$
$$\times J_{2k+1}(\frac{q_o r}{2R}) J_{2n+1}(\frac{\sqrt{3}q_o r}{2R}) \sin(2k-2n)\theta\}$$

This imperfection geometry is only applicable near the pole and is subject to the same assumptions of eq. (1). Thus, two regions exist which are limited by the representation of the behavior of the sphere through shallow shell theory. The shallow spherical shell that has been given much attention is that studied by Ball and Burt [13].

A numerical example is selected for a shallow-spherical cap with an $R/h = 1000$, modulus of elasticity, $E = 1.884 \times 10^8$ kpa (27.3×10^6 psi), and Poisson's ratio, $\nu = 0.3$. The meridional length of the shell, S, is chosen such that the slope of the imperfection in the axisymmetric mode is zero at the edge, ($J_1(r) = 0$), which corresponds to an $S/R = 0.12244$. The classical buckling pressure (p_c) for this shell is determined to be 227.91 kpa (33.03 psi). For the shell subjected to an axisymmetric external pressure, the numerical results of p^*/p_c (r_p^*) ratios are shown in Table I for the axisymmetric imperfection by itself, for a single nonaxisymmetric imperfection, and for a two-term and five-term expansion of eq. (15). The last two cases considered only the cosine term expansion. This corresponds to Hutchinson's condition $\overline{\xi}_1 > 0$, $\overline{\xi}_2 = 0$ represented by eq. (6).

The shell is assumed to have the boundary condition $Q_s=\phi_s=u=v=0$ at its outer edge. The results were derived for a difference spacing of 36 even increments. The convergence criteria used assumed that the radial displacement anywhere on the shell for two successive nonlinear solutions shall agree to within one percent of the maximum radial displacement. For convenience, Hutchinson's results are also given in Table I. It is observed that the numerical results for small imperfections ($\overline{\xi} < 0.2$) are in rather close agreement with Hutchinson. However, as the imperfection is increased in magnitude, the numerical procedure began to

TABLE I. r_p^* FOR SINGLE MODE IMPERFECTION

$\overline{\xi}_1$	From Computer Program NBALL				Hutchinson
	n=0* A_k=00**	n=0,2 A_o=0	n=0,2 2-Term (k=1)	n=0,2,4,6,8 5-Term (k=4)	
0.0001	1.226	0.965	0.917	0.921	0.986
0.05	0.874	0.787	0.746	0.750	0.738
0.1	0.827	0.718	0.663	0.674	0.652
0.2	0.783	0.612	0.550	0.572	0.548
0.4	0.699	0.489	0.369	0.412	0.433
0.6	0 474	0.463	-	-	0.363
0.8	-	0.286	-	-	0.315

*Fourier harmonic in response $w = \sum w_n \cos n\theta$

**Imperfection assumed $\overline{w} = \overline{\xi}_1 h[A_o J_o(\lambda r) + 2 \sum_{k=1} A_k J_{2k}(\lambda r) \cos 2k\theta]$ (16)

deviate and also had difficulty in converging. The reason for this difficulty is twofold: first, the numerical procedure is based upon full coupling of all nonlinear terms that are represented in Sander's shell equations; secondly, the boundary conditions at the edge (ϕ_s=0) are incompatible with the assumed imperfection geometry. Both of these factors tend to reduce the apparent load-carrying capacity of the shell.

For a shallow spherical shell subjected to axisymmetric loads, the governing equation for buckling is of the form:

$$\begin{aligned} \nabla^2 w &= - Aw \\ \nabla^2 f &= - BAw \\ f &= Bw \end{aligned} \qquad (17)$$

where

$$B = - Eh/AR$$
$$A = (q_o/R)^2 = \lambda^2$$

For a polar coordinate system

$$w = \sum_{n=0}^{\infty} w_n J_n(\lambda r) \cos n \qquad (18)$$

is a solution to eq. (17) and is of the same form as eq. (16).

For a rectangular coordinate system, the solution takes the form of eq. (1). If boundaries are of no concern and if natural principal directions of a shell of revolution are of no consequence, then there would be no preference in the selection of a coordinate system. In addition, the particular form of eq. (1) permits complete coupling of the first and second mode for quadratic terms of the strain displacement relations. For shells of revolution, normally two principal directions are associated in the description of the shell. This reference system permits shell buckling into the usual modes observed in experiments. For snap-through problems, the principal mode is axisymmetric. If a rectangular coordinate system is selected, many modes would be required to simulate this response, whereas only a single mode need be used if a polar coordinate system is chosen. The influence of the boundary conditions along the principal radii of curvature can also be easily prescribed.

