On Thermodynamic Restrictions for Some Visco-Plastic Material Models

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ABSTRACT

Thermodynamic restrictions for the constitutive relations of a general internal variable model are derived by evaluating the Clausius-Duhem entropy inequality with a modified version of Müller's concept. This involves the arbitrary modulation of only the initial values on the boundary of the body. The results agree with the classical restrictions obtained by Coleman and Gurtin; their approach involves the arbitrary modulation of the body force and the heat supply which was subject to criticism.

The evaluation of the derived residual entropy inequality is shown for a visco-plastic material model which is a thermodynamically extended version of the Robinson model. Necessary and sufficient conditions for the material data are obtained in an explicit form. These results also demonstrate that the purely mechanical Robinson model is formally consistent with thermodynamics.

A few remarks are added; they concern an evolution equation which contains the temperature rate as an additional variable.

1. INTRODUCTION

It is well known that continuum thermodynamics imposes restrictions on the constitutive equations characterizing the material behavior. These thermodynamic restrictions derive from an evaluation of an entropy principle together with the balance equations and the assumed constitutive relations. Whereas the validity of the balance equations is generally not questioned the entropy principle or the entropy production inequality has raised many controversies which persist at the present time [1-3]. Very often the entropy principle is taken to be the Clausius-Duhem inequality given by

$$\frac{D}{Dt}\int_V \rho\eta \, dV - \oint_A \frac{-q_k n_k}{T} dA - \int_V \frac{\rho r}{T} dV \geq 0 \qquad (1.1)$$

where η is the specific entropy, q_k are components of the heat flux, r is the external heat supply, T is the absolute temperature, ρ is the density, and V, A, n_k are the volume and the surface area of the body and the external unit vector on the surface. The usual summation convention applies to repeated subscripts.

For sufficiently smooth fields the divergence theorem and localization yield the local Clausius-Duhem inequality

$$\sigma := \rho\mathring{\eta} + \left(\frac{q_k}{T}\right)_{,k} - \frac{\rho r}{T} \geq 0; \tag{1.2}$$

here the comma $(.)_{,k}$ denotes partial differentiation with respect to the cartesian coordinate x_k and the dot marks material time differentiation.

In 1963 Coleman and Noll [4] introduced an argument which allowed to draw conclusions from this inequality. The Coleman-Noll argument considers a large variety of admissible thermodynamic processes, i.e., any solutions of the balance equation with proper account for the assumed constitutive relations. Once this process is prescribed a "dissipation postulate" is required to hold: "For all admissible thermodynamic processes and for every part of the body the entropy production must be non-negative." The decisive words are the quantifiers "all" and "every". The quantifier "every" requires that the <u>local</u> form should be satisfied, and the quantifier "all" requires an identical satisfaction of (1.2) which makes the postulate a restrictive condition on the constitutive assumptions.

The concept of Coleman and Noll, applied by many others and promoted by Truesdell [5], has been criticized for different reasons [e.g. 1-3, 6]. One of them is the following. From a physical standpoint an admissible process is controlled by prescribing the body force and heat supply as well as initial and boundary conditions. In the approach of Coleman and Noll the body force and the heat supply are required to be assigneable in any way but initial and boundary conditions need not to be considered explicity. Woods [7] has criticized this aspect: The fictitious body force and heat supply might be supplied only by some imponderable medium permeating the body ("phlogiston axiom"), they lie beyond control at interior points of the body. Green [9] did respond to the criticism but without much success [8].

In 1967 the Coleman-Noll approach was used by Coleman and Gurtin [10] to derive thermodynamic restrictions for the constitutive equations of non-linear materials with internal variables whose temporal evolution is governed by ordinary differential equations. Later this development was the basis for elastic-plastic material models [e.g. 11, 12].

In section 2 the classical result of Coleman and Gurtin is summarized. Then an alternative approach is described which accounts for the criticism of the "phlogiston axiom". This approach is based on an idea of Müller [e.g. 13, 14] where body forces and heat supply are not allowed to be assignable in any way but where the arbitrariness of initial conditions induce a sufficiently large variety of admissible thermodynamic processes. Since Müller's results cannot be simply transfered to the present problem his approach is extended. It is shown that both approaches give the same general thermodynamic restrictions. The evaluation of the derived residual entropy inequality requires more definite assumptions about the structure of the internal variable model.

Such an evaluation is done in section 3 for a visco-plastic material model with internal variables involving discontinuous evolution equations; it is a thermodynamic extension of an early version of the mechanical Robinson model [15]. These discontinuities require due considerations based on the time-integral form of the basic equations. Within the frame of the assumptions it is indicated that the time-integral entropy inequality does not impose additional constitutive restrictions beyond those obtained when smooth processes apply.

The evaluation of the residual entropy inequality for the general three-dimensional state of stress is pointed out. Here the problem is to derive necessary and sufficient conditions for the material parameters such that the entropy production is non-negative. In general this is a formidable problem. However, for the simplified version of the Robinson model the state space can be reduced in dimension such that explicit conditions on the material parameters can be obtained. These conditions apply provided some of the material parameters are within certain ranges relevant to applications.

Finally, for a later version of the Robinson model it is indicated that the presence of a temperature rate term in an evolution equation destroys the classical thermodynamic potential relations but there is an internal variable transformation which restores the classical form.

2. DERIVATION OF THERMODYNAMIC RESTRICTIONS FOR A GENERAL INTERNAL VARIABLE MODEL

As in [10] the constitutive relations of the general internal variable model for the stress σ_{kl}, heat flux q_k, internal energy e and entropy η are assumed to be given by

$$\begin{aligned} \sigma_{kl} &= \hat{\sigma}_{kl}(\varepsilon_{mn}, T, g_k, \alpha_\gamma) = \hat{\sigma}_{lk} \\ q_k &= \hat{q}_k(\varepsilon_{mn}, T, g_k, \alpha_\gamma) \\ e &= \hat{e}(\varepsilon_{mn}, T, g_k, \alpha_\gamma) \qquad k,l = 1,2,3 \\ \eta &= \hat{\eta}(\varepsilon_{mn}, T, g_k, \alpha_\gamma) \end{aligned} \tag{2.1}$$

and ordinary differential equations represent the evolution equations for the internal variables α_γ

$$\mathring{\alpha}_\gamma = f_\gamma(\varepsilon_{mn}, T, g_k, \alpha_\gamma), \quad \gamma = 1,\dots,n\,; \tag{2.2}$$

here ε_{mn} is the infinitesimal strain tensor and $g_k = \partial T/\partial x_k$ is the temperature gradient. It should be noted that the analysis is restricted to infinitesimal strains and rotations.

The local energy balance equation

$$\rho\mathring{e} = \sigma_{kl}\mathring{\varepsilon}_{kl} - q_{k,k} + \rho r \tag{2.3}$$

allows to eliminate the heat supply in the local entropy inequality (1.2) which reduces to the dissipation inequality

$$\gamma := \sigma T = \sigma_{kl}\mathring{\varepsilon}_{kl} - \rho\mathring{\psi} - \rho\eta\mathring{T} - \frac{q_k g_k}{T} \geq 0 \tag{2.4}$$

if the free energy

$$\psi = e - T\eta = \hat{\psi}(\varepsilon_{kl}, T, g_k, \alpha_\gamma) \tag{2.5}$$

is introduced. Thus (2.4) may be written as

$$\left(\hat{\sigma}_{kl} - \rho\frac{\partial\hat{\psi}}{\partial\varepsilon_{kl}}\right)_{sym}\mathring{\varepsilon}_{kl} - \rho\left(\hat{\eta} + \frac{\partial\hat{\psi}}{\partial T}\right)\mathring{T} - \rho\frac{\partial\hat{\psi}}{\partial g_k}g_k - \rho\frac{\partial\hat{\psi}}{\partial\alpha_\gamma}f_\gamma - \frac{\hat{q}_k g_k}{T} \geq 0\,; \tag{2.6}$$

here the subscript *sym* denotes the symmetric part.

A sufficiently smooth displacement and temperature field $u_k(x_m, t)$ and $T(x_m, t)$ as well as initial values g_γ of the internal variables define an admissible thermodynamic process with the associated body force and heat supply obtainable from the balance equations. Therefore the rates (ε_{kl}, T,

g_k) of the external variables are locally arbitrary and independent. Since the rates are only linearly involved, the necessary and sufficient conditions to satisfy equ. (2.6) for all processes are

$$\hat{\sigma}_{kl} - \rho\,\frac{1}{2}\left(\frac{\partial\hat{\psi}}{\partial\varepsilon_{kl}} + \frac{\partial\hat{\psi}}{\partial\varepsilon_{lk}}\right) = 0\,, \quad \hat{\eta} + \frac{\partial\hat{\psi}}{\partial T} = 0\,, \quad \frac{\partial\hat{\psi}}{\partial g_k} = 0 \tag{2.7}$$

and the residual inequality

$$-\rho\,\frac{\partial\hat{\psi}}{\partial\alpha_\gamma}\,f_\gamma - \frac{\hat{q}_k\,g_k}{T} \geq 0\,; \tag{2.8}$$

these are well known classical results.

In the following another approach patterned after Müller's concept [13] is briefly described; for a more detailed analysis see [17].