If we assume the imperfection shape to be the same as eq. (18), then the single-mode and two-mode responses take on a different meaning than that developed by Hutchinson. For each admissible buckling mode for a shallow spherical shell, a continuous imperfection shape can be assumed. For a finite length shell, however, certain modes will couple into "natural" buckling mode shapes which correspond to the boundary conditions, while others will be incompatible. For two-mode coupling (e.g., an axisymmetric mode and one nonaxisymmetric mode) with the boundary condition ϕ_s = 0 imposed, the zeros of the expression

$$\bar{\xi}_0 \lambda J_{-1}(\lambda r) + 2\bar{\xi}_k (-1)^k [\lambda J_{2k-1}(\lambda r) - \frac{2k}{r} J_{2k}(\lambda r)] = 0 \qquad (19)$$

will give the length of shell which produces a minimum critical buckling pressure per Hutchinson's assumptions.

As seen from Table I there are certain combinations which produce lower buckling modes than that suggested by Hutchinson. These modes are associated with the polar coordinate system. Since coupling is possible through the Fourier components, then axial coupling can be developed through the boundary conditions. In order to provide a compatible imperfection shape, the boundary condition at the outer radius, r_0,

$$\frac{\partial \bar{w}}{\partial r} = 0$$

should be satisfied. This condition dictates the roots, $\lambda_n r_0$, for each mode to be determined by

$$\frac{J_{2n-1}(\lambda_n r_o)}{J_{2n}(\lambda_n r_o)} = \frac{2n}{\lambda_n r_o} \tag{20}$$

4. Conclusions

Thus it is possible to consider the imperfection shape that is an admissible buckling mode and that would satisfy the boundary conditions but not necessarily satisfy the minimum pressure requirement. The fact is that even with this discrepancy, a two-mode, Fourier representation in polar coordinates utilizing a fully coupled nonlinear shell theory can produce a buckling pressure less than Hutchinson's predictions. This suggests that lower buckling pressures are possible. This is borne out by Koiter's discussion in his study of complete spherical shells [14].

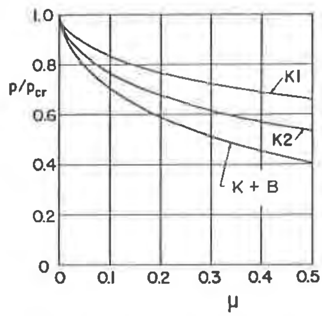

Fig. 1 Critical pressure versus imperfection amplitude by Koiter's theory and analysis of Kalnins and Biricikoglu.

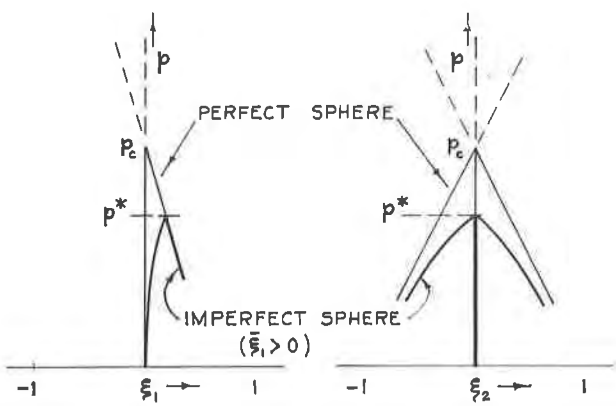

Fig. 2 Pressure-mode deflection behavior

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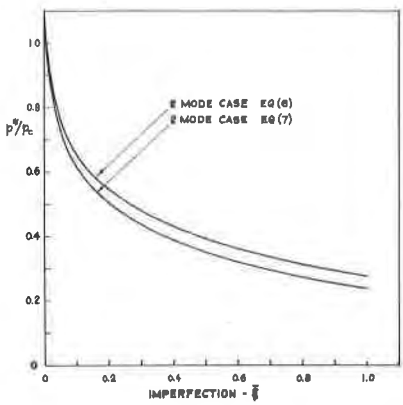

Fig. 3 Buckling pressure for imperfect spherical shells (ν = 0.3)

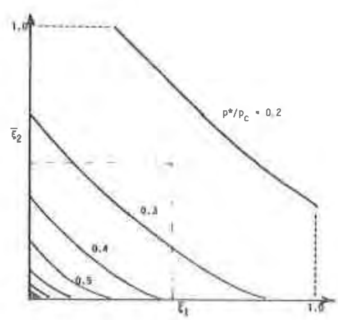

Fig. 4 Buckling pressure ratio (p^*/p_c) vs. $\bar{\xi}_1$ and $\bar{\xi}_2$.