A thermodynamic process is governed by the balance of energy (2.3) and the balance of linear momentum where the heat supply and the body force are prescribed quantities. Initial conditions are assumed to be prescribed for the displacement, velocity, temperature and internal variables, in a certain region D of the body. From these initial conditions the initial values of the strains, strain rates, temperature gradient and gradients of internal variables as well as higher order spatial derivatives are obtainable if required. Then the local balance of energy (2.3) at t = 0 represents a linear P.D.E. of first order for the initial temperature rate

$$y := \left(\frac{\partial T}{\partial t}\right)_{t=0} \tag{2.9}$$

with the following structure

$$A_k\,y_{,k} + By = C \tag{2.10}$$

where

$$A_k = \rho\left(\frac{\partial\hat{e}}{\partial g_k}\right)_{t=0}\,,\quad B = \rho\left(\frac{\partial\hat{e}}{\partial T}\right)_{t=0}$$

$$C = \rho(r)_{t=0} - (\hat{q}_{k,k})_{t=0} - \left(\rho\,\frac{\partial\hat{e}}{\partial\varepsilon_{mn}} - \hat{\sigma}_{mn}\right)_{sym,\,t=0}(\mathring{\varepsilon}_{mn})_{t=0} - \rho\left(\frac{\partial\hat{e}}{\partial\alpha_\gamma}\,f_\gamma\right)_{t=0}\,. \tag{2.11}$$

From the theory of partial differential equations it is known that (2.10) admits a unique solution $y(x_m)$ if the initial temperature rate is prescribed along an open "non-characteristic" boundary surface B, i.e., $y(x^*_m) = y^*(\zeta,\tau) = \partial T^*(\zeta,\tau,t)/\partial t)t{=}0$; here T^* is given along B and (ζ, τ) are surface coordinates on B. B is defined by the parameter representation $x_k = x^*_k$ (ζ, τ). If the surface B does not contain any characteristic base curve defined by the characteristic system of equations associated to (2.10), then B is called non-characteristic. The condition is

$$det\begin{pmatrix} A^*_1 & \dfrac{\partial x^*_1}{\partial\zeta} & \dfrac{\partial x^*_1}{\partial\tau} \\ A^*_2 & \dfrac{\partial x^*_2}{\partial\zeta} & \dfrac{\partial x^*_2}{\partial\tau} \\ A^*_3 & \dfrac{\partial x^*_3}{\partial\zeta} & \dfrac{\partial x^*_3}{\partial\tau} \end{pmatrix} \neq 0 \tag{2.12}$$

where A^*_k are the values of $(2.11)_1$ on B [17]. For given constitutive functions η and q_k and after explicit temporal and spatial differentiation in (1.2) the entropy inequality relates the independent variables (ε_{kl}, T, g_k, α_γ) as well as the first temporal and spatial derivatives collected in the set

$$\mathbb{R} := \{\mathring{\varepsilon}_{mn}, \varepsilon_{mn,k}, \mathring{T}, \mathring{g}_k, g_{k,m}, \alpha_{\gamma,k}\}. \tag{2.13}$$

The entropy inequality is required to be satisfied identically for all admissible processes also at the initial instant $t = 0$. The quantities represented by the set

$$\mathbb{C}_0 := \{\varepsilon_{mn}, \varepsilon_{mn,k}, \varepsilon_{mn}, T, g_k = T_{,k}, g_{k,m} = T_{,mk}, \alpha_\gamma, \alpha_{\gamma,k}\}_{t=0} \tag{2.14}$$

can be obtained from the initial values and they may assume independently arbitrary values locally. However, the initial temperature rate y and its derivative $y_{,k} = (\mathring{g}_k)_{t=0}$ as well as the acceleration $(\mathring{\mathring{u}}_k)_{t=0}$ are not independent. The acceleration is completely determined by the set $\mathbb{C}_0$ for a given body force via the local balance of linear momentum; this needs not to be accounted for since the acceleration does not appear in (1.2). On the other hand, y and $y_{,k}$ are related by (2.10). Therefore, equ. (2.10) is solved for the initial temperature rate y and the result is put into the entropy inequality. For $t = 0$ the inequality takes the following form

$$\begin{aligned}
&\left|\rho \frac{\partial\hat{\eta}}{\partial\varepsilon_{mn}} - \frac{\partial\hat{\eta}/\partial T}{\partial e/\partial T}\left(\rho \frac{\partial\hat{e}}{\partial\varepsilon_{mn}} - \hat{\sigma}_{mn}\right)\right|_{sym} \mathring{\varepsilon}_{mn} + \left[\rho \frac{\partial\hat{\eta}}{\partial g_m} - \frac{\partial\hat{\eta}/\partial T}{\partial\hat{e}/\partial T}\,\rho\, \frac{\partial\hat{e}}{\partial g_m}\right| y_{,m} \\
&+ \left|\frac{1}{T} - \frac{\partial\hat{\eta}/\partial T}{\partial\hat{e}/\partial T}\right| \left\{\left(\frac{\partial\hat{q}_k}{\partial\varepsilon_{mn}}\right)_{sym} \mathring{\varepsilon}_{mn,k} + \frac{\partial\hat{q}}{\partial g_m}\, g_{m,k} + \frac{\partial\hat{q}}{\partial\alpha_\gamma}\, \alpha_{\gamma,k} + \frac{\partial\hat{q}_k}{\partial T}\, g_k - \rho r\right\} \\
&\qquad + \rho\left(\frac{\partial\hat{\eta}}{\partial\alpha_\gamma} - \frac{\partial\hat{\eta}/\partial T}{\partial\hat{e}/\partial T}\, \frac{\partial\hat{e}}{\partial\alpha_\gamma}\right) f_\gamma - \frac{\hat{q}_k\, g_k}{T^2} \geq 0, \quad t = 0 \,.
\end{aligned} \tag{2.15}$$

All quantities in (2.15) except $y_{,m}$ are determined by initial conditions in the region D whereas $y_{,m}$ can be obtained from the solution of the linear P.D.E (2.10). However, if attention is restricted to material points close to the boundary surface B, the derivatives $y_{,m}$ can be expressed uniquely in terms of the initial temperature rate y^* and its surface derivatives $\partial y^*/\partial\zeta$, $\partial y^*/\partial\tau$ on B provided the surface B is non-characteristic [17]. Since this relation is linear and since the surface values are locally arbitrary so are the derivatives $(y_{,m})_B$. The same is true for the initial values of $\mathring{\varepsilon}_{mn}$ as well as $\varepsilon_{mn,k}$, $\alpha_{\gamma,k}$ and especially $g_{m,k}$ close to B. Therefore and because these quantities occur only linearly in the entropy inequality the dissipation postulate requires that their corresponding coefficients in (2.15) vanish and a residual inequality remains.

Actually, these conditions are derived for material points close to the boundary B at t = 0. Since the constitutive relations do not explicitly depend on time, the choice of the initial time is irrelevant. Further, the surface B may be chosen fairly arbitrarily except that it should be non-characteristic. Thus, the conditions apply at any time and any point in the body.

With the introduction of the free energy function $\hat{\psi}$ it may be shown finally that the above conditions agree exactly with (2.7) and (2.8).

It should be noted that the same results are obtained if the method of Lagrange's multipliers is used. This approach has been proposed and used by Liu [18] and Müller [14] to account for the balance equations which represent constraints on the evaluation of an entropy inequality. However, this formal technique conceales essential aspects of the argumentation.

It is obvious that the evaluation of the entropy inequality does not end here: The residual entropy inequality imposes restrictions on various constitutive functions. A concrete example is discussed in the next section.

3. EVALUATION OF THE RESIDUAL ENTROPY INEQUALITY OF A VISCO-PLASTIC MATERIAL MODEL

The Robinson model [15, 16] is essentially a mechanical visco-plastic material model with internal variables which has been extensively used at the Nuclear Research Centre Karlsruhe [19]. It has been developed without embedding it in a thermodynamic frame. However, in general it appears to be desirable that also a mechanical model should be interpretable as a special case of a more embracing thermomechanical model; possibly the thermodynamic requirements may impose restrictions even on the structure of the purely mechanical model. Therefore, a thermodynamic consistency analysis of the Robinson model was done; a detailed description is given in ref. [20]. Of course, such an analysis requires a thermodynamic extension of the model, i.e., an assumption about the structure of the free energy.

The formal structure of an early version [15] of the model is as follows. The elastic response is governed by Hooke's law

$$\sigma_{kl} = 2\mu\,\varepsilon_{kl}^{el} + \lambda\,\varepsilon_{mm}^{el}\,\delta_{kl} \tag{3.1}$$

where $\varepsilon_{kl}^{el} = \varepsilon_{kl} - \varepsilon_{kl}^{p} - \varepsilon_{kl}^{th}$ is the purely elastic strain, ε_{kl} is the total strain, ε_{kl}^{p} is the visco-plastic strain, and $\varepsilon_{kl}^{th} = A^{th}\,(T - T_o)\,\delta_{kl}$ is the thermal strain. The plastic strain is governed by the evolution law

$$\overset{\circ}{\varepsilon}{}_{kl}^{p} = \begin{cases} A\,F^{n}\,\dfrac{\Sigma_{kl}}{\sqrt{J_2}}\,, & F > 0 \;and\; S_{kl}\,\Sigma_{kl} > 0 \\ 0 \;, & F \leq 0 \;or\; S_{kl}\,\Sigma_{kl} \leq 0 \end{cases} \tag{3.2}$$

which involves the deviatoric back stress α_{kl} determined by the evolution law

$$\overset{\circ}{\alpha}_{kl} = H\,\overset{\circ}{\varepsilon}{}_{kl}^{p} - R\left(\frac{I_2}{\underset{\circ}{\kappa}^2}\right)^{m-\beta}\frac{\alpha_{kl}}{\sqrt{I_2}} \tag{3.3}$$

$$H = \begin{cases} \dfrac{h}{(I_2/\underset{\circ}{\kappa}^2)^{\beta}}\,; & I_2 > \underset{\circ}{I}_2 \;and\; S_{kl}\,\alpha_{kl} > 0 \\ \dfrac{h}{(I_2/\underset{\circ}{\kappa}^2)^{\beta}}\,; & I_2 \leq \underset{\circ}{I}_2 \;or\; S_{kl}\,\alpha_{kl} \leq 0\;; \end{cases} \tag{3.4}$$

here S_{kl} is the deviatoric stress, $\Sigma_{kl} = S_{kl} - \alpha_{kl}$ is the effective stress, $J_2 = \Sigma_{kl}\,\Sigma_{kl}/2$ and $I_2 = \alpha_{kl}\,\alpha_{kl}/2$ are the second invariants of the effective and the back stress, $F = J_2/\kappa^2 - 1$ is the yield function, and $\kappa = \underset{\circ}{\kappa} =$ const. is the drag stress assumed to be constant. Note that the second term in (3.3), i.e., the static recovery term, corresponds to a modification by Hornberger and Stamm [19]. The assumption of a constant drag stress implies that isotropic hardening is not modeled in this version. The quantities A, h, R, I_2, $\underset{\circ}{\kappa}$, n, m, β with $(m - \beta) > 0$ are positive temperature dependent material parameters.

Inspection shows that the evolution laws allow for <u>discontinuities</u> in the rates of the plastic strain and back stress. This peculiar property requires some extra considerations. The derivation of thermodynamic restrictions is usually based on the assumption that processes are smooth functions of time; here the local instant form of the Clausius-Duhem entropy inequality is an adequate starting point. However, the Robinson model requires a more general approach, i.e., a time integral formulation of the entropy inequality and the balance equations. In ref. [20] an analysis has been performed for a general internal variable model with <u>continuous</u> constitutive relations (2.1) but allowing for <u>discontinuous</u> evolution equations for the internal variables.

Within the frame of these assumptions it is shown that the time-integral entropy inequality

$$\left| \int_V \rho\eta dV \right|_{t1}^{t2} - \int_{t1}^{t2} \left\{ \int_V \frac{\rho r}{T} dV - \oint_A \frac{q_k n_k}{T} dA \right\} dt \geq 0$$

does not impose additional constitutive restrictions beyond those obtained when smooth processes apply, i.e., equ. (2.7) and (2.8).

Further, it is recalled that the dissipation inequality splits into the two separate inequalities for the mechanical and the thermal dissipation, i.e.,

$$\gamma_M = -\rho \frac{\partial \hat{\psi}}{\partial \alpha_v} f_v \geq 0, \quad \gamma_T = -\frac{\hat{q}_k g_k}{T} \geq 0, \tag{3.5}$$

if the evolution functions are independent of the temperature gradient as usually assumed.

To complete the description of the thermodynamic extension of the Robinson model an assumption about the free energy has to be made compatible with $(2.7)_1$ and (3.1). The most simple assumption is

$$\begin{aligned} \psi = \hat{\psi}(\varepsilon_k^e, \alpha_{mn}, T) = \frac{1}{\rho}\frac{1}{2}(2\mu\,\varepsilon_{kl}^e\,\varepsilon_{lk}^e + \lambda\,\varepsilon_{mm}^e\,\varepsilon_{nn}^e) \\ - \frac{1}{\rho}(3\lambda + 2\mu)A^{th}(T - T_o)\varepsilon_{mm}^e + \frac{1}{\rho} M \frac{1}{2}\alpha_{mn}\alpha_{nm} + \psi^T(T) \end{aligned} \tag{3.6}$$

where $\varepsilon_{kl}{}^e$ is the thermoelastic strain $\varepsilon_{kl}{}^e := \varepsilon_{kl} - \varepsilon_{kl}{}^P$. The purely thermal part ψ^T need not to be specified any further. This assumption essentially involves only one additional material parameter, namely M, which may be temperature dependent. Then it may be shown that the mechanical dissipation inequality $(3.5)_1$ takes the following form

$$\gamma_M = \gamma_P + \gamma_H + \gamma_R$$

$$\gamma_P := S_{mn}\,\mathring{\varepsilon}^P_{nm}, \quad \gamma_H := -MH\,\alpha_{mn}\,\mathring{\varepsilon}^P_{mn}, \quad \gamma_R := 2\,MR\left(\frac{I_2}{\kappa_o^2}\right)^{m-\beta} \sqrt{I_2} \tag{3.7}$$

γ_P is the plastic stress power which is, observing (3.2), a non-negative quantity. The contributions γ_H and γ_P are due to the hardening and the static recovery term in equ. (3.3). In the elastic range ($\mathring{\varepsilon}_{kl}{}^P = 0$) one gets $\gamma_M = \gamma_R \geq 0$ and this yields $MR \geq 0$. Since $R > 0$ one obtains

$$M \geq 0 \tag{3.8}$$

as a necessary and sufficient condition in the elastic range.

With $K_2 := \alpha_{mn}\,\Sigma_{mn}/2$, which is one of the simultaneous invariants of the tensor pair $(\alpha_{mn}, \Sigma_{mn})$, and with (3.2) the hardening contribution γ_H reads in the plastic region

$$\gamma_H = -MHA \frac{F^n}{\sqrt{J_2}} 2K_2 \begin{cases} \geq 0 \ if\ K_2 \leq 0 \\ < 0 \ if\ K_2 > 0 \ . \end{cases} \tag{3.9}$$

It is now obvious that only for $K_2 > 0$ the negative contribution γ_H may cancel out the other two positive contributions and γ_M may become negative. We introduce dimensionless invariants $\tilde{I}_2 = I_2/\kappa_o^2$, $\tilde{J}_2 = J_2/\kappa_o^2$, and $\tilde{K}_2 = K_2/\kappa_o^2$. These invariants are related by the Schwarz inequality

$$|\tilde{K}_2| \leq \sqrt{\tilde{I}_2}\sqrt{\tilde{J}_2} \quad or \quad |\tilde{K}_2| = \xi\sqrt{\tilde{I}_2}\sqrt{\tilde{J}_2},\ 0 < \xi \leq 1 \ . \tag{3.10}$$

Observing that only positive values of $\tilde{K}_2$ are of interest, the mechanical dissipation is a function of the variables $\tilde{I}_2, \tilde{J}_2$, and ξ whereas T is a parameter. The analysis shows that γ_M is definitely non-negative provided $(1 - H_{max} M) \geq 0$, $H_{max} = h/(\tilde{I}_2)^\beta$. However, this yields only a <u>sufficient</u> upper bound condition for the material parameter M. For a given set of values $\tilde{I}_2$, $\tilde{J}_2$ and with $(1 - H_{max} M) < 0$ a lower bound γ_{LB} for γ_M is obtained by setting $\xi = 1$; thus

$$\gamma_{LB} = \hat{\gamma}_M(\tilde{I}_2, \tilde{J}_2, \xi = 1, T) = \hat{\gamma}_{LB}(\tilde{I}_2, \tilde{J}_2, T) \leq \hat{\gamma}_M(\tilde{I}_2, \tilde{J}_2, \xi, T) \tag{3.11}$$

and this bound is actually attained for a state with $\xi = 1$. Therefore, it suffices to analyse the reduced inequality

$$\hat{\gamma}_{LB}(\tilde{I}_2, \tilde{J}_2, T) \geq 0, \quad \forall\, \tilde{I}_2 \geq 0,\ \tilde{J}_2 \geq 1 \ . \tag{3.12}$$

This result greatly simplifies the derivation since the analysis can be done in geometrical terms in a two-dimensional $(\tilde{I}_2, \tilde{J}_2)$-space for constant temperature. With $x := \sqrt{\tilde{I}_2}$, $y := \sqrt{\tilde{J}_2}$ the lower bound dissipation is

$$\gamma_{LB} = 2A\kappa_o(y^2 - 1)^n[y + (1 - H(x)M)x] + 2RM\kappa_o(x^2)^{m-\beta+\frac{1}{2}} = \check{\gamma}_{LB}(x, y),\ x \geq 0, y \geq 1 \ . \tag{3.13}$$

Since $(1 - H_{max}M)$ is assumed to be negative, the square bracket in (3.13) may become negative. Thus, a <u>critical region</u> exists in the (x, y)-plane where γ_{LB} may become negative. If it is assumed that the exponent β is restricted by $\beta \geq \frac{1}{2}$, in accordance with data for the Robinson model [15], then the shape of the critical region AB'B"CA approximates a triangle (Fig. 1) and is located around the "switch value" $x_o = \sqrt{\tilde{I}_{2o}}$. Along and outside the boundary of this region the lower bound dissipation is positive everywhere. The critical region vanishes if point C collapses into B' and this yields another improved sufficient upper bound for M.

An <u>necessary and sufficient condition</u> for the material parameters is derived by requiring that the absolute minimum value of the lower bound dissipation is non-negative in the critical region. However, classical extremal conditions cannot be simply applied since the derivative $\partial\check{\gamma}_{LB}/\partial x$ is discontinuous along B'C. But a qualitative geometrical picture of the function $\check{\gamma}_{LB}$ can be derived separately for the two regions B'B"CB' and AB'CA. A detailed analysis [20] shows that in region B'B"CB' the absolute minimum is located at $D(x_o, y_o)$ where y_o is obtained from the condition $(\partial\check{\gamma}_{LB}/\partial y)_{x_o} = 0$. In the critical region AB'CA the existence of a <u>relative</u> minimum cannot be excluded generally. However, if the exponents satisfy the condition$(m - \beta)/n < 1$, a requirement the data usually obey, a relative minimum does not exist. Consequently, the absolute minimum is either at A or D. But at A the lower bound dissipation is necessarily positive. Therefore, the <u>necessary and sufficient</u> condition for the material parameters is given by

$$\check{\gamma}_{LB}(x_o, y_o) \geq 0 \ . \tag{3.14}$$

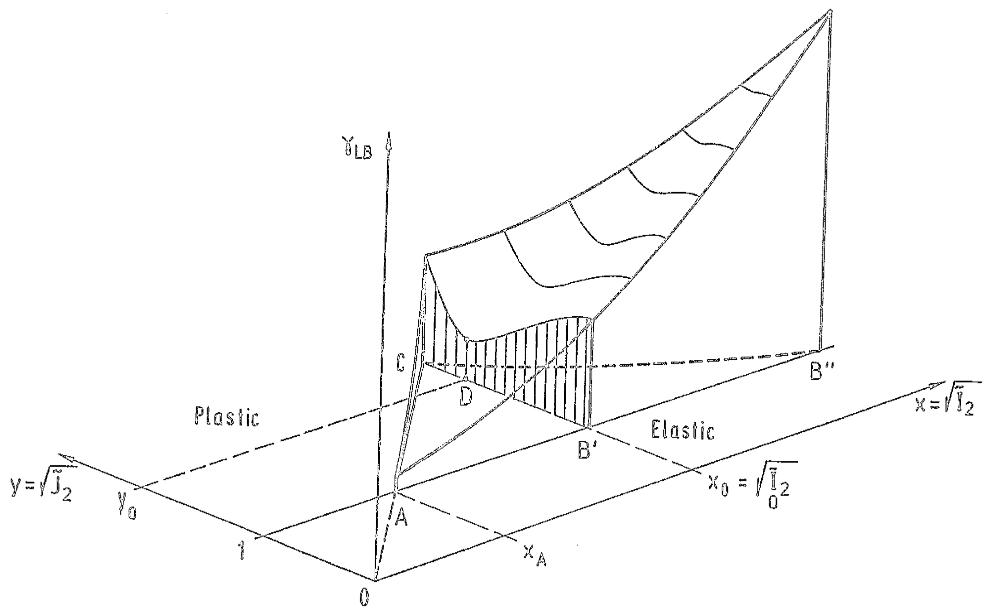

Fig. 1: Distribution of the Lower Bound Dissipation γ_{LB} in the Critical Region AB'B''CA

It should be noted that (3.14) involves only the temperature dependent material parameters.

The derivation of a strict upper bound criterion for the parameter M using (3.14) is a difficult task. However, another <u>improved sufficient</u> upper bound may be obtained from (3.14), which is given by

$$0 \leq M < \frac{x_o^{2\beta}}{h} \left\{ 1 + \frac{1+2n}{x_o 2n} \left| 1 + \left(\frac{R}{Ah}(1+2n) \right)^{1/n} x_o^{2-\frac{m}{n}} \right|^{1/2} - \frac{1}{x_o 2n} \right\} . \qquad (3.15)$$

Note that the r.h. s. contains only the parameters of the mechanical Robinson model. The sufficient upper bound (3.15) encloses the two other sufficient bounds mentioned above.

The results yield the following conclusion. There is a thermodynamically extended version of the Robinson model which satisfies the mechanical dissipation inequality whatever the parameters of the mechanical Robinson model are. The extension chosen here is not the only one but obviously the simplest. Since the Robinson parameters per se are found not to be subject to thermodynamic restrictions, the mechanical Robinson model is formally consistent with the entropy inequality.

For the test of the mechanical Robinson model isothermal purely mechanical experiments (e.g. creep, relaxation, etc.) are sufficient. However, for the test of the thermodynamically extended model caloric measurements are required too, and obviously the extended model may fail in these tests even if mechanical experiments are perfectly described.

These observations demonstrate that a thermodynamic consistency analysis of a purely mechanical model can only yield the assurance that the model is the isothermal derivative of <u>one or the other</u> thermodynamically extended (i.e. non-isothermal) model which is compatible with the entropy inequality.

The example also shows that a thermodynamic consistency analysis may be quite complicated: For more complex models closed form solutions like (3.14) or (3.15) are not likely to be obtained. This fact makes a frequently used assumption rather attractive: The introduction of the concept of thermodynamic forces and the non-negative, convex dissipation potential such that the evolution laws are expressed by a normality property; then the dissipation inequality is automatically satisfied [21]. However, as noted by Stamm [22], the Robinson model does not fit into this frame.

4. REMARKS ON MODELING THE THERMOMECHANICAL HISTORY DEPENCE OF ISOTROPIC HARDENING

A later version of the Robinson model [16] includes isotropic hardening. With $k = 3\kappa^2$ the corresponding evaluation law is assumed to be

$$\overset{\circ}{k} = \Gamma(W^P, T)\,\overset{\circ}{W}{}^P + \Theta(W^P, T)\,\overset{\circ}{T}, \qquad W^P = \int_0^t s_{kl}\,\overset{\circ}{\varepsilon}{}^P_{kl}\,dt \tag{4.1}$$

where Γ and Θ are functions of the absolute temperature T and the plastic work W^P. The temperature rate term models thermomechanical history dependence and thus equ. (4.1) is not integrable, i.e., $\partial\Gamma/\partial T \neq \partial\Theta/\partial W^P$. Equ. (4.1) represents a special case of more general evolution equations which involve the rates of all the external variables in a linear form. This general case has been treated by Lubliner [23] to some extend and a more specific analysis is presently performed by the author; some results with respect to the Robinson model are presented here.

The evaluation of the Clausius-Duhem entropy inequality with the Coleman-Noll argument and with the free energy $\psi = \hat{\psi}(\varepsilon_{kl}, \alpha_{m,n}, \kappa, T)$ yields a residual entropy inequality which is not affected by the temperature rate term in (4.1). However, this additional term destroys the classical potential relation for the entropy since the following restriction applies

$$\hat{\eta} = -\frac{\partial\hat{\psi}}{\partial T} - \frac{\partial\hat{\psi}}{\partial\kappa}\,\frac{\Theta}{6\kappa}$$

This also means that the Legendre-Fenchel transformation is not applicable here. One may raise the question whether a suitable transformation

$$k' = g(k, W^P, T) = 3(\kappa')^2 \tag{4.2}$$

exists which allows to reduce the evolution equation (4.1) to the form $k' = \Gamma'\overset{\circ}{W}{}^P$; then also the classical potential relation for the entropy is retained. Time differentiation of (4.2) yields

$$\overset{\circ}{k}{}' = \left(\frac{\partial g}{\partial k}\,\Gamma + \frac{\partial g}{\partial W^P}\right)\overset{\circ}{W}{}^P + \left(\frac{\partial g}{\partial k}\,\Theta + \frac{\partial g}{\partial T}\right)\overset{\circ}{T}\,. \tag{4.3}$$

If a reduction is possible then the condition

$$\frac{\partial g}{\partial k}\,\Theta + \frac{\partial g}{\partial T} = 0 \tag{4.4}$$

admits a solution. With the new variable

$$\chi = \int_{T_0}^{T} \Theta(W^P, T)\,dT = \hat{\chi}(W^P, T),$$

where T_0 is a reference temperature and W^P plays the part of a parameter, equ. (4.4) reads $\partial\tilde{g}/\partial k + \partial\tilde{g}/\partial\chi = 0$. This admits a general solution of the form $k = \tilde{g}(k,\chi) = w(k-\chi)$ where $w(\cdot)$ is an arbitrary function. Therefore, the simplest transformation is given by

$$k' = w(k - \chi) = g(k, W^P, T) = k - \int_{T_0}^{T} \theta(W^P, T)\, dT \,, \qquad (4.5)$$

and the transformed evolution equation is given by

$$\overset{\circ}{k'} = \left(\frac{\partial g}{\partial k} \Gamma + \frac{\partial g}{\partial W^P} \right) \overset{\circ}{W}{}^P = \left(\Gamma - \int_{T_0}^{T} \frac{\partial \theta}{\partial W^P} \, dT \right) \overset{\circ}{W}{}^P \,. \qquad (4.6)$$

Thus, the suitable transformation simply consists in a temperature and work dependent shift of the isotropic work hardening variable k and the classical potential for the entropy is preserved. It is noted, however, that the free energy then becomes an explicit function of the plastic work, if θ depends on W^P, a property which may raise objections. Finally, the above analysis on the reduceability of a non-standard evolution equation also makes evident that the evolution equations of the form (2.2) are not form-invariant when a transformation of the internal variables is done which involves the external variables.

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